

Interpreting Fuzzy Set Operations and Multi Level Agreement in a Computing with Words Context

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Abstract—Computing with Words (CWW) aims to investigate the possibility of imitating the unique ability of humans for approximate reasoning on the basis of approximately defined classes and concepts in the form of words. Type-2 fuzzy sets have been used to provide an adequate modeling basis for words in a fuzzy logic context. In the context of type-2 fuzzy sets employed as part of CWW, a variety of research efforts have been made to investigate approaches to model the meaning of specific words using type-2 fuzzy sets. In this paper we start by focusing on the interpretation of classical set-theoretical operations (complement, union and intersection) for crisp and type-1 fuzzy sets. We proceed by extending the interpretations to the results of the union and intersection operations of interval type-2 fuzzy sets, specifically indicating their effect on the uncertainty representation in the sets. We note the impact of the choice of t-norms and t-conorms in particular in the context of CWW applications where the interpretation of the resulting sets and its resemblance to the human intuitive meaning of the concept or word is essential. Finally, we provide the interpretation and reasoning behind the Multi Level Agreement (MLA) operation based on zSlices which was previously introduced and discuss the requirement for the selection of the right operations for the amalgamation of individual fuzzy sets and the potential for investigating this choice in particular in a CWW context.

Keywords: *Multi Level Agreement (MLA); zSlices; intersection; union; interpretation; Computing With Words (CWW)*

I. INTRODUCTION

Fuzzy Logic Sets were initially introduced in 1965 by Zadeh [1] in order to provide a framework “which may be of use in dealing with “classes”” that are imprecisely defined such as for example “the class of tall men” or “the class of beautiful women”. Since 1965, fuzzy sets and resulting fuzzy logic systems have become widely popular both in research terms as well as through a large number and variety of applications.

Further to the original concept of what is today referred to as type-1 fuzzy sets, Zadeh in 1975 introduced the concept of a linguistic variable and its application to approximate reasoning [2], providing the basis for what is referred to today as type-2 fuzzy sets and systems.

While the main focus of fuzzy logic (systems) in terms of their applications has been what could be called “traditional areas of fuzzy logic application” such as control applications (e.g.: [3],[4]), medical applications (e.g. [5]) and applications in business (e.g. [6]), in recent years a new body of research

centered around the concept of “Computing With Words” (CWW) has been raising interest (e.g. [7], [8] and [9]).

At the basis of CWW is the renewed effort of investigating the possibility of imitating the unique (in the sense of outstanding, especially compared to computers) ability of humans for approximate reasoning or more specifically the reasoning with, or on the basis of approximately defined classes and concepts in the form of words. Language and the use of “words” encapsulates this human ability and as such, the “word” or more specifically, *the meaning of a word* and how it can be represented, manipulated, stored and in a general sense be computed with (including for inference purposes) is a “hot” topic of research. Type-2 fuzzy sets and specifically interval type-2 fuzzy sets have been used to provide an adequate modeling basis for words in a fuzzy logic context as their Footprint Of Uncertainty (FOU) allows for expressing a degree of uncertainty about a specific degree of membership. In the context of type-2 fuzzy sets employed as part of CWW, a variety of research efforts have been made to investigate approaches to model the meaning of specific words using type-2 fuzzy sets based on (usually survey based) data (e.g. [10], [11] [12]).

In our recent work we have developed the notion of agreement in the sense of zSlices [13] based general type-2 fuzzy sets [14], [15] with a specific focus on its application in the context of Ambient Intelligent Environment (AIE) applications. As part of our research, we have investigated the combination of several interval type-2 fuzzy sets (originating from different sources (e.g.: people) but describing the same linguistic label or word) to form an “agreement” zSlices based set which aims to preserve the interpretation of the specific concept (word) by the individual while assigning higher importance to interpretations shared amongst multiple sources (e.g. individual people). While this work was application-oriented, it is clear that the application of fuzzy sets to model human-centered concepts such as words requires a far more general investigation into the actual meaning of words and how this meaning can be captured based on individual interpretations of a given meaning. Incidentally, the role of classes and concepts in human perception and decision making has been subject to considerable study in sociology and psychology. For example, recent work has considered the investigation into whether humans categorize (make categories) according to exemplar or rule-based theories [16].

If fuzzy logic is to model human concepts and words, it is not farfetched to believe that it is essential to investigate the formation of such concepts in humans in order to replicate fuzzy set based models that truly reflect the human notion of concepts and/or words.

It is clear that the adoption - of existing knowledge in - and cooperation with - other fields such as social and psychological sciences is a long term endeavor. In the current paper we, are aiming to present some early and in comparison basic findings based on the previously mentioned notion of agreement for zSlices based general type-2 fuzzy sets as well the more general question of the interpretation of the “aggregation” of multiple samples of a specific concept (i.e. through finding agreement, intersection, etc.). We are not addressing the question of how individual fuzzy sets can be designed (through manual or automatic means) in order for them to represent the meaning of a specific word or concept. Instead, we focus on the investigation of the operation(s) required to extract or represent meaning based on multiple interpretations (sets) of the same concept. As an example, consider the concept of “tall people” expressed as a fuzzy set. Further, consider that this concept would most likely be expressed differently depending on the subject expressing it, i.e. a basketball player is likely to have a different interpretation of the concept than a normal citizen. The questions we are investigating as part of this paper is the following:

