

# Extensions of Tableau calculi for preference-based conditional logics

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**Abstract.** In this paper we extend the labelled tableau calculus for conditional logic **CE** proposed in [16]. We consider some extensions of **CE** defined by well-known semantic conditions/ conditional axioms. The proposed methods give a decision procedure for the respective logics and can be used to prove complexity bounds.

## 1 Introduction

In this paper we focus on extensions of conditional logic **CE** which can be considered as the basic logic of preferential structures [23, 24, 20, 10, 16]. We extend the calculus for **CE** proposed in [16] in order to deal with its most significant extensions. Our aim is to develop *modular* and uniform tableaux systems, where each semantic condition is captured exactly by one tableau rule (or a small group thereof). This work aims to be a step further into the development of proof systems for conditional logics, which are up to now rather scarce, [22, 19, 4, 2, 12, 8, 16, 26, 25] if compared to the development of proof systems for other families of logics, such as modal logics.

Conditional logics have been studied for a long time: they were first introduced by R. Stalnaker and D. Lewis [28, 23, 24] to formalize counterfactual, hypothetical reasoning (if  $A$  were the case then  $B$ ) that cannot be captured by material implication in classical logic. Conditional logics have been used in knowledge representation to reason about prototypical properties [13], to model database update [18], belief revision [3, 14], causal inference in action planning [27, 15] and diagnosis [13]. Moreover, conditional logics are strictly related to non-monotonic reasoning [20], as it turns out that all forms of inferences studied in the framework of non-monotonic logics are particular cases of conditional axioms [6]. The conditional logic **CE** closely corresponds to preferential logic **P** as defined in [20]: the latter coincides with the first-degree fragment of **CE** whereas rational non-monotonic logic corresponds to **CE**+CV, and so on.

Similarly to modal logics, the semantics of conditional logics can be defined in terms of possible world structures. The intuition is that a conditional  $A \Rightarrow B$  is true in a world  $x$  just in case  $B$  is true in the  $A$ -worlds that are *most similar/most preferred/closest* to  $x$ . This can be expressed by associating to each possible world  $x$  a preference relation  $<_x$  (an irreflexive and transitive relation)<sup>1</sup>. In preferential struc-

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<sup>1</sup> In [16], we took as primitive a pre-order relation  $\leq_x$ , whose strict version is  $<_x$ , clearly there is no significant difference.

tures, a conditional  $A \Rightarrow B$ , is true in  $x$  just in case  $B$  is true in all worlds satisfying  $A$  that are minimal with respect to the relation  $<_x$ . We assume that *minimal worlds always exist*, thereby accepting the so-called Limit Assumption (or Smoothness Condition as in KLM framework). The logic corresponding to this semantics is exactly the logic **CE** [16]. Stronger logics are obtained by imposing further conditions on the relations  $<_x$ . The preference semantics is related to the most popular semantics for conditional logics, namely the *sphere semantics* [23] and the *selection function semantics* [24]. The relation is investigated for some systems by Grahne [18].

In this work we concentrate on the extensions of **CE** that are obtained by adding any combination of conditions from the set of well known axioms **{MP, CV, CS, CEM}**. Our tableau systems give a feasible decision procedure for the respective logics and can be used to prove complexity bounds on them. In [16], we have given a terminating sound and complete tableau procedure for **CE** and we proposed some extensions of it. However these extensions were not entirely uniform and no termination analysis was carried out. The present work extends thus our previous results by giving terminating and uniform tableau calculi for the mentioned extensions of **CE** and we provide a non deterministic exponential upper bound for the time complexity of the tableau based decision procedure.

## 2 The system CE

Let  $\mathcal{L}$  be a language containing a set of propositional variables  $ATM$ , and the set of all formulas of  $\mathcal{L}$  built from  $ATM$  by means of the connectives  $\neg, \wedge, \perp, \Rightarrow$ , where  $\Rightarrow$  is the conditional operator.

**Definition 1 (Semantics of CE).** A **CE** model  $M$  has the form  $(W, \{<_x\}_{x \in W}, I)$ , where  $W$  is a non-empty set (of worlds),  $I$  is a function  $W \rightarrow Pow(ATM)$ , and  $\{<_x\}_{x \in W}$  is a family of relations on  $W$  for each element  $x \in W$ . For  $S \subseteq W$  we define the set of  $<_x$ -minimal elements of  $S$ :

$$Min_x(S) = \{a \in S \mid \neg \exists b \in S : b <_x a\}.$$

For every  $x \in W$ : (1)  $<_x$  is an irreflexive and transitive relation on  $W$ , (2) for every non-empty  $S \subseteq W$ ,  $Min_x(S) \neq \emptyset$ . We define the truth conditions of formulas wrt. worlds in a model  $M$  by the relation  $M, x \models \phi$ , as follows:

1.  $M, x \models p$ , for  $p$  atomic, if  $p \in I(x)$ ,
2.  $M, x \not\models \perp$ ,
3.  $M, x \models \neg \phi$  if  $M, x \not\models \phi$ ,
4.  $M, x \models \phi \wedge \psi$  if  $M, x \models \phi$  and  $M, x \models \psi$ .
5.  $M, x \models \phi \Rightarrow \psi$  if for all  $y \in Min_x(\phi)$ ,  $M, y \models \psi$ , where  $Min_x(\phi)$  stands for  $Min_x(\{y \in W \mid M, y \models \phi\})$ .

We say that  $\phi$  is valid in  $M$  if  $M, x \models \phi$  for every  $x \in W$ .

We say that  $\phi$  is **CE**-valid if it is valid in every **CE**-model.

The axiomatization corresponding to the previous semantics is the following axiom system **CE**.

**Definition 2 (Axiom system CE).** The system **CE** is defined by

(TAUT) All classical tautologies and the Modus Ponens rule.

