

Implicational Relevance Logic is 2-EXPTIME-Complete^{*}

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Abstract. We show that provability in the implicational fragment of relevance logic is complete for doubly exponential time, using reductions to and from coverability in branching vector addition systems.

Key Words. Relevance logic, branching VASS, focusing proofs, complexity

1 Introduction

Relevance logic $\mathbf{R}$ [1, 7] provides a formalisation of ‘relevant’ implication: in such a system, the formula $A \rightarrow B$ indicates that the truth of A is actually useful in establishing B ; an example of an *irrelevant* implication valid in classical logic would be $B \rightarrow (A \rightarrow B)$.

The pure implicational fragment $\mathbf{R}_{\rightarrow}$ of $\mathbf{R}$ was developed independently by Moh [15] in 1950 and Church [4] in 1951, and is as such the oldest of the relevance logics. Kripke already presented in 1959 a decision algorithm for provability in $\mathbf{R}_{\rightarrow}$ [10], which was later extended to larger and larger subsets of $\mathbf{R}$, like the conjunctive-implicational fragment $\mathbf{R}_{\rightarrow, \wedge}$. Several negative results by Urquhart would however foil any hope for elementary algorithms: first in 1984 when he showed the undecidability of the full logic $\mathbf{R}$ [20]; later in 1999 with a proof that $\mathbf{R}_{\rightarrow, \wedge}$ suffers from a non primitive-recursive complexity: it is ACKERMANN-complete [22]. This left a gigantic gap for the implicational fragment $\mathbf{R}_{\rightarrow}$, between an earlier EXPSpace lower bound [21] and the ACKERMANN upper bound shared by the variants of Kripke’s procedure.

In this paper, we close this gap and show that provability in $\mathbf{R}_{\rightarrow}$ is 2-EXPTIME-complete. Our proof relies crucially on a recent result by Demri et al. [6], who show the 2-EXPTIME-completeness of the coverability problem in *branching vector addition systems with states* (BVASS). These systems form a natural generalisation of vector addition systems, and have been defined independently in a variety of contexts (see the survey [18] and Sec. 3 below), notably that of provability in *multiplicative exponential linear logic* (**MELL**, see [9]). More precisely:

- In Sec. 4, we show that so-called *expansive* BVASSs can simulate proofs in $\mathbf{R}_{\rightarrow}$ in a natural manner by exploiting the subformula property of its usual sequent calculus $\mathbf{LR}_{\rightarrow}$. We then show how to reduce reachability in expansive BVASS to coverability, thereby providing a decision procedure in doubly exponential time.

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- The matching hardness proof in Sec. 5 relies on the one hand on *comprehensive* instances of the BVASS coverability problem, and on the other hand on a new *focusing* sequent calculus $FR_{\rightarrow}$ for $R_{\rightarrow}$.
- The reduction from $R_{\rightarrow}$ provability to expansive BVASS reachability is actually a special case of a more general reduction proved in [11] for the multiplicative exponential fragment of intuitionistic contractive linear logic (**IMELLC**), i.e. **IMELL** with structural contraction. Our reduction in Sec. 4.3 from expansive reachability to coverability thus entails that **IMELLC** provability is 2-EXPTIME-complete, as explained in Sec. 6.

Due to space constraints, some material is omitted and can be found in the full paper available from <http://arxiv.org/abs/1402.0705>.

Let us first recall the formal definition of $R_{\rightarrow}$ before turning to that of BVASSs in Sec. 3.

2 The Implicational Fragment $R_{\rightarrow}$

The reader will find in [7, Sec. 4] a nice overview of the decision problem for R , covering in particular Kripke’s solution for $R_{\rightarrow}$ [10] and Urquhart’s lower bound argument for $R_{\rightarrow, \wedge}$ [22].

2.1 A Sequent Calculus

We recall here the formal definition of $R_{\rightarrow}$ as a sequent calculus $LR_{\rightarrow}$ in Gentzen’s style. Let $\mathcal{A}$ be a countable set of atomic propositions; we define the set of formulæ as following the abstract syntax

$$A ::= a \mid A \rightarrow A \quad (\text{implicational formulæ})$$

where a ranges over $\mathcal{A}$. We consider $\rightarrow$ to be right-associative, e.g. $A \rightarrow B \rightarrow C$ denotes $A \rightarrow (B \rightarrow C)$. In the following rules, we use $A, B, C, \dots$ to denote implicational formulæ and $\Gamma, \Delta, \dots$ to denote multisets of such formulæ; commas in e.g. ‘ Γ, A ’ and ‘ Γ, Δ ’ denote multiset unions of Γ with the singleton A and with Δ respectively; finally, a *sequent* is a pair ‘ $\Gamma \vdash A$ ’ stating that the succedent A is valid assuming the antecedent Γ to be relevant:

$$\begin{array}{c} \frac{}{A \vdash A} \text{ (Id)} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \text{ (C)} \\[1em] \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} (\rightarrow_L) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} (\rightarrow_R) \end{array}$$

As we work with multisets, this sequent calculus includes implicitly the structural ‘exchange’ rule. It does however not feature the classical ‘weakening’ rule—which would defeat the very point of relevance—nor the ‘cut’ rule—which is admissible. A visible consequence of this definition is that the calculus enjoys the *subformula property*: all the formulæ in rule premises are subformulæ of the formulæ appearing in the corresponding consequences.