- How can individual interpretations (one, two, ..., many) of a concept be “combined” in a useful way? I.e. how can the two interpretations of “tall men” be combined to get a better understanding of what “tall men” means – and what does “better” mean in this context?
- If the classical set theoretic operations union and intersection are employed, how can the resulting sets be interpreted? I.e. if we were to compute the intersection of the two fuzzy sets representing “tall men” – what would be the interpretation of the resulting set? (The latter is not straightforward – in particular for type-2 sets.)
- How can general type-2 fuzzy sets be employed to preserve information in “combined concepts” and how can for example zSlices based agreement sets be interpreted? I.e. If we were to employ zSlices based agreement to combine the two sets representing “tall men” – what would be the interpretation (meaning) of the resulting set?
- What is the scope to elicit existing / define new operations for (type-2) fuzzy sets that allow the “combination” of individual concept interpretations in different ways in order to imitate human concept creation? I.e. are there operations other than intersection and union that allow us to combine for example the two sets representing individual interpretations of “tall men” and is there a need or case for introducing new operations specifically for CWW?

In order to address the individual points, we will briefly review basic set theoretic operations and their interpretation as

part of Section II. Section III will focus on the interpretation of the intersection and union operations of interval type-2 fuzzy sets while Section IV will focus on the agreement operation based on zSlices and their interpretation, specifically in contrast to the interval type-2 operations. Subsequently, Section V provides a discussion of the individual points and the potential for the elicitation of other operations with the view to generating representative and meaningful fuzzy set representations from series of individual fuzzy set based interpretations, specifically in a CWW context. Section VI finally provides some conclusions and describes future work.

II. BASIC SET OPERATIONS AND THEIR INTERPRETATION

A. Overview

As fuzzy logic is an extension of classical set theory, the most fundamental operations are themselves extensions of the fundamental set theoretic operations of complement, union and intersection. In the current section we revisit each of these operations by referring to both crisp sets as well as type-1 fuzzy sets and briefly discuss their interpretation with the specific aim of investigating their potential for building “agreement fuzzy set model” which incorporate multiple individual fuzzy set sources.

B. The Complement Operation

In classical crisp sets, the standard complement $\bar{A}$ of a set A describes the set of all elements which are not part of A . In fuzzy logic, the complement $\bar{A}$ of a set A describes the membership for any element to not be part of A . Fig. 1 illustrates the complement operation for both crisp and type-1 fuzzy sets. Equation (1) describes the complement operation in terms of the membership values for type-1 fuzzy sets.

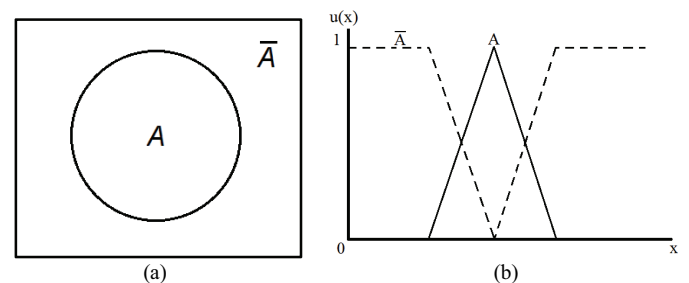


Figure 1. (a) The complement of a crisp set. (b) The complement of a fuzzy set (dashed).

$$u_{\bar{A}} = 1 - u_A \quad (1)$$

The complement operation is straightforward in the logical sense, in terms of its interpretation in a CWW context however it should be considered with caution. For example consider in Fig. 1 the set A describing the concept “tall people” as defined by an individual – a person. It is not clear that $\bar{A}$ in this case reflects that person’s interpretation of the concept “not tall people”. For instance, the person may only consider shorter people as “not tall”, while $\bar{A}$ includes for example includes “very tall people”. The investigation of this question of what the actual meaning is or should be doubtlessly involves the incorporation of existing knowledge from the areas of psychology and social sciences - we are not addressing it in further detail as part of this paper and we merely include the review of the complement operation for completeness.

C. The Union Operation

The union operation in classical set theory results in a new set that comprises all elements of the initial sets as shown in Fig. 2a. The union of two type-1 fuzzy sets A and B is described by taking the t-conorm of the sets' membership values. One possible t-conorm is the maximum operator as used in Equation (2) and shown in Fig. 2b.

$$u_A \vee u_B = \max(u_A, u_B) \quad (2)$$

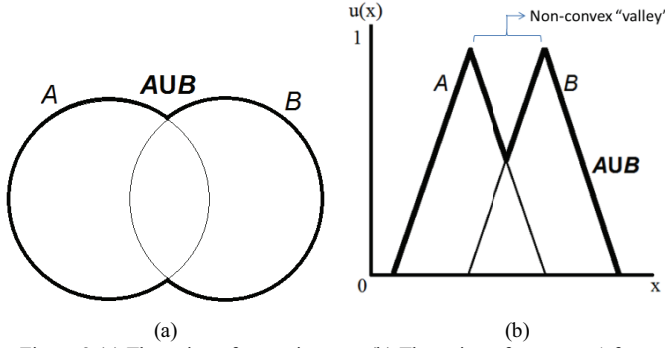


Figure 2 (a) The union of two crisp sets. (b) The union of two type-1 fuzzy sets.