(ID)  $\phi \Rightarrow \phi$

(CA)  $(\phi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi) \rightarrow (\phi \vee \psi \Rightarrow \chi)$

(CSO)  $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \rightarrow (\phi \Rightarrow \chi) \leftrightarrow (\psi \Rightarrow \chi)$

(A0)  $(\neg\phi \Rightarrow \perp) \rightarrow \phi$

(A1)  $(\neg\phi \Rightarrow \perp) \rightarrow \neg(\neg\phi \Rightarrow \perp) \Rightarrow \perp$

(A2)  $\neg(\phi \Rightarrow \perp) \rightarrow (\phi \Rightarrow \perp) \Rightarrow \perp$

(RCEA) if  $\vdash \phi \leftrightarrow \psi$  then  $\vdash (\phi \Rightarrow \chi) \leftrightarrow (\psi \Rightarrow \chi)$

(RCK) if  $\vdash (\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \chi$  then  $\vdash (\psi \Rightarrow \phi_1 \wedge \dots \wedge \psi \Rightarrow \phi_n) \rightarrow (\psi \Rightarrow \chi)$

The following theorems can be derived from the axioms of **CE**:

(AC)  $(\phi \Rightarrow \chi) \wedge (\phi \Rightarrow \psi) \rightarrow (\phi \wedge \chi \Rightarrow \psi)$

(RT)  $(\phi \Rightarrow \chi) \wedge (\phi \wedge \chi \Rightarrow \psi) \rightarrow (\phi \Rightarrow \psi)$

All **CE** axioms and theorems (except (A0), (A1) and (A2)) correspond to well-known properties of nonmonotonic systems: (AC) is called *cumulativity*, (RT) is called *non-monotonic cut* [20]. More precisely, the first-degree fragment of this logic corresponds to preferential logic **P**. Axioms (A0), (A1) and (A2) encode the assumption that each relation  $<_x$  ranges on the same set of worlds. This condition is called *Uniformity* in [10].

We can state the following theorems.

**Theorem 1 (Soundness).** If a formula  $\phi$  is derivable in **CE**, then it is **CE**-valid.

**Theorem 2 (Completeness).** If a formula  $\phi$  is **CE**-valid, then it is derivable in **CE**.

The proof of this last theorem is based on a generalization of Kraus Lehmann and Magidor's completeness proof for the system **P** provided in [20]. Full details of the proof will be provided in a long version of this paper.

## 2.1 Tableau calculus for CE

In [16], a tableau calculus for **CE** is proposed, which is based on the following ideas:

- 1 for all  $w$ , the fact that  $w \in \text{Min}_x(\phi)$  can be represented by extending the language  $\mathcal{L}$  with a kind of *hybrid* formulas. Given a model  $M = (W, \{<_x\}_{x \in W}, I)$ , we introduce pseudo-formulas of the form  $\Box_x \phi$ , for every formula  $\phi$  and world  $x \in W$ ;
- 2 the semantics of hybrid formulas is the following:  $M, y \models \Box_x \phi$  iff  $\forall z \in W$  if  $z <_x y$ , then  $M, z \models \phi$  (this can also be expressed by defining for each world  $x$  the accessibility relation  $R_x$  as the inverse of  $<_x$ :  $yR_x z$  iff  $z <_x y$ , and then adopting the standard semantics for  $\Box_x$ :  $M, y \models \Box_x \phi$  iff  $\forall z \in W$  if  $yR_x z$ , then  $M, z \models \phi$ );
- 3 from definition of  $\text{Min}_x$  in 1 above, it follows that for any formula  $\phi$ ,  $y \in \text{Min}_x(\phi)$  iff  $M, y \models \phi \wedge \Box_x \neg\phi$ .
- 4 We use these modal formulas in our tableau calculus for **CE** as illustrated in Figure 1. For instance: if  $M, x \models \phi \Rightarrow \psi$ , then for all  $y \in \text{Min}_x(\phi)$ ,  $M, y \models \psi$  (see definition 1 above). By point [3], this can be captured by saying that for all  $y$ , if  $M, y \models \phi \wedge \Box_x \neg\phi$  then  $M, y \models \psi$ . This is equivalent to saying that for all  $y$  either  $M, y \not\models \phi$  or  $M, y \not\models \Box_x \neg\phi$  or  $M, y \models \psi$ . This is exactly what is expressed by rule  $(T \Rightarrow)$  in Figure 1.

The tableau rules for **CE** are illustrated in Figure 1. The tableau formulas are of the following kinds: (a)  $x : \phi$ , where  $\phi$  is a formula or a pseudo-formula; (b)  $x <_y z$ . A branch is a set of tableau formulas. Given a branch  $B$ , we denote by  $W_B$  the set of labels occurring in  $B$ . A branch is closed if it contains both  $x : \phi$  and  $x : \neg\phi$ , or it contains  $x : \perp$ , otherwise it is open. A tableau is *closed* if every branch is closed. A formula  $\phi$  is *T-provable* if there exists a closed tableau for  $x : \neg\phi$ .

|                  |                                                                                                           |                  |                                                                                                               |
|------------------|-----------------------------------------------------------------------------------------------------------|------------------|---------------------------------------------------------------------------------------------------------------|
| $(T\wedge)$      | $\frac{x : \phi \wedge \psi}{x : \phi \quad x : \psi}$                                                    | $(F\wedge)$      | $\frac{x : \neg(\phi \wedge \psi)}{x : \neg\phi \mid x : \neg\psi}$                                           |
| $(T\vee)$        | $\frac{x : \phi \vee \psi}{x : \phi \mid x : \psi}$                                                       | $(F\vee)$        | $\frac{x : \neg(\phi \vee \psi)}{x : \neg\phi \quad x : \neg\psi}$                                            |
| $(T\rightarrow)$ | $\frac{x : \phi \rightarrow \psi}{x : \neg\phi \mid x : \psi}$                                            | $(F\rightarrow)$ | $\frac{x : \neg(\phi \rightarrow \psi)}{x : \phi \quad x : \neg\psi}$                                         |
| $(NEG)$          | $\frac{x : \neg\neg\psi}{x : \psi}$                                                                       |                  |                                                                                                               |
| $(T\Rightarrow)$ | $\frac{x : \phi \Rightarrow \psi}{y : \neg\phi \mid y : \neg\Box_x\neg\phi \mid y : \psi} \text{ old } y$ | $(F\Rightarrow)$ | $\frac{x : \neg(\phi \Rightarrow \psi)}{y : \phi \quad y : \Box_x\neg\phi \quad y : \neg\psi} \text{ new } y$ |
| $(T\Box)$        | $\frac{z : \Box_x\phi \quad y <_x z}{y : \phi}$                                                           | $(F\Box)$        | $\frac{y : \neg\Box_x\phi}{z <_x y \quad z : \neg\phi \quad z : \Box_x\phi} \text{ new } z$                   |
| $(Trans)$        | $\frac{y <_x z \quad z <_x u}{y <_x u}$                                                                   |                  |                                                                                                               |