2.2 Decidability and Complexity

With hindsight, the decision procedure of Kripke [10] for provability in the implicational fragment of relevance logic can be seen as a precursor for many later algorithms that rely on the existence of a *well quasi ordering* (wqo) for their termination [17]. This decision procedure can be understood as an application of Dickson's Lemma to prove the finiteness of 'irredundant' proof trees for the target sequent $\vdash A$. Furthermore, combinatorial analyses of Dickson's Lemma as e.g. in [8] provide explicit upper bounds on the size of those irredundant proofs, in the form of the Ackermann function in the size of A , yielding an **ACKERMANN** upper bound for $\mathbf{R}_{\rightarrow}$ provability, as shown by Urquhart [22].

Regarding lower bounds, Urquhart in [21, Sec. 9] explains how to derive **EXSPACE**-hardness for $\mathbf{R}_{\rightarrow}$, using model-theoretic techniques to reduce from the word problem for finitely presented commutative semigroups [14].

2.3 Strict λ -Calculus

The implicational fragment $\mathbf{R}_{\rightarrow}$ is in bijection with the typing rules of the simply typed λI -calculus, where abstracted terms $\lambda x.t$ are well-formed only if x appears free in t ; see [5, Sec. 9F]. This means that $\mathbf{R}_{\rightarrow}$ provability can be restated as the *type inhabitation* problem for the simply typed λI -calculus. Our complexity results should then be contrasted with the **PSPACE**-completeness of the same problem for the simply typed λ -calculus [19].

3 Branching VASS

Branching vector addition systems with states (BVASS) have been independently defined in several contexts; see [18] for a survey.

3.1 Formal Definitions

Given d in $\mathbb{N}$, we write $\bar{0}$ for the null vector in $\mathbb{N}^d$, and for $0 < i \leq d$, $\bar{e}_i$ for the unit vector in $\mathbb{N}^d$ with 1 on coordinate i and 0 everywhere else. Let $U_d \stackrel{\text{def}}{=} \{\bar{e}_i, -\bar{e}_i \mid 0 < i \leq d\}$. Syntactically, an *ordinary BVASS* is a tuple $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$ where Q is a finite set of *states*, d is a *dimension* in $\mathbb{N}$, and $T_u \subseteq Q \times U_d \times Q$ and $T_s \subseteq Q^3$ are respectively finite sets of *unary* and *split* rules. We denote unary rules $(q, \bar{u}, q_1)$ in T_u with $\bar{u}$ in U_d by ' $q \xrightarrow{\bar{u}} q_1$ ' and split rules (q, q_1, q_2) in T_s by ' $q \rightarrow q_1 + q_2$ '.

We define the semantics of an ordinary BVASS through a deduction system over *configurations* $(q, \bar{v})$ in $Q \times \mathbb{N}^d$:

$$\frac{q, \bar{v}}{q_1, \bar{v} + \bar{e}_i} \text{ (incr)} \quad \frac{q, \bar{v} + \bar{e}_i}{q_1, \bar{v}} \text{ (decr)} \quad \frac{q, \bar{v}_1 + \bar{v}_2}{q_1, \bar{v}_1 \quad q_2, \bar{v}_2} \text{ (split)}$$

respectively for unary rules $q \xrightarrow{\bar{e}_i} q_1$ and $q \xrightarrow{-\bar{e}_i} q_1$ in T_u and a split rule $q \rightarrow q_1 + q_2$ in T_s ; in (split) '+' denotes component-wise addition in $\mathbb{N}^d$. Such a deduction

system can be employed either *top-down* or *bottom-up* depending on the decision problem at hand (as with tree automata); the top-down direction will correspond in a natural way to goal-directed *proof search* in the sequent calculus of Sec. 2.

Ordinary BVASSs are a slight restriction over BVASSs, which would in general allow any vector in $\mathbb{Z}^d$ in unary rules. Because they often lead to more readable proofs, we only employ ordinary BVASSs in this paper. This is at no loss of generality, since one can build an ordinary BVASS ‘equivalent’ to a given BVASS in logarithmic space, where equivalence should be understood relative to the reachability and coverability problems; see [18] for details.