In terms of interpretation, the union implements the logical OR, thus reflecting that any member of the union belongs to either (one or many) of its constituting sets. In fuzzy logic the union operation is generally applied to individual sets qualifying a specific linguistic variable such as *height of people*, where for example the union of the adjacent sets *tall* and *very tall* could be computed to reflect the membership of people who are *tall or very tall*.

In contrast to computing the union on different sets describing a linguistic label, in this paper we are focusing on the union of different instances of the same fuzzy set or more precisely, different interpretations (expressed through a fuzzy set) of the same linguistic label, for example: *tall people*. Further, we are specifically aiming to investigate how a useful and meaningful model of a specific label can be extracted from a series of individual models/interpretations. In other words – what is an appropriate general model for *tall people* based on the sample interpretations gathered?

In this context, where for example the type-1 fuzzy sets A and B in Fig. 2b describe different interpretations of the same concept (e.g. *tall people*), the union can be interpreted as a summary of all individual interpretations or opinions in terms of the meaning of the actual concept. While this may be useful, it is clear that as the number of “opinions” (i.e. fuzzy set interpretations of the concept) increases, the resulting model (the union of the individual sets) tends to grow in terms of its support. This in turn means that the precision of the model for *tall people* decreases or perhaps more appropriately the generality of the model increases. This effect may not be desirable as part of CWW applications as a very general concept is often less useful when attempting to reason based on specific data. For example, what information can be gathered from *tall people* if the fuzzy set model covers a very large range from 1 meter – 2.5 meters? Further, applying the union operation does not allow for the modeling of where more candidate sets are in agreement. For example, the resulting

union membership function does not allow differentiating between a union set formed by many similar sets or potential outliers. While one could aim to remedy this effect by pre-selecting candidate sets for *tall people* and removing outliers, we are aiming to investigate a more transparent approach to aggregating all individual interpretations while also modeling “agreement” between individual sets (i.e. to model where many sets overlap, rather than just few).

Finally, in terms of employing the union operation in CWW applications, it can be noted that in terms of its meaning / interpretation, it can be difficult to explain the usually resulting area of non-convexity or “valley” which is shown for example in Fig. 2b. Referring to our example of multiple interpretations of the concept “*tall people*”, one may expect that if A and B depict two interpretations, that their union should cover the entire space between the sets (thus incorporating the range between the individual interpretations). Interestingly, replacing the most common t-conorm *maximum* to implement the union operation by the equally valid but less popular Łukasiewicz t-conorm [17] (more precisely the dual to the Łukasiewicz t-norm) for strong disjunction shown in Equation (3) results in sets which generally more closely represent the expected result as shown for example for sets A and B in Fig. 3.

$$u_A \vee_L u_B = \min(1, (u_A + u_B)) \quad (3)$$

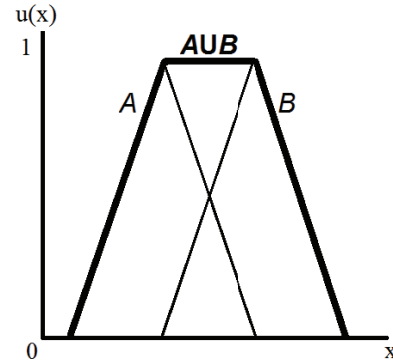


Figure 3 The union of two fuzzy sets when employing the Łukasiewicz t-conorm.

It should be noted that we are not recommending the replacement of the *t-conorm* maximum by the *Łukasiewicz t-conorm* in a general fashion. However, the example above serves the purpose to show that when considering the modeling of human concepts and words (as part of CWW), it may be appropriate to revisit the choice of t-conorms (and t-norms) with the aim of producing fuzzy sets which closely mimic the human interpretation of a given concept or word. In particular in type-2 fuzzy logic, less common t-conorms and t-norms have so far only been explored in a very limited fashion, in particular in CWW applications.

D. The Intersection Operation

While the union operation implements the logical OR, the intersection operation implements the logical AND, i.e. the conjunction operation. The intersection operation in classical set theory results in a new set that comprises the elements in common between the initial sets as shown in in Fig. 4a. The intersection of two type-1 fuzzy sets A and B is described by taking the t-norm of the sets' membership values. One possible

t-norm is the minimum operator as used in Equation (4) and shown in in Fig. 4b.

In terms of interpretation, the intersection operation extracts the “common ground” between two concepts or words. As such, following our example of two interpretations of the concept “*tall people*” expressed as sets A and B in Fig. 4, their intersection (also in Fig. 4) describes the set representing the area where both interpretations A and B “agree”, i.e. where both interpretations overlap.

$$u_A \vee u_B = \min(u_A, u_B) \quad (4)$$

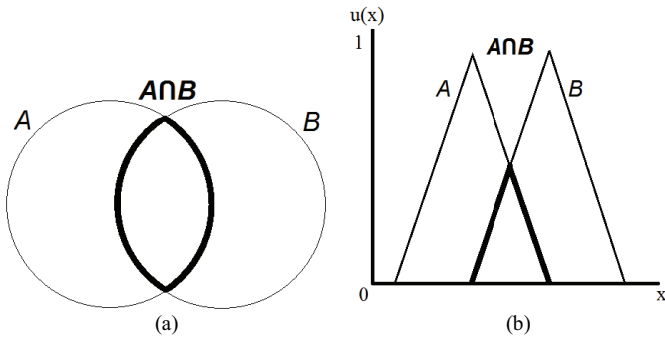


Figure 4 (a) The intersection of two crisp sets. (b) The intersection of two type-1 fuzzy sets.