**Fig. 1.** Tableau rules for **CE**

In [16], the following theorems are proved.

**Theorem 3 (Soundness).** *If  $\phi$  is T-provable then it is CE-valid.*

**Theorem 4 (Completeness).** *If a formula  $\phi$  is CE-valid then  $\phi$  is T-provable.*

The tableau calculus for **CE** as given in Figure 1 does not grant termination, since repeated applications of the rules  $(F\Rightarrow)$  and  $(F\Box)$  may generate infinite labels. In [16]

suitable restrictions on the applications of the rules  $(F \Rightarrow)$  and  $(F \Box)$  are introduced, which are shown to preserve the completeness of the calculus. Under such restrictions, the calculus is proved to be terminating.

### 3 Extensions of CE

The tableau method for **CE** can be extended uniformly to deal with some wellknown extensions of **CE** defined by combinations of the following axioms and corresponding semantic properties. We provide a set of extra rules for each extra axiom.

In order to obtain a calculus for conditional logics containing more than one axiom, it will be enough to consider the calculus containing all the rules introduced for the concerned axioms. Thus, for instance, in order to obtain a calculus for the logic obtained by adding to **CE** axioms (CV) and (MP) below, we will have to consider all the rules introduced for **CE** + (CV) and **CE** + (MP). In order to deal with Stalnaker's logic C2, containing **CE** plus axioms (CV), (MP) and (CEM) we will further have to consider the rules introduced to deal with (CEM), and so on.

We consider here the extensions of **CE** by axioms **(MP)**, **(CS)**, **(CEM)**, and **(CV)**:

- (MP)**  $((\phi \Rightarrow \psi) \wedge \phi) \rightarrow \psi$ ,  
its corresponding semantic property is (S-MP):  
 $\forall x, y \quad \neg(y <_x x)$ .
- (CS)**  $\phi \wedge \psi \rightarrow (\phi \Rightarrow \psi)$ ,  
its corresponding semantic property is (S-CS):  
 $\forall x, y \quad x <_x y \vee x = y$ .
- (CV)**  $(\phi \Rightarrow \psi) \wedge \neg(\phi \Rightarrow \neg\chi) \rightarrow (\phi \wedge \chi \Rightarrow \psi)$ ,  
its corresponding semantic property is (S-CV):  
 $\forall x, y, z, u \quad y <_x z \rightarrow (y <_x u \vee u <_x z)$ .
- (CEM)**  $(\phi \Rightarrow \psi) \vee (\phi \Rightarrow \neg\psi)$ ,  
its corresponding semantic property is (S-CEM):  
 $\forall x, y, z \quad y = z \vee y <_x z \vee z <_x y$ .

These axioms/semantic conditions have been variously considered in the literature on conditional logics. We give some intuitive justification. According to (S-MP) no world is preferred to  $x$ , with respect to  $<_x$ .

According to (S-CS), which is stronger than (S-MP),  $x$  is preferred to all other possible worlds with respect to its associated relation  $<_x$ . This condition is controversial. It is for instance the condition that differentiates Lewis's systems  $VC$  and  $VW$  [23], namely (S-CS) belongs to the first system whereas it does not belong to the second one.

The condition (S-CV) makes  $<_x$  a *modular* (or ranked) relation. This condition is required by most of the conditional logics based on preference semantics (or the related sphere based semantics), with the exception of **CE**. It is contained, for instance in Lewis's logics  $VC$  and  $VW$  [23].

The property (S-CEM) requires that given two different worlds  $y$  and  $z$ , either  $y$  is strictly preferred to  $z$  or  $z$  is strictly preferred to  $y$  w.r.t.  $<_x$ . In turn, this entails that for each formula  $\phi$ ,  $Min_{<}(\phi)$  contains at most one single possible world. This property

is the well-known Uniqueness Assumption that characterizes Stalnaker's logic  $C'2$  [28, 24].

Let  $\Sigma \subseteq \{(\mathbf{S-MP}), (\mathbf{S-CS}), (\mathbf{S-CV}), (\mathbf{S-CEM})\}$ , to ease the reading, we sometime say that a model  $M$  is a  $(\mathbf{S-\Sigma})$ -model if it satisfies the semantic conditions in  $\Sigma$ .

In order to provide the calculi for the extensions of  $\mathbf{CE}$ , we have to extend the language of the tableau calculus by allowing equalities of the form  $x = y$  in branches, where  $x, y$  are labels. This extension is needed for systems containing (CS) or (CEM). Consequently, we enrich the notion of closed branch as follows.

**Definition 3 (Closed Branch).** We say that a branch is closed if it contains: (i)  $x : \phi$  and  $x : \neg\phi$ , or  $x : \perp$ ; (ii)  $x = y, x : \phi, y : \neg\phi$ ; (iii)  $y <_x y$ . A tableau is closed if every branch is closed.

