

Proof Terms for Infinitary Rewriting

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Abstract. We generalize the notion of proof term to the realm of transfinite reduction. Proof terms represent reductions in the first-order term format, thereby facilitating their formal analysis. Transfinite reductions can be faithfully represented as infinitary proof terms, unique up to infinitary associativity. We use proof terms to define equivalence of transfinite reductions on the basis of permutation equations. A proof of the compression property via proof terms is presented, which establishes permutation equivalence between the original and the compressed reductions.

Keywords: infinitary rewriting, proof terms, permutation equivalence

1 Introduction

We study infinitary left-linear rewriting, based on the notion of strong convergence, as developed in [5]. Proof terms denoting reductions for finitary rewriting have been introduced in [11], Chap. 8. The use of proof terms marks a shift of attention from reduction *as a relation* between terms, towards reductions *as objects* by themselves. Proof terms represent reductions in the first-order term format, thereby facilitating their formal analysis. For example, proof-theoretic analysis, equational reasoning, and even rewriting techniques, can now be applied in the study of properties of rewriting such as standardisation and the like. This use of proof terms in finitary rewriting has been well developed. See [11], Chaps. 8 and 9.

The main objective of our work is to generalize the notion of proof term to the realm of infinite reduction, and to use this to define and study the notion of permutation equivalence for transfinite reduction. The representation of transfinite reductions by proof terms can be exploited to yield transparent proofs of results like standardisation and compression. As a matter of fact, strong versions of these theorems: the standardised or compressed reductions thus obtained are permutation equivalent to the original ones. The compression proof will be presented here, standardisation is beyond the scope of this paper.

We will prove in Section 5 that any transfinite reduction can be faithfully represented as an infinitary, so-called *stepwise* proof term. Moreover, the original reduction can be reconstructed from the stepwise proof term: the representation is injective. The proof terms formalism is broader, however. Apart from sequential reduction, also parallel and multi-step reduction can be represented.

An alternative, *co-inductive* approach to the study of infinitary rewriting, was proposed in [3]. Proof objects emerge there as witnesses in the co-inductive

characterisation of the reduction relation. Their focus, however, is on techniques for proving properties of the reduction *relation*, rather than the fine structure of the space of transfinite reductions, which is our primary interest. We use *inductive* techniques. Although a term can be infinite, the distance of each symbol to its root is finite, allowing for inductive reasoning about occurrences in a term. Likewise, transfinite induction can be used to reason about infinite reduction sequences, of which the length can always be expressed as an ordinal.

An extended version of the present article, containing full details of the definitions and proofs, is available as [8].

2 Preliminaries

We assume acquaintance with term rewriting, both finitary and infinitary, and with ordinal numbers. As background sources on rewriting we mention [1], [11], [5], [7]. The following survey fixes some notations and terminology.

Ordinals Properties of countable ordinals (cf. e.g. [10]) are extensively used in this work. Especially we will make use of the following cofinality result.

Lemma 1. *Let α be a countable limit ordinal. Then there exists a sequence of ordinals $\langle \alpha_i \rangle_{i < \omega}$ such that $0 < \alpha_i < \alpha$ for all $i < \omega$, and $\alpha = \sum_{i < \omega} \alpha_i$.*

Positions, terms A *position* is a finite sequence of elements of $\mathbb{N}_{>0}$. We will use p, q, r for positions. The length of a position p is called its *depth*, denoted $|p|$. The empty position is denoted by ϵ . Concatenation of positions is denoted by an infix dot, i.e. $p \cdot q$. The dot will be mostly omitted. The *prefix order* on positions is denoted by $\leq$, i.e. $p \leq q$ iff $q = pq'$.

A *tree domain* (cf. [2]) is a set of positions P , such that $P \neq \emptyset$, P is prefix-closed, and $pj \in P$ and $i < j$ imply $pi \in P$. Note that a tree domain can be infinite. However, the depth of any position in a tree domain is finite. A *signature* Σ is a function from a set of symbols to $\mathbb{N}_{\geq 0}$; the value of this function for a symbol f is called its *arity*, denoted as $ar(f)$. We write $f/m \in \Sigma$ to indicate $ar(f) = m$. We assume a countably infinite set $\mathbf{Var}$ of *variables*. An *infinitary* (i.e. finite or infinite) *term* over Σ is a pair $\langle P, F \rangle$ where P is a tree domain, and $F : P \rightarrow (\Sigma \cup \mathbf{Var})$ verifying the following: if $p \in P$, then $pi \in P$ iff $F(p) \in \Sigma$ and $ar(F(p)) \geq i$. We denote the set of infinitary terms over Σ as $Ter^\infty(\Sigma)$. If $t = \langle P, F \rangle$, then we define $\mathbf{pos}(t) := P$ and $t(p) := F(p)$; we say that t is *finite* iff $\mathbf{pos}(t)$ is. If f is a unary symbol, then f^ω denotes the term $f(f(f \dots))$.

Given t, u terms and p a position in t , we let $t|_p$ denote the *subterm* of t at p , and $t[u]_p$ the term that results by replacing that subterm of t by u .

The *distance* between two terms t and u is defined as follows: $d(t, u) := 0$ if $t = u$; otherwise, $d(t, u) := 2^{-|p|}$ where t and u coincide on q if $|q| < |p|$, and $t(p) \neq u(p)$. Given this notion of distance, the *limit* of any Cauchy-convergent sequence of terms is defined.

Term rewriting systems, reduction A *rule* over a signature Σ is a pair $\langle l, h \rangle$ of terms over Σ such that $l \notin \mathbf{Var}$, l is a finite term, and all the variables

occurring in h occur also in l . We denote the rule $\langle l, h \rangle$ as $\mu : l \rightarrow h$; here μ is the *name* of the rule, a symbol from a signature disjoint with Σ . A *term rewriting system* (TRS) is a pair $T = \langle \Sigma, R \rangle$, where Σ is a signature and R a set of rules over Σ . We only consider *left-linear* TRSs, i.e., if $\langle l, h \rangle \in R$ then l has no multiple occurrences of the same variable. If the rules are used for *infinitary rewriting*, as in this paper, then T is also called an *infinitary TRS* (iTRS).

A *reduction step* over a TRS $T = \langle \Sigma, R \rangle$ is a triple $a = \langle t, p, \mu \rangle$ where t is a term, $p \in \text{pos}(t)$, and $\mu : l \rightarrow h \in R$, iff $t|_p = \sigma l$ for some substitution σ . We define the source term, target term, activity position and depth of $a = \langle t, p, \mu \rangle$ as follows: $\text{src}(a) := t$, $\text{tgt}(a) := t[\sigma h]_p$, $\text{rpos}(a) := p$, $d(a) := |p|$.

A *reduction sequence* is: either Id_t , the *empty reduction sequence* for the term t , or else a non-empty sequence of reduction steps $\delta := \langle \delta[\alpha] \rangle_{\alpha < \beta}$, where $\beta > 0$ and the following conditions are met: (1) for all α such that $\alpha + 1 < \beta$, $\text{src}(\delta[\alpha + 1]) = \text{tgt}(\delta[\alpha])$; and for all limit ordinals $\beta_0 < \beta$: (2.a) the sequence $\langle \text{tgt}(\delta[\alpha]) \rangle_{\alpha < \beta_0}$ has a limit, (2.b) that limit coincides with $\text{src}(\delta[\beta_0])$, and (2.c) for all $n < \omega$, there exists $\beta' < \beta_0$ such that $d(\delta[\alpha]) > n$ if $\beta' < \alpha < \beta_0$. We say that a reduction sequence δ is *strongly convergent* if either $\delta = \text{Id}_t$ for some term t , or else $\delta = \langle \delta[\alpha] \rangle_{\alpha < \beta}$, and either β is a successor ordinal, or else β is a limit ordinal and conditions (2.a) and (2.c) hold for β as well. By *convergence*, we will always mean strong convergence.