Incidentally, Fig. 4 also indicates the common tendency of intersections to decrease generality in terms of support. In other words, the support of the intersection of two sets is always smaller or equal to the support of either set. This in turn can be interpreted as a “homing in” on the core meaning of the concept. In other words, the intersection in Fig. 4b can be interpreted as the core meaning of “*tall people*” in the sense that the interpretations of two individuals overlap (agree). If one imagines further sets (interpretations by other individuals), the intersection’s support will tend to decrease further in size.

As exemplified in Fig. 4, the intersection of two type-1 fuzzy sets tends to result in a set with low degrees of membership if the sets are significantly different. This reflects the intuitive notion that the common ground between two people with vastly differing opinions tends to be small. Reciprocally, in terms of employing the intersection to extract a global meaning of many individual interpretations of a specific concept (e.g. *tall people*) rapidly tends to result in an empty set if the sample sets show any significant degree of variation. Of course, one could identify and eliminate so called outliers but by doing so one specifically excludes individual interpretations which while convenient, may not reflect the globally valid meaning. In this context, it should be noted that what “the globally valid meaning” actually is, is itself an open question and quickly leads to research in democracy and the role of an individual in a society. We are not making any claims in the direction of what a “globally valid meaning” is in reality, but as part of this paper we are considering the “globally valid meaning” as a fuzzy set which incorporates all individual interpretations of a concept and at the same time reflects the potential common ground between several interpretations where it exists.

For completeness it should be noted that, analogous to the union operation, a selection of different t-norm operations are

available which slightly alter the resulting intersection and thus allow for the capturing of different interpretations.

E. Discussion

While the review of the complement, union and intersection operations can be considered trivial, their interpretation, in particular in a CWW context becomes highly significant. It lays the foundations of the idea of “agreement” as established using zSlices [15]. Specifically, considering the generalization properties of the union and the constraining properties of the intersection (in terms of the support of the resulting sets), it is clear how the latter two operations capture two very important aspects in the definition of a “globally valid meaning” in the form of a fuzzy set.

In the following section we proceed by considering the union and the intersection of interval type-2 fuzzy sets. We briefly discuss some at first glance non-intuitive properties of these basic operations when applied to type-2 fuzzy sets and provide our attempt at interpreting the sets resulting from these operations in a CWW context.

III. INTERPRETATION OF THE UNION AND INTERSECTION OPERATIONS FOR INTERVAL TYPE-2 FUZZY SETS

A. Overview

Both the union and intersection operations of interval type-2 fuzzy sets can be reduced to applying t-conorms and t-norms respectively to the upper and lower membership functions of the sets upon which the operations are to be performed [18]. Thus, it is reasonable to expect similar interpretations of the resulting sets as were described in Section II for type-1 fuzzy sets. However, while the basic properties such as the generalizing effect of the union (in terms of the support) hold both in the type-1 and in the interval type-2 case, the resulting effects on the fuzzy set Footprints Of Uncertainty (FOUs) are far less intuitive and warrant a directed effort to establish their interpretation. In the following two sections we review both the union and intersection operations for interval type-2 fuzzy sets and provide our attempt at their interpretations with a specific focus on the resulting FOUs.

B. The Union Operation for Interval Type-2 Fuzzy Sets

As noted, the union operation for interval type-2 fuzzy sets is based on computing the chosen t-conorm on the upper and lower membership functions of the specific sets. As expected, the resulting sets exhibit the same generalization tendency (i.e. a widening of the support) as was identified for their type-1 counterparts in Section II. This observation can also clearly be made in the two examples of union operations performed on different interval type-2 fuzzy sets provided in Fig. 5 and Fig. 6.

However, while it may be intuitive that the union of two interval type-2 fuzzy sets (i.e. the logical OR) should include the whole FOU of both initial interval type-2 fuzzy sets, Fig. 5 and Fig. 6 clearly indicate (by example) that this is clearly not the case.

The interpretation of this observation is not immediately straightforward. By definition, the FOU of an interval type-2 fuzzy set describes the uncertainty in terms of the degree of

membership for a given input. Consider for example the sets $\tilde{A}$ and $\tilde{B}$ as two interpretations of the concept *tall people*. As we noted, the union operation can generally be interpreted as a summary of both “opinions”, i.e. in classical sets the union includes all elements of the underlying sets. In interval type-2 fuzzy sets however, the result of the union operation does not necessarily include the lower membership functions (and thus associated values) of both sets (see Fig. 5 and Fig. 6). The most obvious interpretation of this is that the union operation does not extract a “summary” of the individual sets’ memberships which could be expected to include the full range from the lowest to the highest witnessed degrees of membership but actually extracts the highest degrees of membership.