In order to deal with equality we need the following rules:

$$\begin{array}{cc}
 (\mathbf{E-R}) \frac{}{x=x} (*) & (\mathbf{E-S}) \frac{x=y}{y=x} \\
 (\mathbf{E-T}) \frac{x=y, y=z}{x=z} & (\mathbf{E-<}) \frac{x=x', y=y', z=z', y <_x z}{y' <_{x'} z'}
 \end{array}$$

**Fig. 2.** Rules for Equality. (\*)  $x$  is a label in the branch.

The rules corresponding to the mentioned extensions of  $\mathbf{CE}$  are the following ones:

$$\begin{array}{cc}
 (\mathbf{R-MP}) \frac{y <_x x}{x : \perp} & (\mathbf{R-CS}) \frac{}{x <_x y \mid x = y} \text{ old } x, y \\
 (\mathbf{R-CV}) \frac{y <_x z}{y <_x u \mid u <_x z} \text{ old } u & (\mathbf{R-CEM}) \frac{}{y = z \mid y <_x z \mid z <_x y} \text{ old } x, y, z
 \end{array}$$

**Fig. 3.** Rules for the extensions of  $\mathbf{CE}$ .

**Definition 4.** Let  $\Sigma \subseteq \{(\mathbf{R-MP}), (\mathbf{R-CS}), (\mathbf{R-CV}), (\mathbf{R-CEM})\}$  a  $\Sigma$ -tableau is a tableau built according to: (i) the rules for  $\mathbf{CE}$  of Figure 1, (ii) the rules in  $\Sigma$  of Figure 3, (iii) the rules for Equality in case  $(\mathbf{R-CS})$  or  $(\mathbf{R-CEM})$  belongs to  $\Sigma$ .

A formula  $\phi$  is  $\mathbf{T} + (\Sigma)$ -provable if there exists a closed  $\Sigma$ -tableau tableau for  $x : \neg\phi$ .

The soundness of the rules can be proved in a standard way (see Theorem 2 in [16]). To this purpose, we need to give a formal meaning to the notion of satisfiability of a

branch. Let  $M = (W, \{\leq_x\}_{x \in W}, I)$  be a model. Given a branch  $B$ , a mapping is a function  $f$  from  $W_B$  to  $W$  such that (i) if  $y <_x z \in B$  then  $f(y) <_{f(x)} f(z)$ , (ii) if  $x = y \in B$  then  $f(x) = f(y)$ .

**Definition 5.** Let  $\Sigma \subseteq \{(\mathbf{R-MP}), (\mathbf{R-CS}), (\mathbf{R-CV}), (\mathbf{R-CEM})\}$ , given a branch  $B$  of a  $\Sigma$ -tableau, a  $(\mathbf{S-\Sigma})$ -model  $M$ , and a mapping  $f$  from  $W_B$  to  $W$ , we say that  $B$  is *satisfiable under  $f$  in  $M$*  if the following holds:

1. if  $x : \phi \in B$  then  $M, f(x) \models \phi$ ,
2. if  $x : \neg\phi \in B$  then  $M, f(x) \not\models \phi$ .

where  $\phi$  is a conditional formula or a pseudo formula.

Moreover,  $B$  is  $(\mathbf{S-\Sigma})$ -satisfiable if it is satisfiable in some  $(\mathbf{S-\Sigma})$ -model  $M$  under some mapping  $f$ . A tableaux is  $(\mathbf{S-\Sigma})$ -satisfiable if one of its branches is  $(\mathbf{S-\Sigma})$ -satisfiable.

One can prove the following lemma<sup>2</sup>.

**Lemma 1.** Let  $\Sigma \subseteq \{(\mathbf{R-MP}), (\mathbf{R-CS}), (\mathbf{R-CV}), (\mathbf{R-CEM})\}$ , let  $T$  be a  $(\mathbf{S-\Sigma})$ -satisfiable tableau and let  $T'$  be obtained from  $T$  by applying one of the rules of  $\Sigma$ -tableaux. Then  $T'$  is also  $(\mathbf{S-\Sigma})$ -satisfiable.

**Theorem 5 (Soundness of  $T - \Sigma$ ).** If  $\phi$  is  $\mathbf{T} + (\Sigma)$ -provable, then it is valid in all  $(\mathbf{S-\Sigma})$ -models.

**Proof.** Suppose that  $\phi$  is not valid, then  $\neg\phi$  is satisfiable in a  $(\mathbf{S-\Sigma})$ -model  $M$ , thus the tableau beginning with  $x : \neg\phi$  is  $(\mathbf{S-\Sigma})$ -satisfiable. By the previous lemma any expansion of the tableaux contains a  $(\mathbf{S-\Sigma})$ -satisfiable branch  $B$ ;  $B$  cannot be closed, for otherwise we would have  $z : \psi, z : \neg\psi \in B$ , whence  $M, f(z) \models \psi$  and  $M, f(z) \models \neg\psi$ , a contradiction.  $\square$

### 3.1 Completeness of the calculi

First of all, notice that since none of the rules above introduce in the tableau new labels or new formulas that create new labels when decomposed, from Lemma 3, Section 4.2 of [16], we have the following lemma.

**Lemma 2.** Let  $B$  be a branch of a tableau.  $B$  does not contain any infinite descending chain of labels s.t.  $y_1 <_x y_0, y_2 <_x y_1, \dots, y_{i+1} <_x y_i, \dots$

We can now show the completeness of the calculi. To this purpose, we have to enrich the notion of saturated set used in the proof of the completeness of the calculus in Definition 5 in Section 4, of [16].

**Definition 6 (Saturated Branch).** We say that a branch  $B$  of a tableau is:

- saturated w.r.t.  $T$  if:
  1. If  $x : \phi \wedge \psi \in B$  then  $x : \phi \in B$  and  $x : \psi \in B$ .
  2. If  $x : \neg(\phi \wedge \psi) \in B$  then either  $x : \neg\phi \in B$  or  $x : \neg\psi \in B$ .

<sup>2</sup> The proof will be included in a full paper.