The symbols $\rightarrow$, $\Rightarrow$ and $\twoheadrightarrow$ denote the one-step, finite rewrite, and convergent infinitary rewrite relations respectively.

We will denote the step $a = \langle t, p, \mu \rangle$ as $t \xrightarrow{a} u$, or $t \xrightarrow{\langle p, \mu \rangle} u$ where $u = \text{tgt}(a)$. Analogously for $\Rightarrow$ and $\twoheadrightarrow$. We indicate the TRS below the arrow if needed.

WN^∞ is the property of having an infinitary normal form and UN^∞ of unicity of normal forms. SN^∞ is more complicated, but boils down to the absence of non-convergent reduction sequences. All orthogonal iTRSs are UN^∞ , see [11], Chap. 13 or [7].

3 Finitary and Infinitary Proof Terms

The idea motivating the definition and application of proof terms, as they were introduced in [11], Chap. 8, is to denote reductions of a TRS T as *terms* over an extended signature. For each reduction rule $\rho : l \rightarrow r$ of T a *rule symbol* is introduced, which we will also denote by ρ . The arity of the rule symbol ρ equals the number of different variables occurring in l . So e.g., the signature of proof terms for a TRS $T = \langle \Sigma, R \rangle$ with rules $\mu : f(x) \rightarrow g(x)$, $\rho : h(m(x), m(y)) \rightarrow k(x)$ and $\nu : g(x) \rightarrow k(x)$ adds to Σ rule symbols $\mu/1$, $\rho/2$ and $\nu/1$.

We describe some valid proof terms along with the T -reductions they denote. Proof terms with only one occurrence of a rule symbol denote single reduction steps: $\mu(a) : f(a) \rightarrow g(a)$ and $g(\rho(a, b)) : g(h(m(a), m(b))) \rightarrow g(k(a))$. With more occurrences of rule symbols, they denote multi-steps, like $h(\mu(a), \mu(b)) : h(f(a), f(b)) \twoheadrightarrow h(g(a), g(b))$ and $\rho(\mu(a), b) : h(m(f(a)), m(b)) \twoheadrightarrow k(g(a))$. A proof term ψ without rule-symbol occurrences denotes an empty step. It will also be denoted by the symbol 1_ψ , or just briefly 1 .

The beginning and end terms of the corresponding reductions are called the source and target of the proof term. For the proof terms considered so far, they can be obtained via rewriting in two companion TRSs, denoted as *SRC* and *TGT* respectively. For each rule symbol $\rho : l \rightarrow r$, *SRC* includes a rule $\rho(x_1, \dots, x_m) \rightarrow l[x_1, \dots, x_m]$ and *TGT* a rule $\rho(x_1, \dots, x_m) \rightarrow r[x_1, \dots, x_m]$. Source and target of a proof term are its normal forms in *SRC* and *TGT*, respectively. It is easy to see that *SRC* and *TGT* are orthogonal and terminating.

To complete the definition of the set of finitary proof terms, we add a new binary function symbol $\cdot$ (written infix), expressing concatenation, or composition, of reductions. There is a restriction on term formation, though: $\psi \cdot \phi$ is a valid proof term only if $\text{tgt}(\psi) = \text{src}(\phi)$. Computing source and target for proof terms containing the symbol $\cdot$ is made possible by schematically adding rewrite rules $\psi \cdot \phi \rightarrow \psi$ to *SRC* and $\psi \cdot \phi \rightarrow \phi$ to *TGT*.

Just to give a simple example, the proof term $f(\mu(a)) \cdot f(\nu(a))$ denotes the two-step reduction $f(f(a)) \rightarrow f(g(a)) \rightarrow f(k(a))$. The same reduction is represented by the proof term $f(\mu(a) \cdot \nu(a))$.

Proof terms without the symbol $\cdot$ are called *multi-steps*, with *one-step* proof terms, multi-steps with precisely one rule-symbol occurrence, as a special case.

3.1 Infinitary Proof Terms

We extend the concept of proof terms to the setting of infinitary rewriting. *Infinitary multi-steps* are finite or infinite terms over the signature extended with rule symbols. A multi-step may now contain infinitely many rule symbol occurrences, as e.g. in the proof term $\mu^\omega : f^\omega \rightarrow g^\omega$. As in the finite case, infinitary multi-steps with just one rule-symbol occurrence are called *one-steps*.

Source $\text{src}(\psi)$ and target $\text{tgt}(\psi)$ for an infinitary multi-step ψ can again be defined using the TRSs *SRC* and *TGT*, now considered as *infinitary* TRSs, as the respective infinitary normal forms of ψ . Of course there are the questions of existence and uniqueness. First note that UN^∞ holds for both *SRC* and *TGT*, since they are orthogonal iTRSs. It is also not hard to verify WN^∞ for *SRC*. For *TGT*, however, we have WN^∞ only if the object TRS does not include collapsing rules. We conclude that the source of an infinitary multi-step ψ is always uniquely defined. The target is only defined if ψ is WN^∞ , but if so, it is also unique. If ψ is not WN^∞ for *TGT*, then we say that $\text{tgt}(\psi)$ is undefined.

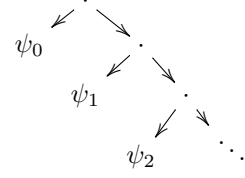
The set of redexes in $\text{src}(\psi)$ corresponding to the rule symbol occurrences in ψ admits at least one convergent development (respectively, all developments are convergent) precisely if ψ is WN^∞ (respectively SN^∞) in the TRS *TGT*. An infinitary multi-step is called *convergent*, if its target can be computed.

Composition is expressed, as before, by the binary symbol $\cdot$ of which a proof term may now contain an infinite number of occurrences. Not all terms over the thus extended signature are valid proof terms though, but only those that can be constructed starting from the infinitary multi-steps by the following three inductive clauses.

First, closure under application of function or rule symbols: if $\psi_1, \dots, \psi_n$ are proof-terms, then so are $f(\psi_1, \dots, \psi_n)$ and $\mu(\psi_1, \dots, \psi_n)$.

Secondly, *binary composition*: if ψ, ϕ are proof terms, then so is $\psi \cdot \phi$, provided that $\text{tgt}(\psi) = \text{src}(\phi)$. This presupposes convergence of ψ . The proof term $\psi \cdot \phi$ is convergent iff ϕ is.

Thirdly, *infinite composition*: the term corresponding to the figure is a proof term, if $\psi_0, \psi_1, \psi_2, \dots$ are, provided that for each $i < \omega$ we have convergence of ψ_i and $\text{tgt}(\psi_i) = \text{src}(\psi_{i+1})$. A linear rendering would be $\psi_0 \cdot (\psi_1 \cdot (\psi_2 \cdot \dots))$. We use $\cdot_{i < \omega} \psi_i$ as shorthand for this proof term.



For $\psi = \cdot_{i < \omega} \psi_i$, we define $\text{src}(\psi) = \text{src}(\psi_0)$. That leaves convergence for $\cdot_{i < \omega} \psi_i$ to be defined, and the target, if it exists. For this we need the notion of *minimal activity depth* of a proof term ψ , notation $\text{mind}(\psi)$. This can be defined as the minimal depth of a rule symbol occurrence, where dot occurrences are not counted when computing the depth of a symbol. (If ψ does not have any rule symbol occurrence, then we call ψ *trivial* and define $\text{mind}(\psi) = \omega$.) For example: if $\psi = \mu(a) \cdot \nu(a) : f(a) \rightarrow g(a) \rightarrow k(a)$, then $\text{mind}(\psi) = 0$. This is the correct value since ψ denotes a reduction including contractions at the root.