In terms of our example of the union of two sets describing *tall*, the resulting set $\tilde{A} \cup \tilde{B}$ combines both individual interpretations. As a consequence the resulting model of *tall* becomes more general (wider support) and has a new uncertainty model (FOU) associated with it. Incidentally, the uncertainty (FOU) of the resulting set $\tilde{A} \cup \tilde{B}$ is generally neither the uncertainty of any one of the original interpretations (sets) but a combination of both uncertainty models.

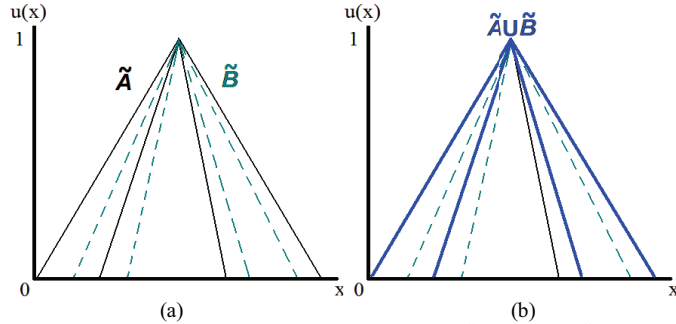


Figure 5 (a) Two example interval type-2 fuzzy sets $\tilde{A}$ (solid) and $\tilde{B}$ (dashed). (b) The union of $\tilde{A}$ and $\tilde{B}$ (bold).

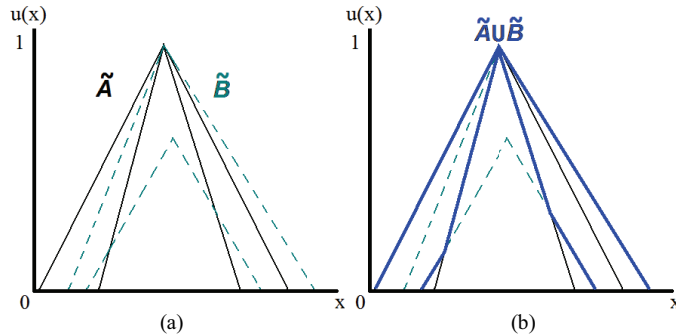


Figure 6 (a) Two example interval type-2 fuzzy sets $\tilde{A}$ (solid) and $\tilde{B}$ (dashed). (b) The union of $\tilde{A}$ and $\tilde{B}$ (bold).

In terms of CWW, the interesting question that arises is if such a recombination of “opinions” (expressed as interval type-2 fuzzy sets) indeed reflects the expected interpretation of “OR-ing” the two opinions. In the same context, to which extent can the union operation with its described results be employed as part of CWW to reflect the notion of agreement (where the incorporation of all “opinions” can be considered crucial) when computed over multiple instances (sets) describing the same concept?

The following section will review the intersection operation for interval type-2 fuzzy sets before we proceed and review the

zSlices based “agreement” operation in Section IV in order to address some of the posed questions.

C. The Intersection Operation for Interval Type-2 Fuzzy Sets

Analogous to the union operation, the intersection operation for interval type-2 fuzzy sets can be achieved by applying a t-norm operation to the upper and lower membership functions respectively of the given candidate interval type-2 fuzzy sets.

As noted in Section II, the intersection operation, reflecting the logical AND, can be interpreted as extracting the “common ground” between the individual interval type-2 fuzzy sets. We provide two examples of the intersection operation on two interval type-2 fuzzy sets as illustrated in Fig. 7 and Fig. 8.

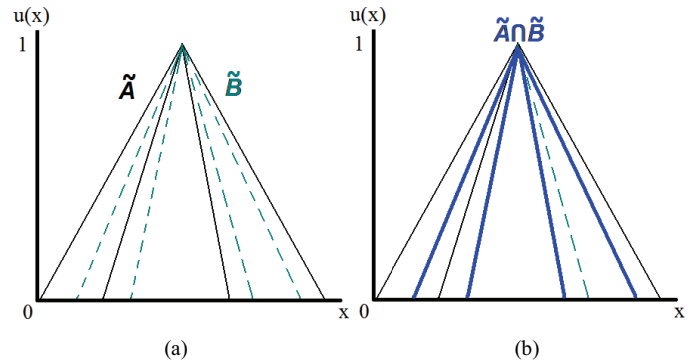


Figure 7 (a) Two example interval type-2 fuzzy sets $\tilde{A}$ (solid) and $\tilde{B}$ (dashed). (b) The intersection of $\tilde{A}$ and $\tilde{B}$ (bold).

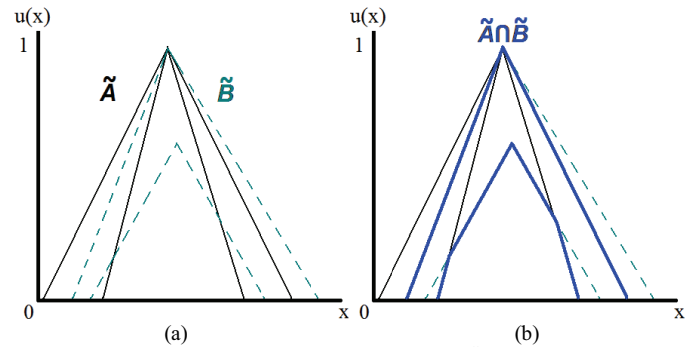


Figure 8 (a) Two example interval type-2 fuzzy sets $\tilde{A}$ (solid) and $\tilde{B}$ (dashed). (b) The intersection of $\tilde{A}$ and $\tilde{B}$ (bold).