3. If  $x : \neg\neg\phi \in B$  then  $x : \phi \in B$ .
  4. If  $x : (\phi \Rightarrow \psi) \in B$  then for any label  $y \in W_B$ , either  $y : \neg\phi \in B$  or  $y : \neg\Box_x\neg\phi \in B$  or  $y : \psi \in B$ .
  5. If  $x : \neg(\phi \Rightarrow \psi) \in B$  then there is a label  $y$  such that  $y : \phi \in B$  and  $y : \neg\psi \in B$  and  $y : \Box_x\neg\phi \in B$ .
  6. If  $y : \Box_x\phi \in B$  and  $z <_x y \in B$  then  $z : \phi \in B$ .
  7. If  $y : \neg\Box_x\phi \in B$  then there is a label  $z$  such that  $z <_x y \in B$  and  $z : \neg\phi \in B$  and  $z : \Box_x\phi \in B$ .
  8. If  $y <_x z \in B$  and  $z <_x u \in B$  then  $y <_x u \in B$ .
- saturated w.r.t. = if:
    1. for all  $x \in B$ ,  $x = x \in B$ ;
    2. if  $x = y \in B$ , also  $y = x \in B$ ;
    3. if  $x = y \in B$  and  $y = z \in B$ , also  $x = z \in B$ ;
    4. if  $x = x' \in B$ ,  $y = y' \in B$ ,  $z = z' \in B$ , and  $y <_x z \in B$ , then also  $y' <_{x'} z' \in B$ ;
  - saturated with respect to **CE** + **(MP)** if it is saturated w.r.t. to  $T$ , and for no  $x, y$   $y <_x x \in B$ ;
  - saturated with respect to **CE** + **(CS)** if it is saturated w.r.t. to  $T$ , to  $=$ , and for all  $x, y \in B$ , either  $x <_x y \in B$  or  $x = y \in B$ ;
  - saturated with respect to **CE** + **(CV)** if it is saturated w.r.t. to  $T$ , and for all  $x, y, z, u \in B$ , if  $y <_x z \in B$ , then also  $y <_x u \in B$  or  $u <_x z \in B$ ;
  - saturated with respect to **CE** + **(CEM)** if it is saturated w.r.t. to  $T$ , to  $=$ , and for all  $x, y, z \in B$ , either  $y = z \in B$  or  $y <_x z \in B$  or  $z <_x y \in B$ .

Given a set of axioms  $\Sigma \subseteq \{(\mathbf{MP})(\mathbf{CS})(\mathbf{CV})(\mathbf{CEM})\}$ , we can now show the following lemma and completeness theorem.

**Lemma 3.** *If  $\phi$  is not  $\mathbf{T} + (\Sigma)$ -provable, then there exists a tableau which contains an open branch starting with  $x : \neg\phi$ , and which is saturated with respect to the rules corresponding to the axioms contained in  $\Sigma$ .*

**Proof.**(Sketch) Since  $\phi$  is not provable, all tableaux starting with  $\neg\phi$  will contain an open branch. Thus also the tableau built according to the following systematic procedure contains an open branch: **(step a:)** apply all the static rules as far as possible; in case of branching, make the choice which preserves non-closure (there is always one choice, by hypothesis); **(step b:)** apply the dynamic rules to the new formulas generated in the previous step.

By this strategy, every formula in the branch is eventually considered. It can be easily shown that the branch generated by this systematic procedure is saturated with respect to the rules corresponding to the axioms contained in  $\Sigma$ .  $\square$

**Theorem 6 (Completeness of  $T - \Sigma$ ).** *If  $\phi$  is valid in all  $(\mathbf{S}-\Sigma)$ -models, then it is  $\mathbf{T} + (\Sigma)$ -provable.*

**Proof.**(Sketch) We show the contrapositive: that if  $\phi$  is not  $\mathbf{T} + (\Sigma)$ -provable, then  $\neg\phi$  is satisfiable by a **CE** model satisfying **(S- $\Sigma$ )**. If  $\phi$  is not provable, then the tableau starting with  $x : \neg\phi$  contains an open branch  $B$ , saturated with respect to the rules corresponding to the axioms contained in  $\Sigma$ . Starting from this open, saturated branch,

we build a canonical model. We have two alternative constructions, one for the case in which neither axiom **(CS)** nor axiom **(CEM)** belong to  $\Sigma$ , and another corresponding to the case in which (at least) one of the two axioms belongs to  $\Sigma$  (and therefore the symbol  $=$  appears in  $B$ ).

**Case 1: (CS)  $\notin \Sigma$  and (CEM)  $\notin \Sigma$ .** We can build the canonical model  $M_C$  satisfying  $\neg\phi$ , as follows.  $M_C = \langle W, \{\sqsubset_x\}_{x \in W}, I \rangle$ , where: 1)  $W = W_B$ ; 2) For each  $x \in W$ ,  $y \sqsubset_x z$  iff  $y <_x z \in B$ ; 3) For each  $x \in W$ ,  $I(x) = \{p \mid p \in ATM \text{ and } w : p \in B\}$ . We prove that:

- 1  $\sqsubset_x$  is an irreflexive and transitive relation on  $W$ . This follows from definitions 3 and 6 above;
- 2 For all  $S \subseteq W$ ,  $S \neq \emptyset$ ,  $Min_x(S) \neq \emptyset$ . This follows from Lemma 2 above.
- 3 By induction over the complexity of the formulas, we can show that  $M_C$  satisfies all formulas and pseudo-formulas in  $B$ . As an example, we consider  $x : \phi \Rightarrow \psi \in B$ . Let  $y \in W_B$ . Then either (i)  $y : \neg\phi \in B$  or (ii)  $y : \neg\Box_x\neg\phi \in B$ , or (iii)  $y : \psi \in B$ . If (i) by the induction hypothesis, we have  $M_C, y \models \neg\phi$ , thus  $y \notin Min_x(\phi)$ . If (ii) there is a label  $z$  such that  $z <_x y \in B$  and  $z : \phi \in B$ . By construction of  $M_C$ , we have  $z \sqsubset_x y$  and by the induction hypothesis  $M_C, z \models \phi$ , thus  $y \notin Min_x(\phi)$ . If (iii) by the induction hypothesis we get  $M_C, y \models \psi$ . Thus, if  $y \in Min_x(\phi)$  then  $M_C, y \models \psi$ .
- 4 If **(MP)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-MP)**. For a contradiction, suppose  $y \sqsubset_x x$ . By definition of  $\sqsubset_x$ ,  $y <_x x \in B$ . But this contradicts definition 6 above.
- 5 If **(CV)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-CV)**. Let  $y \sqsubset_x z$ . By definition of  $M_C$ ,  $y <_x z \in B$ , and by definition 6, for all  $u \in W_B$ , either  $y <_x u$  or  $u <_x z$ . In the first case,  $y \sqsubset_x u$ ; in the second case  $u \sqsubset_x z$ .

**Case 2: (CS)  $\in \Sigma$  or (CEM)  $\in \Sigma$ .** We build the canonical model as follows: for all  $x, y \in W_B$ , we let  $x =_M y$  iff  $x = y \in B$ . Notice that  $=_M$  is an equivalence relation. Indeed: it is symmetric (all the rules that introduce  $x = y$  also introduce  $y = x$ ); it is transitive, by definition 6; it is reflexive by symmetry + transitivity.

We let  $[x] = \{y : x =_M y\}$  and  $W = \{[w] \mid w \in W_B\}$ , the set of all the equivalence classes of  $W_B$  with respect to  $=_M$ .

We define  $I([x]) = \bigcup_{y \in [x]} \{p : p \in ATM \text{ and } y : p \in B\}$ .

Furthermore, we let  $[y] \sqsubset_{[x]} [z]$  iff  $y' <_{x'} z' \in B$  for  $x' = x$ ,  $y' = y$ , and  $z' = z$ . We show that  $M_C$  is a **CE** model of  $B$  satisfying **(S-CS)**.

- 1  $\sqsubset_{[x]}$  is irreflexive and transitive. Irreflexive, by 3 and by definition of  $\sqsubset_{[x]}$ , and transitive by 6.
- 2 if  $S \subseteq W$  and  $S \neq \emptyset$ , then  $Min_x(S) \neq \emptyset$ . This follows from Lemma 2.
- 3 Reasoning by induction on the complexity of the formulas, we show that for all formula or pseudo-formula  $F$  (a) if  $x : F \in B$ , then  $M_C, [x] \models F$ , and (b) if  $x : \neg F \in B$ , then  $M_C, [x] \not\models F$ :
  - for  $F \in ATM$ : (a) follows from definition of  $I$ , and (b) follows from definition of  $I$  together with the fact that there is no  $x'$  s.t.  $x' = x \in B$ , and  $x : F \in B$  (otherwise  $B$  would be closed);
  - if  $F$  is a boolean combination of formulas, both (a) and (b) can be easily shown;

- if  $F$  is a pseudo-formula  $\Box_y \phi$ . (a): for all  $[z] \sqsubset_{[y]} [x]$ , by definition of  $M_C$  and by definition 6 above,  $z <_y x \in B$ . Hence,  $z : \phi \in B$ , and by inductive hypothesis,  $M_C, [z] \models \phi$ . (b): for some  $z, z <_y x \in B, z : \neg \phi$ . By definition of  $M_C, [z] \sqsubset_{[y]} [x]$ , and by inductive hypothesis,  $M_C, [z] \not\models \phi$ ;
  - if  $F$  is  $\phi \Rightarrow \psi$ . (a): for all  $y \in W_B$ , either (i)  $y : \neg \phi \in B$  or (ii)  $y : \neg \Box_x \neg \phi \in B$  or (iii)  $y : \psi \in B$ . In all cases, the result follows by inductive hypothesis. (b): there is a  $z$  s.t.  $z : \phi \in B, z : \Box_x \neg \phi \in B$ , and  $z : \neg \psi \in B$ . By inductive hypothesis,  $M_C, [z] \models \phi, M_C, [z] \models \Box_x \neg \phi, M_C, [z] \not\models \psi$ , and the result follows.
- 4 If **(CS)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-CS)**. Indeed, by definition 6, either  $x = y \in B$  (and  $[x] = [y]$ ) or  $x <_x y \in B$ , and  $[x] <_{[x]} [y]$ .
- 5 If **(CEM)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-CEM)**. Indeed, by definition 6, for all  $[x], [y], [z]$ , either  $y = z \in B$ , and by construction of  $M_C$   $[y] = [z]$ , or  $y <_x z \in B$ , and by construction of  $M_C, [y] <_{[x]} [z]$  or  $z <_x y \in B$ , and by construction of  $M_C, [z] <_{[x]} [y]$ .
- 6 If **(MP)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-MP)**. We reason analogously to Case 1.
- 7 If **(CV)**  $\in \Sigma$ , then  $M_C$  satisfies **(S-CV)**. We reason as for Case 1.  $\square$

### 3.2 Termination and Complexity of the Calculi

As in the case of logic **CE**, the calculi presented above for **CE** extensions can lead to non-terminating computations due to the interplay among the rules  $(F \Rightarrow)$  and  $(F \Box)$ , generating new labels, and rule  $(T \Rightarrow)$ , whose applications to the new labels may generate new formulas to which  $(F \Rightarrow)$  or  $(F \Box)$  can again be applied. In this section we show that the calculi for **CE** extensions can be made terminating. The solution we propose is similar to the one we have adopted to prove the termination of **CE** in [16]. More precisely, in order to avoid the generation of infinite branches, we introduce a systematic procedure to build a tableau and we put suitable restrictions on the applications of rules  $(F \Rightarrow)$  and  $(F \Box)$ . We show that the systematic procedure terminates, whence it does not lead to generate infinitely many labels. Moreover, we show that the completeness of the calculi is not lost if we adopt the systematic procedure with the mentioned restrictions.