Now we define $\psi = \cdot_{i < \omega} \psi_i$ to be convergent if the sequence $\langle \text{mind}(\psi_i) \rangle_{i < \omega}$ tends to infinity, and if so, $\text{tgt}(\psi)$ is defined as the limit of the sequence $\text{tgt}(\psi_i)$.

In [8] infinitary proof terms are defined in layers corresponding to ordinal numbers. We implicitly refer to these layers when applying induction on proof terms. Let $\nu(\psi)$ denote the ordinal layer for ψ . We have $\nu(\psi_1 \cdot \psi_2) = \nu(\psi_1) + \nu(\psi_2) + 1$, and $\nu(\cdot_{i < \omega} \psi_i) = \sum_{i < \omega} \nu(\psi_i)$. Hence $\nu(\psi_i) < \nu(\psi_1 \cdot \psi_2)$ for $i = 1, 2$, $\nu(\psi_0 \cdot \dots \cdot \psi_n) < \nu(\cdot_{i < \omega} \psi_i)$ for all $n < \omega$, and $\nu(\psi)$ is a limit ordinal iff ψ is an infinite composition.

By the above definition of proof terms an infinite composition is also a binary composition: $\cdot_{i < \omega} \psi_i = \psi_0 \cdot (\cdot_{i < \omega} \psi_{i+1})$. However, since proof terms have unique layers, in the layered definition the proof term is constructed only once, as infinite composition, and with layer a limit ordinal. We still have unique constructibility.

3.2 Examples

The definition of infinitary proof terms extends that of the finitary ones so that the examples given earlier are infinitary proof terms as well.

Let us show some infinite examples. Let $\mu : f(x) \rightarrow g(x)$ and $\nu : g(x) \rightarrow k(x)$. Consider $f^\omega \twoheadrightarrow g^\omega$ taken as the simultaneous reduction of all the μ redexes present in the term f^ω . Such a simultaneous reduction can be denoted by the infinitary multi-step $\psi_1 := \mu^\omega$. To compute $\text{src}(\psi_1) = f^\omega$ and $\text{tgt}(\psi_1) = g^\omega$ we observe that the “companion TRSs” *SRC* and *TGT* include the rules $\mu(x) \rightarrow f(x)$ and $\mu(x) \rightarrow g(x)$ respectively, leading to the convergent reduction sequences $\mu^\omega \rightarrow f(\mu^\omega) \rightarrow f^2(\mu^\omega) \twoheadrightarrow f^\omega$ and $\mu^\omega \rightarrow g(\mu^\omega) \rightarrow g^2(\mu^\omega) \twoheadrightarrow g^\omega$.

Consider the reduction sequence $f^\omega \rightarrow g(f^\omega) \rightarrow g^2(f^\omega) \twoheadrightarrow g^\omega$ having length ω . The i -th step of this sequence, namely $g^i(f^\omega) \rightarrow g^{i+1}(f^\omega)$, can be described by the proof term $g^i(\mu(f^\omega))$. It is straightforward to check that the sequence formed by these proof terms verifies the conditions of the infinitary composi-

tion rule, and that the depth of the denoted activity tends to infinity. Therefore $\psi_2 := \cdot_{i < \omega} g^i(\mu(f^\omega))$ is a valid and convergent proof term; we observe $\text{src}(\psi_2) = f^\omega$. In order to obtain $\text{tgt}(\psi_2) = g^\omega$, it is enough to observe that the sequence of targets of each $g^i(\mu(f^\omega))$, namely $g(f^\omega), g^2(f^\omega), \dots$, converges to that term. Analogously, the reduction sequence $f^\omega \rightarrow g(f^\omega) \rightarrow k(f^\omega) \rightarrow k(g(f^\omega)) \rightarrow k^2(f^\omega) \rightsquigarrow k^\omega$ can be denoted by either $\psi_3 := \cdot_{i < \omega} (k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega)))$ or $\psi_4 := \cdot_{i < \omega} (k^i(\mu(f^\omega) \cdot \nu(f^\omega)))$.

The rules defining the set of proof terms can be combined in different ways. E.g., the reduction $f^\omega \rightsquigarrow g^\omega \rightsquigarrow k^\omega$, can be denoted by either $\cdot_{i < \omega} g^i(\mu(f^\omega)) \cdot \cdot_{i < \omega} k^i(\nu(g^\omega))$ (if taken as a sequence having length $\omega \times 2$) or $\mu^\omega \cdot \nu^\omega$ (if considered as the composition of two infinite simultaneous reductions). The reduction $f^\omega \rightarrow f(g(f^\omega)) \rightarrow f(k(f^\omega)) \rightarrow f(k(g(f^\omega))) \rightarrow f(k^2(f^\omega)) \rightsquigarrow f(k^\omega) \rightarrow g(k^\omega)$ can be denoted by $\cdot_{i < \omega} f(k^i(\mu(f^\omega) \cdot \nu(f^\omega))) \cdot \mu(k^\omega)$.

Particularly, infinite composition can be combined with itself. Let us consider a reduction sequence having length ω^2 , and ϕ_{ij} be a proof term denoting its $\omega * i + j$ -th step, so that for each $i < \omega$, $\cdot_{j < \omega} \phi_{ij}$ denotes the subsequence including the steps from the $\omega * i$ -th up to the $\omega * (i + 1)$ -th. Then $\cdot_{i < \omega} \cdot_{j < \omega} \phi_{ij}$ is a proof term denoting the entire reduction sequence. By iteration of this pattern, proof terms can be built denoting reduction sequences of any countable ordinal length. This claim is proved in Sec. 5.

4 Permutation Equivalence

Two proof terms can be the result of arranging the same contraction activity in different ways, regarding parallelism/nesting degree, sequential order, and/or localisation of contractions. Such proof terms should be recognised as being *permutation equivalent*.

In this section we give the definition of equivalence between infinitary proof terms. We do this by extending *permutation equivalence*, as it is defined in [11] Sec. 8.3, to the infinitary setting. Permutation equivalence, which will be denoted $\approx$ henceforth, is defined there for finitary proof terms as the congruence generated by the following six basic equivalences.

$$\begin{array}{ll}
(\text{IdLeft}) & 1 \cdot \psi \approx \psi \\
(\text{IdRight}) & \psi \cdot 1 \approx \psi \\
(\text{Assoc}) & \psi \cdot (\phi \cdot \chi) \approx (\psi \cdot \phi) \cdot \chi \\
(\text{Struct}) & f(\psi_1, \dots, \psi_m) \cdot f(\phi_1, \dots, \phi_m) \approx f(\psi_1 \cdot \phi_1, \dots, \psi_m \cdot \phi_m) \\
(\text{OutIn}) & \mu(\psi_1, \dots, \psi_m) \approx \mu(s_1, \dots, s_m) \cdot r[\psi_1, \dots, \psi_m] \\
(\text{InOut}) & \mu(\psi_1, \dots, \psi_m) \approx l[\psi_1, \dots, \psi_m] \cdot \mu(t_1, \dots, t_m)
\end{array}$$

where $\mu : l \rightarrow r$, $s_i = \text{src}(\psi_i)$ and $t_i = \text{tgt}(\psi_i)$ in (InOut) and (OutIn).

In the present, infinitary setting, three elements must be added in order to obtain an adequate characterisation of infinitary permutation equivalence. Two of these elements are related with infinitary composition.