While the fundamental interpretation of the intersection operation is straightforward and analogous to that of the intersection operation of type-1 fuzzy sets described in Section II, it is noteworthy that the results in terms of the FOU of the interval type-2 intersection are far less straightforward/intuitive to interpret. As can easily be seen in the examples in Fig. 7 and Fig. 8, the intersection operation does not result in a FOU that solely includes areas of membership that belong to both sets $\tilde{A}$ and $\tilde{B}$ but moreover it results in an FOU that ranges from the lowest lower membership function to the lowest upper membership function. To illustrate the difference in what could be considered an intuitive result of the intersection operation, consider Fig. 9 which depicts the “common ground” in terms of FOU of both fuzzy sets $\tilde{A}$ and $\tilde{B}$, while Fig. 7 shows the actual results of the intersection operation.

If we refer to our example of *tall people*, the computation of the intersection of two interpretations of this concept (i.e.

two interval type-2 fuzzy sets) could be expected to result in the overlap of the FOU's of the specific sets (as in Fig. 9). As shown, the actual result (Fig. 7) however is a set that while sharing the upper membership function with the expected result, preserves the lowest witnessed (in an input set) degree of membership.

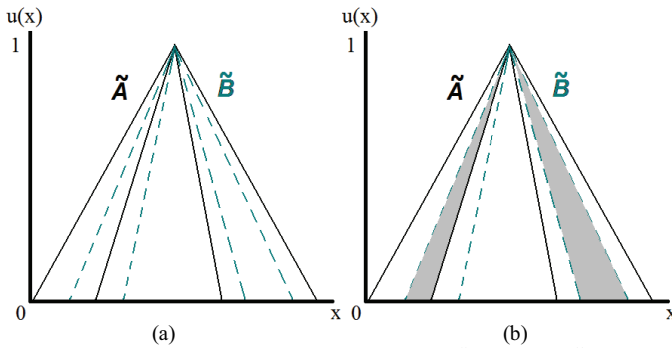


Figure 9 (a) Two example interval type-2 fuzzy sets $\tilde{A}$ (solid) and $\tilde{B}$ (dashed). (b) The potentially intuitive result of an intersection of the FOU's of $\tilde{A}$ and $\tilde{B}$ (shaded).

As for the union operation, the reason for pointing out this potential difference in the actual result of the intersection and the expected result is not to question the intersection operation as a mathematically correct operation but to provide an example of the increased complexity that results from using and particularly amalgamating (through set theoretic operations) higher order fuzzy sets. If such sets are used as part of CWW to represent and encode human-centered concepts such as words, it is essential that:

- The operations (such as intersection) employed result in sets that match our expectation – respectively reality (i.e. the concepts that are being modeled).
- We are able to accurately interpret the resulting sets in a given context.

Having explored the basic operations of union and intersection for interval type-2 fuzzy sets and briefly reflecting on their respective interpretations, we proceed in the following section to providing the underlying reasoning for introducing the agreement operation based on zSlices, followed by an overview of the actual operation and its interpretation.

IV. THE AGREEMENT OPERATION BASED ON zSLICES

A. Overview

zSlices based agreement was conceptually introduced in [14] (where it was largely based around overlapping FOU's as for example in Fig. 9) and recently developed further by basing it firmly on classical set theoretical operations of union and intersection [15] (as for example in Fig. 5 and Fig. 7). While both publications addressed the operation in terms of its functionality and applicability, in this paper we are reviewing the underlying concept and need for expanding on the “basic” amalgamation operations offered by interval type-2 fuzzy set operations (specifically the union and intersection as described in Section III) which underpins the creation of the “zSlices based agreement” operation. We briefly review the operation itself in Section IV.B and proceed by focusing on its interpretation in Section IV.C.

B. Review of zSlices based notion of Agreement.

While an initial approach to developing agreement focused on overlapping FOU's was mentioned in [14] (as shown in Fig. 9), in this paper we exclusively focus on the notion of Multi Level Agreement (MLA) based firmly on the sole use of the classical set-theoretic operations of union and intersection as detailed in [15]. It is worth mentioning that the two slightly different approaches were developed as a result of the discussion raised in this paper – the requirement to employ classical firmly established set theoretical operations versus the need to develop interpretable models of agreement.

MLA describes an operation which transforms a series of interval type-2 fuzzy sets into a zSlices based set. Each of the initial interval type-2 sets is generally *one individual interpretation* of a specific concept or word, such as *tall people*. MLA subsequently aims to provide a general model which reflects all individual interpretations while considering the agreement (common ground) between multiple individual interpretations (modeled in the third dimension). In other words, it aims to preserve all “interpretation inputs” but to weight specific areas on the final MLA output set in terms of the number of input sets that intersect. An MLA zSlices based set $\tilde{Z}_{MLA}$ based on K source interval type-2 fuzzy sets $\tilde{S}_k$, where $k \in \{1, \dots, K\}$ has a number of zLevels I equal to K (and as such zSlices $\tilde{Z}_i$, where $i \in \{1, \dots, I\}$). At low zLevels, the MLA set models the most general information provided by the individual interpretations while with increasing zLevel (third dimension), higher levels of agreement are represented. We do not repeat the full detail of the operation here for space reasons, but for a full account of the detailed steps of computing an MLA set, please see [15].