First of all let us assume that the tableau construction does not allow any redundant application of the rules ([16]). In particular, we assume that the  $(T \Rightarrow)$  applies exactly once to each formula and label on the branch. Also, we assume that a branch does not contain repetitions of labelled formulas: labelled formulas which are already on the branch are not added again when applying new rules.

Following the terminology of [17], we refer to the rules generating new worlds as *dynamic rules*, and to all other rules as *static rules*. Observe that all the rules we have introduced are static except for  $(F \Rightarrow)$  and  $(F \Box)$  which are dynamic.

We define a *systematic procedure* for constructing the tableau for a given formula  $x : \alpha$ . This is the same procedure described in the proof of Lemma 3. The procedure executes repeatedly two steps: **(step a)** applies all the static rules as far as possible; **(step b)** applies the rules  $(F \Box)$  and  $(F \Rightarrow)$  to the new formulas generated in the previous step.

We observe that (step a) terminates after a finite number of rule applications. Indeed, none of the static rules adds new labels and the number of pseudo-formulas built from a

finite set of formulas (subformula property) and from a finite number of labels is finite. As we require that the tableau does not contain redundant applications of rules, each application of a rule in step (a) must at least add a new pseudo-formula to the branch, and thus the number of rule applications must be finite. After (step a) is terminated the branch is downward saturated except for the formulas of the form  $w : \neg(\phi \Rightarrow \psi)$  and  $w : \neg\Box_x\phi$ . In (step b) we apply the rule  $(F \Rightarrow)$  to all the formulas  $y : \neg(\gamma \Rightarrow \delta)$  on the branch and rule  $(F\Box)$  to all the formulas  $y : \neg\Box_x\gamma$  on the branch, where  $y$  must be a label generated in the previous (step a).

To avoid that (step a) and (step b) in the systematic procedure repeat forever, by continuing to generate new worlds, we put the following restrictions on the application of the rules  $(F \Rightarrow)$  and  $(F\Box)$ , which are essentially loop checking conditions.

- **(Restriction 1)** the rule  $(F \Rightarrow)$  can be applied to the formula  $y : \neg(\gamma \Rightarrow \delta)$  on a branch only if there is no label  $z$  on the branch such that:
  - (1a) the branch contains the formula  $z : \neg(\gamma \Rightarrow \delta)$  and the rule  $(F \Rightarrow)$  has already been applied to that formula;
  - (1b) the positive conditional formulas labelled by  $y$  are a subset of the positive conditional formulas labelled by  $z$ .
- **(Restriction 2)** the rule  $(F\Box)$  can be applied to the formula  $y : \neg\Box_x\gamma$  on a branch only if we cannot find two labels  $z$  and  $w$  on the branch such that:
  - (2a) the branch contains a formula  $z : \neg\Box_w\gamma$  and the rule  $(F\Box)$  has already been applied to that formula;
  - (2b) the positive conditional formulas labelled by  $x$  on the branch are a subset of the positive conditional formulas labelled by  $w$ ;
  - (2c) for each (positive) modal pseudo-formula  $y_1 : \Box_x\phi$  occurring on the branch, such that either  $y_1$  coincides with label  $y$  or one of the two relations  $y <_x y_1$  and  $y_1 = y$  occurs on the branch, there is a (positive) modal pseudo-formula  $z_1 : \Box_w\phi$  on the branch, such that either  $z_1$  coincides with label  $z$  or one of the two relations  $z <_w z_1$  and  $z_1 = z$  occurs on the branch.

The idea behind (Restriction 1) is that applying the rule  $(F \Rightarrow)$  to  $y : \neg(\gamma \Rightarrow \delta)$  cannot add anything more on the branch than what has been obtained by applying it to  $z : \neg(\gamma \Rightarrow \delta)$ , if all the positive conditional formulas that hold at  $y$  also hold at  $z$ . In such a case we can avoid to generate a new label  $y'$  from  $y$ . The idea behind (Restriction 2) is that applying the rule  $(F\Box)$  to  $y : \neg\Box_x\gamma$  cannot add anything more on the branch than what has been obtained by applying it to  $z : \neg\Box_w\gamma$ , under the assumptions that: all the positive conditional formulas that hold at  $x$  also hold at  $w$ ; and all the positive modal pseudo-formulas on the branch that would be applicable to the world  $y'$  generated from  $y$ , can also be applied to the world  $z'$  generated from  $z$ .

The systematic procedure with the two restrictions preserves completeness and guarantees termination.

**Lemma 4.** *If a branch  $B$  closes, it still closes under (Restriction 1) and (Restriction 2).*

**Theorem 7.** *The systematic procedure with (Restriction 1) and (Restriction 2) terminates.*

The proofs of Lemma 4 and of Theorem 7 are similar to those reported in [16] for logic **CE**. The intuition is the following one. If a systematic attempt at proving a formula goes forever, there must be an infinite branch  $S$  containing infinitely many different labels because it does not contain repetitions and only a finite number of formulas can appear in the tableau (subformula property). Therefore, the branch must either contain infinitely many applications of rule  $(F \Rightarrow)$  or infinitely many applications of rule  $(F \Box)$  to some formula (or both). By (Restrictions 1) and (Restriction 2) above, this is not possible. For instance, each time the rule  $(F \Rightarrow)$  is applied to  $y : \neg(\phi \Rightarrow \psi)$ , by (Restriction 1) for all possible  $z : \neg(\phi \Rightarrow \psi)$  on the branch to which  $(F \Rightarrow)$  has already been applied there must be some positive conditional formula occurring on the branch labelled by  $y$  and not by  $z$ . This can only occur a finite number of times, since (by the subformula property) there is only a finite number of positive conditionals.