First, we add one basic equation, the analogue of (Struct) for infinite composition:

$$(\text{InfStruct}) \cdot_{i < \omega} f(\psi_i^1, \dots, \psi_i^m) \approx f(\cdot_{i < \omega} \psi_i^1, \dots, \cdot_{i < \omega} \psi_i^m)$$

Secondly, we need an infinitary equational logic rule in order to ensure that $\approx$ behaves as a congruence also w.r.t. infinite composition:

$$\frac{\psi_i \approx \phi_i \quad \text{for all } i < \omega}{\cdot_{i < \omega} \psi_i \approx \cdot_{i < \omega} \phi_i} \text{ InfComp}$$

Lastly, we need a limit rule, by which permutation equivalence of two convergent reductions can be concluded, roughly, from permutation equivalence up to any finite depth. In the following **Lim**-rule this is implemented by requiring that for any k the reductions can be factorised into prefixes that coincide and “tails” for which all activity occurs at depth greater than k .

$$\frac{\left. \begin{array}{l} \psi \approx_1 \chi_k \cdot \psi'_k \quad \text{mind}(\psi'_k) > k \\ \phi \approx_1 \chi_k \cdot \phi'_k \quad \text{mind}(\phi'_k) > k \end{array} \right\} \text{ for all } k < \omega}{\psi \approx \phi} \text{ Lim}$$

Here $\approx_1$ is the congruence generated by the six basic equations, (**InfStruct**) and **InfComp**, but excluding the **Lim**-rule itself. Variations of the **Lim**-rule can be considered; we will briefly discuss this point in the conclusions.

The equational logic that we use to define permutation equivalence is infinitary in two ways. It involves the infinite proof terms, and two of the rules, **InfComp** and **Lim**, require an infinite number of premises. In order to obtain a sound (transfinite) induction principle to reason about infinitary permutation equivalence, its definition can be *layered* by ordinals, analogously to the definition of the set of proof terms. For details we refer to [8].

In the remainder of this section, we will give some examples of infinitary permutation equivalence. Let $\mu : f(x) \rightarrow g(x)$, $\nu : g(x) \rightarrow k(x)$, $\sigma : k(x) \rightarrow m(x)$, $\rho : h(x, y) \rightarrow x$, $\pi : a \rightarrow b$, $\tau : i(x) \rightarrow x$, and $\kappa : f(x) \rightarrow g(f(x))$.

Consider $\psi = \psi_1 \cdot \psi_2$ where $\psi_1 = \cdot_{i < \omega} g^i(\mu(f^\omega))$ and $\psi_2 := \cdot_{i < \omega} k^i(\nu(g^\omega))$, and $\phi = \cdot_{i < \omega} \chi_i$ where $\chi_i = k^i(\mu(f^\omega) \cdot \nu(f^\omega))$. These proof terms denote different sequentialisations of the same contraction activity, namely, the transformation of each occurrence of f in f^ω to g and subsequently to k , by means of the μ - and ν -rules respectively.

Using the augmented congruence, including the **Lim** rule, the assertion $\psi \approx \phi$ can be justified as follows. Let $n < \omega$. Then

$$\begin{aligned} \psi &\approx_1 \mu(f^\omega) \cdot g(\psi_1) \cdot \nu(g^\omega) \cdot k(\psi_2) &\approx_1 \mu(f^\omega) \cdot \nu(f^\omega) \cdot k(\psi_1) \cdot k(\psi_2) \\ &\approx_1 \chi_0 \cdot k(\psi) &\approx_1 \chi_0 \cdot k(\chi_0 \cdot k(\psi)) \\ &\approx_1 \chi_0 \cdot k(\chi_0) \cdot k^2(\psi) &= \chi_0 \cdot \chi_1 \cdot k^2(\psi) \\ &\dots &\approx_1 \chi_0 \cdot \chi_1 \cdot \dots \cdot \chi_n \cdot k^{n+1}(\psi) \end{aligned}$$

by applying (**InfStruct**), (**InOut**), (**OutIn**), (**Struct**), repeating this for the copy of ψ in $k(\psi)$ followed by (**Struct**), and so on. For ϕ we have $\phi \approx_1 \chi_0 \cdot \chi_1 \cdot \dots \cdot \chi_n \cdot \cdot_{i < \omega} \chi_{n+1+i}$. Hence **Lim** yields $\psi \approx \phi$.

We point out the infinitary aspects of this transformation. An *infinite* number of ν -steps must be permuted, each one over an *infinite* number of μ -steps. Permuting a ν -step requires equations (**InOut**) and (**OutIn**). The equation (**InfStruct**)

is used to extract the pattern $g(\square)$ of ν , obtaining $g(\psi_1)$, thereby allowing permutation of a ν step over an infinite number of μ -steps, using (InOut) and (OutIn) just once. The infinite number of ν -permutations take place at ever-bigger depths. This makes it possible to apply Lim.

4.1 The Role of the InfComp Rule

We consider the following variation of the first permutation equivalence example discussed: let $\psi = \psi_1 \cdot \psi_2$ where $\psi_1 = \cdot_{i < \omega} k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega))$ and $\psi_2 = \cdot_{i < \omega} m^i(\sigma(k^\omega))$, and $\phi = \cdot_{i < \omega} \chi_i$ where $\chi_i = m^i(\mu(f^\omega) \cdot \nu(f^\omega) \cdot \sigma(f^\omega))$. Let $\psi'_1 := \cdot_{i < \omega} k^{i+1}(\mu(f^\omega)) \cdot k^{i+1}(\nu(f^\omega))$. The proof terms ψ and ϕ denote the transformation of f^ω to m^ω , where at each depth level the rules μ , ν and σ are applied successively.

The idea of a judgement allowing to assert $\psi \approx \phi$ is similar to that of the previous example: in order to transform ψ into ϕ , ψ'_1 must be transformed into $k(\psi_1)$, so that the first σ step can be permuted. But in this case, for any $i < \omega$, the (Struct) equation gives $k^{i+1}(\mu(f^\omega)) \cdot k^{i+1}(\nu(f^\omega)) \approx_1 k(k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega)))$. Consequently, InfComp yields $\psi'_1 \approx_1 \cdot_{i < \omega} k(k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega)))$. Therefore,

$$\begin{aligned} \psi &\approx_1 \mu(f^\omega) \cdot \nu(f^\omega) \cdot \psi'_1 \cdot \sigma(k^\omega) \cdot \cdot_{i < \omega} m^{i+1}(\sigma(k^\omega)) \\ &\approx_1 \mu(f^\omega) \cdot \nu(f^\omega) \cdot (\cdot_{i < \omega} k(k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega)))) \cdot \sigma(k^\omega) \cdot \cdot_{i < \omega} m^{i+1}(\sigma(k^\omega)) \\ &\approx_1 \mu(f^\omega) \cdot \nu(f^\omega) \cdot k(\cdot_{i < \omega} k^i(\mu(f^\omega)) \cdot k^i(\nu(f^\omega))) \cdot \sigma(k^\omega) \cdot m(\cdot_{i < \omega} m^i(\sigma(k^\omega))) \\ &= \mu(f^\omega) \cdot \nu(f^\omega) \cdot k(\psi_1) \cdot \sigma(k^\omega) \cdot m(\psi_2) \\ &\approx_1 \mu(f^\omega) \cdot \nu(f^\omega) \cdot \sigma(f^\omega) \cdot m(\psi_1) \cdot m(\psi_2) \\ &\approx_1 \chi_0 \cdot m(\psi) \end{aligned}$$

by (Assoc), the previous argument, (InfStruct), (InOut) followed by (OutIn), and (Struct) respectively. An iteration of this argument yields $\psi \approx_1 \chi_0 \cdot \dots \cdot \chi_n \cdot m^{n+1}(\psi)$ for any $n < \omega$. From this result, it is easy to conclude $\psi \approx \phi$ by resorting to the Lim rule.