It is crucial to note the difference between MLA and employing interval type-2 fuzzy set based operations (such as union or intersection). The latter only allow for the focus on the common ground (intersection) or a generalization (union). Further, they do not allow for a modeling of the information in terms of the number of interpretations (interval type-2 sets) that are in agreement (i.e. a single outlier is weighted the same as a series of highly similar sets). MLA addresses both issues by combining union and intersection and utilizing the third dimension to weight areas of agreement where many interval type-2 sets (interpretations) are similar. As such it could be said that MLA aims to model the interpretation which the majority can agree on (and weighting it accordingly) but still preserving other interpretations (with an associated lower weight).

C. Interpretation of zSlices based Agreement.

In order to facilitate the interpretation of MLA we provide an example detailing the computation of an MLA model based on three individual “input” interval type-2 fuzzy set interpretations. The example sets have been reproduced from [15] as the in-depth review of the MLA sets in terms of their interpretation completes the theoretical background introduced in [15]. Fig. 10 depicts the three example “input” sets.

Fig. 11 shows the results of applying the basic operations of union and intersection to the “input” sets shown in Fig. 10a and Fig. 10b. The results of both operations have been included to reflect the review of the said operations in Section III and to

allow the contrasting of the resulting sets with the MLA set (Fig. 12a) computed on two interval type-2 “input” sets shown in Fig. 10a and Fig. 10b.

Fig. 12b finally shows the MLA set based on computing MLA on all three “input” sets shown in Fig. 10.

In order to interpret the MLA based sets shown in Fig. 12 it is worth recapitulating the MLA based sets are zSlices based and that higher zLevels (i.e. the third dimension) are to represent higher levels of agreement.

Considering the example of a MLA set in Fig. 12a computed on the “input” sets in Fig. 10a and Fig. 10b, it first of all becomes clear that the MLA set indeed reflects the union of both “input” sets as primary zSlices (front / dark blue). The union as a simple interval type-2 fuzzy set is provided for comparison in Fig. 11a. Further, as we have two “input” sets, the second zLevel should represent the agreement of both “input” sets which as part of MLA is modeled through the intersection operation. Considering the second zLevel (rear / light blue) in Fig. 12a and comparing it to the basic intersection shown in Fig. 11b it is clear that the MLA set reflects the expected intersection at the second zLevel.

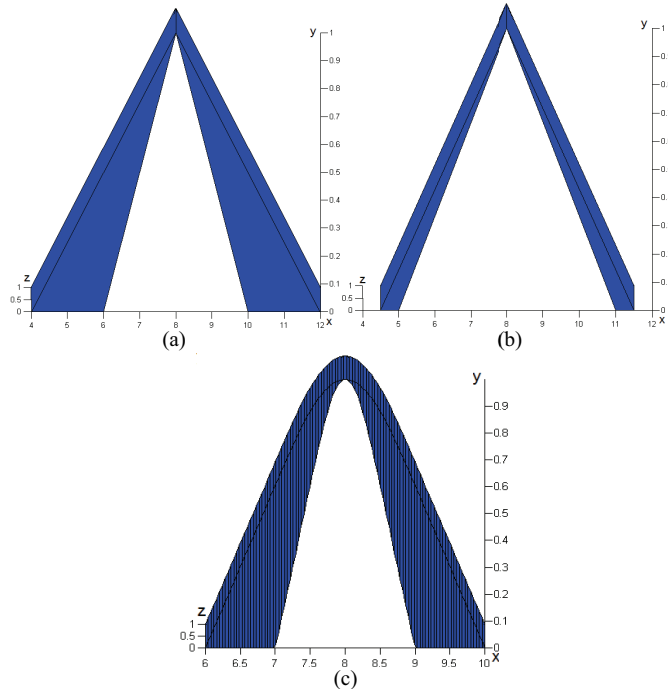


Figure 10 Three example interval type-2 fuzzy sets: (a), (b) and (c).

In terms of interpretation, the resulting MLA sets reflect the generalized amalgamation (union – see Section III.B) of the “input” sets at the lower zLevels while the common ground of the individual “input” sets is modeled (through the intersection – see Section III.C) at higher zLevels. With the zLevels acting as weights it is clear that “common ground” is weighted more heavily which is in line with the intuition that common ground between individuals should be reflected strongly in any model of agreement. The fact that levels of low “common ground”, i.e. the very general low zLevels incorporate all “input set” interpretations is also in line with the intuition that an

agreement model should not disregard interpretations made by only one or a few members.

Considering the common operations of union and intersection performed on interval type-2 fuzzy sets (Fig. 11), it is clear that the resulting interval type-2 sets cannot achieve both aims – the union by itself allows only to generalize while the intersection solely focuses on the common ground.