Observe that the number of positive conditionals is finite, and at most linear in the size  $n$  of the input formula. Hence, the number of distinct sets of positive conditionals is  $O(2^n)$ . Therefore, the number of applications of rule  $(F \Rightarrow)$  to the same formula (but different labels) on a branch is  $O(2^n)$ . As there are  $O(n)$  distinct positive conditionals, the number of applications of rule  $(F \Rightarrow)$  on a branch is  $O(2^n)$ . By similar considerations, we can conclude that the number of applications of rule  $(F \Box)$  on a branch is  $O(2^n)$ . We can thus say that the number of different labels on the branch is  $O(2^n)$ .

In order to establish an upper bound on the complexity of the systematic procedure above, we need to estimate the overall number of rule applications, including those in step (a). To this concern, we can observe that the number of pseudo-formulas (including equalities and disequalities) which can be built from  $O(n)$  formulas and  $O(2^n)$  labels is  $O(2^n)$ . As we do not admit redundant applications of rules, on each branch the overall number of tableau rule applications determined by the systematic procedure is  $O(2^n)$ . Thus, our systematic procedure allows us to define a non-deterministic algorithm for testing satisfiability in **CE**: at each step the algorithm first saturates with respect to the static rules (step a), then guesses a branch and saturates it w.r.t. the dynamic rules (step b). By what said just above, the algorithm eventually terminates, and returns "satisfiable" if it finds an open branch and "unsatisfiable" if it does not. Since the number of rules applied is  $O(2^n)$ , we can conclude that:

**Theorem 8. Theorem 9.** *For all extensions  $\mathbf{CE} + \Sigma$  of **CE**, the systematic procedures decides the satisfiability of a formula in non-deterministic exponential time with respect to the size of the input.*

The above result seems to be in partial accordance with [10], where the authors show that a formula is satisfiable iff it is satisfiable in an exponentially sized structure. Moreover, the authors argue that deciding satisfiability can be done in EXPTIME (that we understand as Deterministic Exponential Time), even if the algorithm they suggest is not described in full details (Theorem 0.13). In future investigation we will study how we can obtain a *deterministic* EXPTIME decision procedure based on our tableau calculus.

## 4 Conclusions

In this work we have presented tableaux systems for the most important extensions of **CE**, the logic of preferential structures. The proof methods are uniform in the sense

that each specific semantic condition is captured by a tableau rule. Moreover, the proof method is based on the introduction of pseudo-formulas, that are modalities in a hybrid language indexed on worlds. We have been able to obtain terminating procedures by a loop-checking mechanism. These methods can also be used to establish complexity bounds of the logics, in a constructive way.

In future work, we intend to carry on a finer grained complexity analysis of **CE** extensions by means of tableaux methods. Moreover, we intend to investigate several improvements on the tableau systems, including loop-checking mechanisms, a better treatment of equality via label-substitution and specific proof-search strategies.

We briefly remark on some related works. De Swart [8] and Gent [12] give sequent/tableaux calculi for conditional logics VC (= **CE**+CV+MP+CS) and VCS. The kind of systems they propose are based on the entrenchment connective  $\leq$ , from which the conditional operator can be defined. Their systems are analytic and comprise an infinite set of rules  $\leq F(n, m)$ , with a uniform pattern, to decompose each sequent with  $m$  negative and  $n$  positive entrenchment formulas.

Crocco and Farinas [4] present sequent calculi for some conditional logics including minimal CK, CEM, CO (= **CE** without CA) and others. Their calculi comprise two levels of sequents: principal sequents with  $\vdash_P$  corresponds to the basic deduction relation, whereas auxiliary sequents with  $\vdash_a$  corresponds to the conditional operator: thus the constituents of  $\Gamma \vdash_P \Delta$  are sequents of the form  $X \vdash_a Y$ , where  $X, Y$  are sets of formulas.

Artosi, Governatori, and Rotolo [2] develop labelled tableau for the *first-degree* fragment (i.e. without nested conditionals) of conditional logic CO (they call it CU and it corresponds to cumulative non-monotonic logics). Formulas are labelled by path of worlds containing also variable worlds. Since they adopt a selection function semantics, they have to cope with the problem of equivalent antecedents: i.e. if  $[A]^M = [A']^M$  then  $f(A, w) = f(A', w)$ . They use an efficient unification procedure to propagate positive conditionals, and the unification procedure takes care of checking the equivalence of antecedents. Their tableau system is not analytical, as it contains a cut-rule, called PB, which is not eliminable. Moreover it is not clear how to extend it to **CE** and stronger systems on the one hand, and to nested conditionals, on the other.

Broda, Gabbay, Lamb, and Russo develop an elegant natural deduction system for Boutilier's conditional logic of normality and some variants of it [3], [1]. Their proof system uses labels following the methodology of *Labelled Deductive Systems* [1], where the objects involved in the proofs are structured configurations of formulas, worlds, and relations thereof. In this respect their approach is rather similar to ours. However, as we already observed, Boutilier's conditional logic has a simpler semantics defined in terms of standard modal logic without world-indexed relations or modalities (and thus it cannot handle nested and iterated conditionals). Moreover, it is not evident if one can extract a decision procedure for Boutilier's logic from their natural deduction system.

Lamarre [22] presents tableaux systems for conditional logics V(= **CE**+CV), VN, VC and VW(= **CE**+CV+MP). Lamarre's method is a consistency-checking procedure which tries to build a system of sphere falsifying the input formulas. The method makes use of a subroutine to compute the *core*, that is defined as the set of formulas characterizing the minimal sphere. The computation of the core needs in turn the consistency

checking procedure. Thus there is a mutual recursive definition between the procedure for checking consistency and the one to compute the core.

Gronebner and Delgrande [19] have developed a tableau method for conditional logic VN which is based on the translation of this logic into modal logic S4.3.

In [26], it is presented a labelled sequent calculus minimal normal conditional logic **CK** and some extensions of it. The calculi are based on the selection function semantics for these logics. In case of **CK**, the calculus is used to provide a polynomial-space complexity bound for this logic.

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