4.2 Infinitary Erasure

As for the finitary setting, this characterisation of permutation equivalence models adequately the phenomenon of *erasure* of some contraction activity by step permutation. A simple example follows: by applying twice the (OutIn) equation, we obtain $\rho(\mu(a), \tau^\omega) \approx \rho(f(a), i^\omega) \cdot \mu(a) \approx \rho(\mu(a), i^\omega)$.

Now consider $\psi = f(\pi) \cdot \cdot_{i < \omega} g^i(\kappa(b))$ and $\phi = \cdot_{i < \omega} g^i(\kappa(a))$. These proof terms are sequential descriptions of the reduction sequences $f(a) \rightarrow f(b) \rightarrow g(f(b)) \rightarrow g^2(f(b)) \twoheadrightarrow g^\omega$ and $f(a) \rightarrow g(f(a)) \rightarrow g^2(f(a)) \twoheadrightarrow g^\omega$ respectively. Observe that the π step in the former sequence can be permuted with each κ step in turn, yielding $f(a) \rightarrow g(f(a)) \rightarrow \dots \rightarrow g^n(f(a)) \rightarrow g^n(f(b)) \twoheadrightarrow g^\omega$. The latter sequence can be seen as the result of taking the limit of this sequence of step permutations.

This example shows the existence of a form of erasure unique to infinitary rewriting: some contraction activity (in this case the π step) is erased as the result of a limit of step permutations. We call this phenomenon *infinitary erasure*.

According to this view, ψ and ϕ denote equivalent contraction activities, and they therefore should be stated as permutation equivalent. Observe that

$$\begin{aligned}\psi &\approx_1 f(\pi) \cdot \kappa(b) \cdot g(\cdot_{i<\omega} g^i(\kappa(b))) \\ &\approx_1 \kappa(a) \cdot g(f(\pi)) \cdot g(\kappa(b)) \cdot g^2(\cdot_{i<\omega} g^i(\kappa(b))) \\ &\approx_1 \kappa(a) \cdot g(\kappa(a)) \cdot g^2(f(\pi)) \cdot g^2(\kappa(b)) \cdot g^3(\cdot_{i<\omega} g^i(\kappa(b))) \dots\end{aligned}$$

where the first component of each successive infinitary concatenation is extracted by (Assoc), (Struct) and (InfStruct), and each inversion involves (Struct) (except for the first one), (InOut) and (OutIn). Consequently, for any $n < \omega$, we can obtain $\psi \approx_1 \kappa(a) \cdot \dots \cdot g^n(\kappa(a)) \cdot g^{n+1}(f(\pi)) \cdot g^{n+1}(\cdot_{i<\omega} g^i(\kappa(b)))$. On the other hand, it is straightforward to get $\phi \approx_1 \kappa(a) \cdot \dots \cdot g^n(\kappa(a)) \cdot g^{n+1}(\phi)$. Hence, the Lim rule yields $\psi \approx \phi$.

This example adequately describes infinitary erasure: the equivalence between ψ and ϕ is obtained by a limit argument.

5 Denotation

In this section we show that *any* reduction sequence can be faithfully denoted by a proof term. Since also simultaneous reduction can be expressed by proof terms, in the form of infinitary multi-steps, not every proof term uniquely denotes a reduction sequence. However, we will define a subclass of so-called *stepwise* proof terms, which do have unique denotations. We will show that any reduction can be represented by a *stepwise* proof term, which is unique up to an equivalence which we baptized “rebracketing”, and which results from applying associativity of composition in this infinitary setting.

5.1 Existence

A *stepwise proof term* is inductively defined as: either a one-step proof term, or a binary or infinite concatenation of stepwise proof terms. A *stepwise-or-nil proof term* is either a stepwise proof term or a term in $Ter^\infty(\Sigma)$.

Given a stepwise-or-nil proof term ψ , we define the *number of steps* and the α -th step of ψ , notation $ns(\psi)$ and $\psi[\alpha]$ respectively. The definitions are straightforward. For number of steps: $ns(\psi) := 0$ if $\psi \in Ter^\infty(\Sigma)$, $ns(\psi) := 1$ if ψ is a one-step, $ns(\psi_1 \cdot \psi_2) := ns(\psi_1) + ns(\psi_2)$, $ns(\cdot_{i<\omega} \psi_i) := \sum_{i<\omega} ns(\psi_i)$. Moreover, α -th step is defined for $\alpha < ns(\psi)$ by: $\psi[0] := \psi$ if ψ is a one-step, $\cdot_{i<\omega} \psi_i[\alpha] := \psi_k[\gamma]$ where $\alpha = \sum_{i<k} ns(\psi_i) + \gamma$ and $\gamma < ns(\psi_k)^\dagger$, and analogously for binary concatenation.

Observe that the steps of stepwise proof terms are one-step proof terms. A stepwise proof term is a representation of the sequence of its steps, organised by means of binary and infinite compositions. In this way, one-step proof terms denote reduction steps, while stepwise proof terms denote reduction sequences.

Properties that support this view of stepwise proof terms can be proved by analysing their form. Specifically, if ψ is a stepwise proof term: $src(\psi) =$

[†] Properties of ordinal infinitary sum imply uniqueness of k and γ here.

$src(\psi[0]), tgt(\psi) = tgt(\psi[\alpha])$ if $ns(\psi) = \alpha + 1$, $tgt(\psi) = \lim_{\alpha \rightarrow ns(\psi)} tgt(\psi[\alpha])$ if $ns(\psi)$ is a limit ordinal, $src(\psi[\alpha + 1]) = tgt(\psi[\alpha])$ if $\alpha + 1 < ns(\psi)$, and $mind(\psi) = \min\{d(\psi[\alpha]) \mid \alpha < ns(\psi)\}$.

We formalise the relation between stepwise proof terms and reduction sequences as follows: a stepwise-or-nil proof term ψ denotes a reduction sequence δ iff $ns(\psi) = \text{length}(\delta)$, $src(\psi) = src(\delta)$, and for all $\alpha < ns(\psi)$, $\psi[\alpha]$ coincides in source term, redex position and rule (symbol) with $\delta[\alpha]$. This relation preserves convergence: if ψ denotes δ , then ψ is convergent if and only if δ is, cf. [8].

In this setting, the existence requirement amounts to the following:

Proposition 1. *Let δ be a reduction sequence having countable length. Then there exists a stepwise-or-nil proof term ψ such that ψ denotes δ .*

Proof. By induction on $\text{length}(\delta)$. If $\text{length}(\delta) = 0$, i.e. $\delta = \text{Id}_t$, then just take $\psi := t$. If $\text{length}(\delta) = 1$, then replacing the pattern of the contracted redex in the source term by the corresponding rule symbol yields a one-step proof term denoting δ . If $\text{length}(\delta)$ is a successor (resp. limit) ordinal, then it is easy to split δ in two (resp. ω , cf. Lem. 1) parts, so that IH can be applied on each part and then the results joined by binary (resp. infinite) concatenation.

5.2 Uniqueness

We say that two stepwise-or-nil proof terms ψ and ϕ are *denotationally equivalent* iff they contain exactly the same sequence of steps, organised in possibly different ways. Formally, $\psi \equiv \phi$ iff $ns(\psi) = ns(\phi)$ and either $ns(\psi) = 0$ and $\psi = \phi$, or $ns(\psi) > 0$ and $\psi[\alpha] = \phi[\alpha]$ if $\alpha < ns(\psi)$. It is straightforward to verify that two stepwise-or-nil proof terms are denotationally equivalent iff they denote the same reduction sequence.