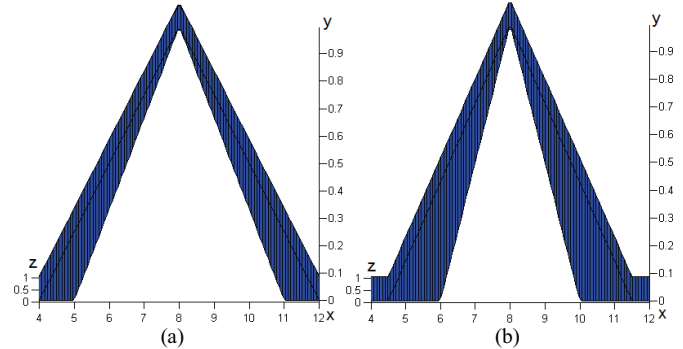


Figure 11 Interval type-2 sets: (a) Union of sets shown in Fig. 10a and Fig. 10b. (b) Intersection of sets shown in Fig. 10a and Fig. 10b.

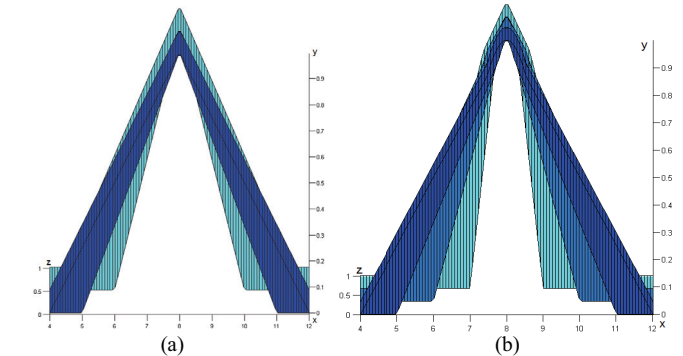


Figure 12 MLA zSlices based general type-2 sets. (a) MLA computed on sets shown in Fig. 10a and Fig. 10b. (b) MLA computed on sets shown in Fig. 10a, Fig. 10b Fig. 10c.

Finally, it is worth noting that the shape of the MLA sets as shown for two and three “input” interval type-2 fuzzy sets in Fig. 12a and Fig. 12b respectively reflects the basic properties of the underlying union and intersection operations (see Section III): a generalization effect and as such a widening of the support at lower zLevels and an increasing precision in terms of the concept (narrowing of support) as the zLevels increase.

V. DISCUSSION AND SCOPE FOR OTHER OPERATIONS.

This paper has mainly focused on the classical set theoretical operations of intersection and union and on their most common implementations – using the minimum t-norm and maximum t-conorms. These operations and their effect and have been shown and MLA has been interpreted being based on these classical implementations. However, as we indicated with the example of the Łukasiewicz t-conorm in Section II.C, it is clear that the common t-norms/t-conorms are by far not the only possible ones. Indeed, as was shown for the example of the Łukasiewicz t-conorm, other operations, or at the very least, the use of other t-norms and t-conorms to implement the classical set-theoretical operations of union and intersection promises great potential in the aim of actually reflecting the meaning of concepts as they are perceived by humans. In terms

of interval type-2 operations the use of different t-norm / t-conorm operations for the upper and lower membership functions could also be investigated.

As such, we believe that the choice of operation (union, intersection, ...) as well as the choice of the underlying t-norm and t-conorm can be an essential component in designing CWW applications. The selection of what the “right” operations are for which application and if there are common indicators across CWW applications promises to be a fruitful ground for future multi-disciplinary research, firmly based on collaboration between human-centred sciences such as the social sciences and the fuzzy logic field.

VI. CONCLUSIONS AND FUTURE WORK

As part of this paper we have reviewed the classical set-theoretical operations of complement, union and intersection for classical and type-1 fuzzy sets. Specifically, we have focused on investigating the interpretation of the latter two operations, laying the groundwork for subsequently considering the interpretation of said operations for interval type-2 fuzzy sets. We have briefly reviewed the concept of MLA based zSlices sets and have proceeded to detail their interpretation in the light of the interval type-2 operation interpretations that form the basis of MLA. We have indicated how MLA allows a more comprehensive representation of agreement compared to interval type-2 based operations through the weighting of highly similar interpretations (sets) in the third dimension while still preserving all, including infrequent interpretations (sets) at lower weights. By employing the third dimension, MLA can combine both the concept “narrowing” and “generalization” provided by the intersection and union operations.

Importantly, our efforts of interpreting the given operations and sets were focused on their use in CWW applications. As CWW aims to “compute with words”, it is paramount to investigate whether the individual sets and operations really reflect words, or more explicitly, whether they reflect the interpretation that is intuitively made by humans.

As such, this paper is not aimed at providing novel results or operations, it is merely aimed at:

- Providing details on the motivation of using MLA sets based on zSlices rather than interval type-2 fuzzy sets (based on either the union or intersection) for agreement modeling.
- Providing the interpretation of MLA sets in the context of the underlying operations.
- Indicating that the choice of operations and indeed the choice of implementation of operations (t-norm / t-conorm) has a significant impact on the resulting sets and interpretations. This is of particular importance in the context of CWW application which aim to imitate the human.

The scope of this paper has only allowed for a comparatively shallow investigation of the interpretations of the sets and the additional questions raised (such as the choice of t-norm / t-conorm). A remaining question is for example the application of different t-norms and t-conorms as part of MLA. Further, in the future we aim to deepen our investigation, in

particular through multi-disciplinary collaboration with the aim of advancing the goal of “imitating” human concepts and reasoning though the application of fuzzy logic in a CWW context.

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