We define *rebracketing equivalence*, notation $\approx_{()}$, as the congruence generated using (Assoc) as the only basic equation, and including InfComp and Lim as equational logic rules.[‡] The corresponding base rebracketing equivalence, notation $\approx_{(1)}$, is defined analogously but not including Lim as a rule. In the premises of Lim for $\approx_{()}$, the occurrences of $\approx_1$ must be replaced by $\approx_{(1)}$.

We claim that the denotation of reduction sequences by using proof terms is unique modulo rebracketing. To verify this claim, we show that the relations $\equiv$ and $\approx_{()}$ coincide.

Lemma 2. *Let ψ be a stepwise proof term, and α an ordinal verifying $0 < \alpha < ns(\psi)$. Then there exist stepwise proof terms ϕ, χ such that $\psi \approx_{(1)} \phi \cdot \chi$ and $ns(\phi) = \alpha$. Moreover, $\nu(\phi) < \nu(\psi)$ and $\nu(\chi) \leq \nu(\psi)$.*

Proof. Easy induction on ψ . Cf. the stated properties about ν in Sec. 3.1.

Proposition 2. *Let ψ, ϕ be stepwise-or-nil proof terms such that $\psi \approx_{()} \phi$. Then $\psi \equiv \phi$.*

[‡] This is part of what in [11], Table 8.1, are called the *reduction identities*, adapted to the infinitary setting.

Proof. Induction on the judgement $\psi \approx_{()} \phi$, observing the rule used to conclude it. We give some details w.r.t. the Lim rule. We prove $ns(\psi) = ns(\phi)$ by contradiction. Assume wlog that $ns(\phi) > ns(\psi)$, implying that the step $\phi[ns(\psi)]$ exists. Let $n := d(\phi[ns(\psi)])$, observe $n < \omega$. Then the premises of the Lim rule yield $\psi \approx_{(1)} \chi \cdot \psi'$ and $\phi \approx_{(1)} \chi \cdot \phi'$, where $mind(\psi') > n$ and $mind(\phi') > n$. In turn, $\psi \approx_{(1)} \chi \cdot \psi'$ implies $ns(\psi) \geq ns(\chi)$, while $\phi \approx_{(1)} \chi \cdot \phi'$, IH, and $mind(\phi') > n = d(\phi[ns(\psi)])$, imply $\phi[ns(\psi)] = \chi[ns(\psi)]$, then $ns(\chi) > ns(\psi)$; hence, a contradiction. Let $\gamma < ns(\psi)$, and consider $\psi \approx_{(1)} \chi \cdot \psi'$ and $\phi \approx_{(1)} \chi \cdot \phi'$ where $mind(\psi') > d(\psi[\gamma])$ and $mind(\phi') > d(\psi[\gamma])$. Then IH on these premises and the definition of $\equiv$ imply that $\psi[\gamma] = (\chi \cdot \psi')[\gamma]$ and $\phi[\gamma] = (\chi \cdot \phi')[\gamma]$. In turn, the conditions on $mind(\psi')$ imply that $\gamma < ns(\chi)$, so that $\psi[\gamma] = \phi[\gamma] = \chi[\gamma]$.

Proposition 3. *Let ψ, ϕ be convergent stepwise-or-nil proof terms such that $\psi \equiv \phi$. Then $\psi \approx_{()} \phi$.*

Proof. We proceed by induction on $\langle \psi, \phi \rangle$. The interesting cases are when ψ is either a binary or an infinite concatenation.

Assume $\psi = \psi_1 \cdot \psi_2$. It is easy to see that also ϕ must be of this form: $\phi = \phi_1 \cdot \phi_2$. If $ns(\psi_1) = ns(\phi_1)$, then $\psi_i \equiv \phi_i$ for $i = 1, 2$, so that IH and then congruence suffice to conclude. Assume $ns(\psi_1) < ns(\phi_1)$. Then Lem. 2 implies $\phi_1 \approx_{()} \chi_1 \cdot \chi_2$ where $ns(\chi_1) = ns(\psi_1)$, and therefore $\phi \approx_{()} (\chi_1 \cdot \chi_2) \cdot \phi_2 \approx_{()} \chi_1 \cdot (\chi_2 \cdot \phi_2)$. Hence hypotheses and Prop. 2 imply $\psi_1 \cdot \psi_2 = \psi \equiv \phi \equiv \chi_1 \cdot (\chi_2 \cdot \phi_2)$. From this, since $ns(\chi_1) = ns(\psi_1)$, it is easy to show $\psi_1 \equiv \chi_1$, and also $\psi_2 \equiv \chi_2 \cdot \phi_2$. The result then follows by applying IH twice, congruence and transitivity. Finally, if $ns(\psi_1) > ns(\phi_1)$, an analogous argument yields $\chi_1 \cdot (\chi_2 \cdot \psi_2) \equiv \psi \equiv \phi = \phi_1 \cdot \phi_2$ where $ns(\chi_1) = ns(\phi_1)$, and hence $\chi_1 \equiv \phi_1$ and $\chi_2 \cdot \psi_2 \equiv \phi_2$. Again using $\psi_1 \approx_{()} \chi_1 \cdot \chi_2$, it is now easy to obtain $\nu(\chi_1) < \nu(\psi)$ and $\nu(\chi_2 \cdot \psi_2) \leq \nu(\psi)$. Hence the result follows again by IH, congruence and transitivity.

Assume $\psi = \cdot_{i < \omega} \psi_i$. Then also ϕ is an infinite concatenation: $\phi = \cdot_{i < \omega} \phi_i$. By recombining the steps of ψ , a proof term $\chi = \cdot_{i < \omega} \chi_i$ can be built satisfying: (i) $\chi \equiv \psi$, (ii) $ns(\chi_i) = ns(\phi_i)$ for all $i < \omega$, and (iii) for all $p < \omega$, $\chi_0 \cdot \dots \cdot \chi_p \approx_{(1)} \psi_0 \cdot \dots \cdot \psi_j \cdot \xi$, where $ns(\chi_0 \cdot \dots \cdot \chi_p) = ns(\psi_0 \cdot \dots \cdot \psi_j) + \alpha$, $\alpha < ns(\psi_{j+1})$, $\psi_{j+1} \approx_{(1)} \xi \cdot \xi'$ and $ns(\xi) = \alpha$; cf. Lem. 2. E.g. if $ns(\phi_0) = ns(\psi_0) + \alpha$ such that $\alpha < ns(\psi_1)$, then we define $\chi_0 := \psi_0 \cdot \xi_1$ and χ_1 such that $\chi_1[0] = \xi_2[0]$, where $\psi_1 \approx_{(1)} \xi_1 \cdot \xi_2$ and $ns(\xi_1) = \alpha$.

Let $n < \omega$, p such that $mind(\chi_i) > n$ if $i > p$, and j, ξ, ξ' as in condition (iii) above. Let us call $\xi_n := \chi_0 \cdot \dots \cdot \chi_p$, $\psi'_n := \xi' \cdot \cdot_{i < \omega} \psi_{j+2+i}$, and $\chi'_n := \cdot_{i < \omega} \chi_{p+1+i}$. Then $\xi_n \approx_{(1)} \psi_0 \cdot \dots \cdot \psi_j \cdot \xi$, implying $\psi \approx_{(1)} \xi_n \cdot \psi'_n$. Prop. 2 yields $\xi_n \equiv \psi_0 \cdot \dots \cdot \psi_j \cdot \xi$, from this and (i) it is easy to obtain $\chi'_n \equiv \psi'_n$, and therefore $mind(\psi'_n) = mind(\chi'_n) > n$. Hence we can obtain $\psi \approx_{()} \chi$ by applying the Lim equation.

Moreover, (i) and $\psi \equiv \phi$ imply $\phi \equiv \chi$. From this and considering (ii), it is easy to obtain $\chi_i \equiv \phi_i$ for all i . Furthermore, for any $i < \omega$ it can also be proved that $\nu(\chi_i) < \nu(\psi)$, so that we can apply IH to obtain $\chi_i \approx_{()} \phi_i$.

We conclude by building the following rebracketing equivalence judgement.

$$\frac{\frac{\dots \frac{\psi \approx_{(1)} \xi_n \cdot \psi'_n}{\chi \approx_{(1)} \xi_n \cdot \chi'_n} \dots}{\psi \approx_{()} \chi} \text{Lim} \quad \frac{\frac{\dots IH \dots}{\chi_n \approx_{()} \phi_n} \dots}{\chi \approx_{()} \phi} \text{InfComp}}{\psi \approx_{()} \phi}$$

6 Compression

The compression lemma, [5, 11, 6] states that the full power of (strongly) convergent reduction can be achieved by reductions having length at most ω . Formally: if $t \xrightarrow{\delta} u$, then there exists a convergent reduction $t \xrightarrow{\gamma} u$, such that $\text{length}(\gamma) \leq \omega$. In [5] a more precise statement is given: for orthogonal TRSs, γ can be chosen such that it is Lévy-equivalent (cf. [4]) to δ .

We present the compression result for arbitrary proof terms. Moreover, it establishes permutation equivalence between the original and the compressed proof terms. Via the faithful representation of reduction sequences as stepwise proof terms, our compression result extends to the original reductions as well.

As in [6], we do not require orthogonality for compression (although our general assumption of left-linearity is crucial here). However, as far as we know the combination of not assuming orthogonality and yet obtaining equivalence between original and compressed form is new.

The compression proof uses as a key technical result, that any proof term can be *factorised* into a leading part denoting *finite* contraction activity, composed with a tail denoting activity at *arbitrary depths*. That is, for any proof term ψ and $n < \omega$, we obtain two proof terms χ and ϕ , such that $\psi \approx_1 \chi \cdot \phi$, χ is a finite stepwise-or-nil proof term, and $\text{mind}(\phi) > n$.

Collapsing rules make factorisation non-trivial for infinitary multi-steps. We need the notion of a *collapsing chain* for an infinitary multi-step ψ : a sequence of positions $\langle p_i \rangle_{i \leq n}$ (resp. $\langle p_i \rangle_{i < \omega}$), such that for all $i < n$ ($i < \omega$), $\psi(p_i) : l[x_1, \dots, x_m] \rightarrow x_j$ is a collapsing rule symbol, and $p_{i+1} = p_i \cdot j$. We consider particularly chains *starting at ϵ* , i.e. such that $p_0 = \epsilon$.

Lemma 3. *Let ψ be an infinitary multi-step such that an infinite collapsing chain for ψ starting at ϵ exists. Then ψ is not TGT-weakly normalising.*

Proof. It is enough to verify that if $\psi \xrightarrow{\delta} \phi$ by using the TGT companion TRS, then ϕ includes an infinite collapsing chain starting at ϵ , implying that $\phi(\epsilon)$ is a (collapsing) rule symbol, and therefore it is not a TGT-normal form.

Lemma 4. *Let ψ be an infinitary multi-step and n , $1 < n < \omega$, such that there is no collapsing chain of length n for ψ , starting at ϵ . Then there exists a TGT-reduction sequence δ , such that $\psi \xrightarrow[\text{TGT}]{\delta} \phi$, $\text{length}(\delta) < n$, $d(\delta[i]) = 0$ for all $i < \text{length}(\delta)$, and $\phi(\epsilon) \in \Sigma$.*

Proof. By induction on n . If $n = 2$, then either $\psi(\epsilon) \in \Sigma$ so that we conclude immediately, or $\psi(\epsilon) = \mu$ where $\mu : l[x_1, \dots, x_m] \rightarrow f(t_1, \dots, t_k)$, and then $\psi \xrightarrow[\text{TGT}]{(\epsilon, \mu)} f(t'_1, \dots, t'_k)$, thus we conclude. Assume $n = n' + 1$. Then $\psi(\epsilon)$ not being a

collapsing rule symbol allows an argument similar to that of the previous case. Otherwise, i.e. if $\psi(\epsilon) = \mu$ where $\mu : l[x_1, \dots, x_m] \rightarrow x_k$, and $\psi \xrightarrow[\text{TGT}]{\langle \epsilon, \mu \rangle} \psi|_k$, it is not difficult to verify that a collapsing chain having length n' in $\psi|_k$ would imply the existence of a collapsing chain having length n in ψ , contradicting the lemma hypotheses. Then IH can be applied on $\psi|_k$, which suffices to conclude.

Lemma 5. *Let ψ be a convergent infinitary multi-step. Then there exist χ, ϕ such that $\psi \approx_1 \chi \cdot \phi$, χ is a finite stepwise-or-nil proof term, $d(\chi[i]) = 0$ for all $i < ns(\chi)$, and ϕ is a convergent infinitary multi-step verifying $mind(\phi) > 0$.*

Proof. Convergence of ψ implies that ψ is WN^∞ in TGT , so that Lem. 3 implies the existence of some $n < \omega$ such that ψ does not have a collapsing chain of length n starting with ϵ . Therefore, Lem 4. implies $\psi \xrightarrow[\text{TGT}]{\delta} \phi$, $\text{length}(\delta) < n$, $d(\delta[i]) = 0$ for all $i < \text{length}(\delta)$, and $\phi(\epsilon) \in \Sigma$. Then, it is not difficult to prove that $\psi \xrightarrow[\text{TGT}]{\delta} \phi$ implies $\psi \approx_1 \chi \cdot \phi$, where χ is a stepwise-or-nil proof term verifying $ns(\chi) = \text{length}(\delta)$ and $d(\chi[i]) = d(\delta[i])$ if $i < \text{length}(\delta)$: induction on $d(\delta[0])$ yields the property for one-steps, and then induction on $\text{length}(\delta)$ suffices. Observe that $\phi(\epsilon) \in \Sigma$ implies $mind(\phi) > 0$. To verify that ϕ is convergent, the strip lemma or parallel moves lemma, valid for TGT since it is an orthogonal iTRS, yields that ψ convergent implies ϕ convergent. Thus we conclude.

To extend factorisation to any proof term we need to be able to *swap* its components. The (InOut) and (OutIn) equations allow for this swapping in simple cases, e.g. as in the following permutation equivalence judgement: $f(\mu(a)) \cdot \mu(g(a)) \approx_1 \mu(\mu(a)) \approx_1 \mu(f(a)) \cdot g(\mu(a))$, where $\mu : f(x) \rightarrow g(x)$. The following lemma shows that such an operation can be performed in a general case.

Lemma 6. *Let ψ be a finite stepwise-or-nil proof term. Then there exist two numbers $n, n' < \omega$ such that, for any convergent proof term ξ verifying $\text{tgt}(\xi) = \text{src}(\psi)$ and $mind(\xi) \geq n + n'$, a finite stepwise-or-nil proof term ψ' and a convergent proof term ξ' can be found, such that $\xi \cdot \psi \approx_1 \psi' \cdot \xi'$, $ns(\psi') = ns(\psi)$, $d(\psi'[i]) = d(\psi[i])$ for all i , and $mind(\xi') \geq mind(\xi) - n' \geq n$.*

Proof. Induction on $ns(\psi)$, the interesting case being when ψ is a one-step.

Let μ be the rule symbol occurring in ψ , and p the position of the occurrence of μ in ψ . We consider $n := |p|$ and n' to be the depth of the deepest variable occurrence in the left-hand side of μ . It is not difficult to obtain that $\psi = C[t_1, \dots, t_{j-1}, \mu(u_1, \dots, u_m), t_{j+1}, \dots, t_k]$ where the depth of all the holes in C is exactly n . By carefully defining the *compressed to fixed prefix form* of a proof term for which no activity is performed at any position in a prefix-closed set, we can obtain that $\xi \approx_1 C[\xi_1, \dots, \xi_{j-1}, l[\phi_1, \dots, \phi_m], \xi_{j+1}, \dots, \xi_k]$. Therefore

$$\begin{aligned} \xi \cdot \psi &\approx_1 C[\xi_1 \cdot t_1, \dots, \xi_{j-1} \cdot t_{j-1}, l[\phi_1, \dots, \phi_m] \cdot \mu(u_1, \dots, u_m), \xi_{j+1} \cdot t_{j+1}, \dots, \xi_k \cdot t_k] \\ &\approx_1 C[\xi_1, \dots, \xi_{j-1}, \mu(\phi_1, \dots, \phi_m), \xi_{j+1}, \dots, \xi_k] \\ &\approx_1 C[s_1 \cdot \xi_1, \dots, s_{j-1} \cdot \xi_{j-1}, \mu(w_1, \dots, w_m) \cdot h[\phi_1, \dots, \phi_m], s_{j+1} \cdot \xi_{j+1}, \dots, s_k \cdot \xi_k] \\ &\approx_1 C[s_1, \dots, s_{j-1}, \mu(w_1, \dots, w_m), s_{j+1}, \dots, s_k] \cdot C[\xi_1, \dots, \xi_{j-1}, h[\phi_1, \dots, \phi_m], \xi_{j+1}, \dots, \xi_k] \end{aligned}$$

where $\mu : l \rightarrow h$, $s_i = \text{src}(\xi_i)$ and $w_i = \text{src}(\phi_i)$.

We consider $\psi' = C[s_1, \dots, s_{j-1}, \mu(w_1, \dots, w_m), s_{j+1}, \dots, s_k]$ and $\xi' = C[\xi_1, \dots, \xi_{j-1}, h[\phi_1, \dots, \phi_m], \xi_{j+1}, \dots, \xi_k]$. Convergence of ξ' stems from convergence of each ξ_i and each ϕ_i . Finally, a careful analysis of minimal activity depth yields the desired condition about $\text{mind}(\xi')$.

Now we can prove the general factorisation result.

Proposition 4. *Let ψ be a convergent proof term and $n < \omega$. Then there exist χ and ϕ such that $\psi \approx_1 \chi \cdot \phi$, χ is a finite stepwise-or-nil proof term, ϕ is convergent and $\text{mind}(\phi) > n$.*

Proof. We proceed by induction on ψ .

If ψ is an infinitary multi-step, then we proceed by induction on n . Lem. 5 implies $\psi \approx_1 \chi_0 \cdot f(\psi_1, \dots, \psi_m)$. If $n = 0$ then we are done. Otherwise convergence of all ψ_i can be obtained from convergence of $f(\psi_1, \dots, \psi_m)$ by a careful study of projections. Then IH applied to the ψ_i 's yields $\psi \approx_1 \chi_0 \cdot f(\chi_1 \cdot \phi_1, \dots, \chi_m \cdot \phi_m)$. From this proof term, a permutation equivalence argument allows to conclude.

Assume $\psi = \psi_1 \cdot \psi_2$. Then IH on ψ_2 yields $\psi \approx_1 \psi_1 \cdot \chi_2 \cdot \phi_2$. IH applies also on ψ_1 w.r.t. a sufficiently big n' , yielding $\psi \approx_1 \chi_1 \cdot \phi_1 \cdot \chi_2 \cdot \phi_2$. Then Lem. 6 on $\phi_1 \cdot \chi_2$ implies $\psi \approx_1 \chi_1 \cdot \chi'_2 \cdot \phi'_1 \cdot \phi_2$, which suffices to conclude.

If $\psi = \cdot_{i < \omega} \psi_i$, then it is enough to observe that IH can be applied to $\psi_0 \cdot \dots \cdot \psi_k$, where $\text{mind}(\psi_i) > n$ if $i > k$.

If $\psi = f(\psi_1, \dots, \psi_m)$ then an easy inductive argument suffices.

Finally, if $\psi = \mu(\psi_1, \dots, \psi_m)$, then two cases must be considered: μ is collapsing or not. In either case an easy argument suffices. If μ is not collapsing, then observe that the (Struct) equation can be extended to arbitrary contexts having a finite number of holes.

Theorem 1. *Let ψ be a convergent proof term. Then there exists some stepwise proof term ϕ verifying $\psi \approx \phi$ and $\text{ns}(\phi) \leq \omega$.*

Proof. We define the sequences of proof terms $\langle \psi_i \rangle_{i < \omega}$ and $\langle \phi_i \rangle_{i < \omega}$ as follows. We start defining $\psi_0 := \psi$. Then, for each $i < \omega$, we define ϕ_i and ψ_{i+1} to be proof terms such that $\psi_i \approx_1 \phi_i \cdot \psi_{i+1}$, ϕ_i is a finite stepwise-or-nil proof term and either $\text{mind}(\psi_{i+1}) > \text{mind}(\psi_i)$ or $\text{mind}(\psi_{i+1}) = \text{mind}(\psi_i) = \omega$; cf. Prop. 4. Observe that $\text{mind}(\psi_i) < \omega$ implies $\text{mind}(\phi_i) = \text{mind}(\psi_i)$, so in that case ϕ_i is a stepwise proof term, i.e. it is not trivial. Moreover, an easy induction on n yields $\psi \approx_1 \phi_0 \cdot \dots \cdot \phi_n \cdot \psi_{n+1}$ for all n .

There are two cases to consider. If there is some n such that ψ_n is trivial, then it is enough to take $\phi := \phi_0 \cdot \dots \cdot \phi_{n-1}$ for the minimal such n (or just $\phi := \text{src}(\psi)$ if $\psi_0 = \psi$ is trivial). Otherwise, it is easy to prove that $\psi \approx \cdot_{i < \omega} \phi_i$ using the Lim rule, and, moreover, ϕ_i being finite for all i implies $\text{ns}(\cdot_{i < \omega} \phi_i) \leq \omega$.

7 Conclusions and Future Work

The reasoning in the compression proof in Sec. 6 can be extended in order to obtain *standardisation* results. As noted in [6], a concept of standard reduction

being adequate for infinitary rewriting should be used, leftmost-outermost reduction does not fit in this setting. We claim that it is possible to prove the existence of a *unique* standard reduction in each permutation equivalence class, using depth-leftmost standardness as defined in [6].

It seems natural to consider a variant of permutation equivalence in which the Lim-rule can be used at most once in a derivation, and only as its last step. The derivations of permutation equivalence in our examples in Sec. 4 are all of this form. By a proof-theoretic analysis using the purported standardisation result, one can show that the restricted variant is equivalent to the more general version used in this paper. We plan to present these results in a future paper.

Another area for further exploration is the comparison of permutation equivalence as defined in this paper with other notions of equivalence of reduction. Compare [11], Ch. 8 and [9], where the equivalence of several such notions is established for finitary rewriting. An obvious first candidate would be Lévy equivalence as defined in [5] and [11], Ch. 12, via projections of reductions.

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