Reachability. Branching VASSs are associated with a natural decision problem: *reachability* asks, given a BVASS $\mathcal{B}$, a root state q_r , and a leaf state q_ℓ , whether there exists a deduction tree with root label $(q_r, \bar{0})$ and every leaf labelled $(q_\ell, \bar{0})$; such a deduction tree is called a *reachability witness*. De Groote et al. [9] have shown that this problem is recursively equivalent to **MELL** provability, and it is currently unknown whether it is decidable—both problems are however known to be of non-elementary computational complexity [11].

Let us introduce some additional notation that will be handy in proofs. We write ‘ $\mathcal{B}, T, q_\ell \triangleright q, \bar{v}$ ’ if there exists a deduction tree of $\mathcal{B}$ with root label $(q, \bar{v})$ and leaves labelled by $(q_\ell, \bar{0})$, which uses each rule in $T \subseteq T_u \uplus T_s$ at least once. Such *root judgements* can be derived through the deduction system

$$\frac{}{\mathcal{B}, \emptyset, q_\ell \triangleright q_\ell, \bar{0}} \quad \frac{\mathcal{B}, T, q_\ell \triangleright q_1, \bar{v} + \bar{e}_i}{\mathcal{B}, T \cup \{q \xrightarrow{\bar{e}_i} q_1\}, q_\ell \triangleright q, \bar{v}} \\[1em] \frac{\mathcal{B}, T, q_\ell \triangleright q_1, \bar{v}}{\mathcal{B}, T \cup \{q \xrightarrow{-\bar{e}_i} q_1\}, q_\ell \triangleright q, \bar{v} + \bar{e}_i} \quad \frac{\mathcal{B}, T_1, q_\ell \triangleright q_1, \bar{v}_1 \quad \mathcal{B}, T_2, q_\ell \triangleright q_2, \bar{v}_2}{\mathcal{B}, T_1 \cup T_2 \cup \{q \rightarrow q_1 + q_2\}, q_\ell \triangleright q, \bar{v}_1 + \bar{v}_2}$$

We write more simply ‘ $\mathcal{B}, q_\ell \triangleright q, \bar{v}$ ’ if there exists $T \subseteq T_u \uplus T_s$ such that $\mathcal{B}, T, q_\ell \triangleright q, \bar{v}$. With these notations, the reachability problem asks whether $\mathcal{B}, q_\ell \triangleright q_r, \bar{0}$.

3.2 Root Coverability

Our interest in this paper lies in a relaxation of the reachability problem, where we ask instead to *cover* the root: given as before $\langle \mathcal{B}, q_r, q_\ell \rangle$, we ask whether there exists a *coverability witness*, i.e. a deduction tree with root $(q_r, \bar{v})$ for some $\bar{v}$ in $\mathbb{N}^d$ and leaves $(q_\ell, \bar{0})$; in other words whether $\mathcal{B}, q_\ell \triangleright q_r, \bar{v}$ for some $\bar{v}$ in $\mathbb{N}^d$.

This problem was shown decidable by Verma and Goubault-Larrecq [23], and was later proven 2-EXPTIME-complete by Demri et al. [6] in a slight variant called *branching vector addition systems* (BVAS):

Fact 1 (6, Thm.8 and Thm.21). *BVAS coverability is 2-EXPTIME-complete.*

Branching VAS are not equipped with a state space Q . Their coverability problem is stated slightly differently, but is easy to reduce in both directions to

BVASS coverability. This would not be worth mentioning here if it were not for the following instrumental corollary of their proof, which exploits an encoding of d -dimensional BVASSs into $(d + 6)$ -dimensional BVASSs:

Corollary 2. *Coverability in a BVASS $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$ can be solved in deterministic time $2^{2^{O(n \cdot \log(n \cdot \log |Q|))}}$, where n denotes the size of the representation of $\langle d, T_u \rangle$.*

Corollary 2 entails that coverability remains in 2-EXPTIME for BVASSs with double exponential state space. This is an easy result, which we show in the full paper.

4 Upper Bound

In order to show a 2-EXPTIME upper bound for $\mathbf{R}_{\rightarrow}$ provability, we introduce as an intermediate decision problem the *expansive reachability* problem for BVASS (Sec. 4.1). Then, the first step of our proof in Sec. 4.2 takes us from the sequent calculus $LR_{\rightarrow}$ to reachability in expansive BVASS. This is a simple construction that relies on the subformula property of $LR_{\rightarrow}$, and is actually a particular case of a more general reduction shown in [11, Prop.9]. The new technical result here is the second step: a reduction from expansive BVASS reachability to BVASS coverability, which is shown in Sec. 4.3. This new reduction also entails new upper bounds for provability in extensions of $LR_{\rightarrow}$ studied in [11]; see Sec. 6.

4.1 Expansive Reachability

An *expansive BVASS* is a BVASS with an additional deduction rule:

$$\frac{q, \bar{v} + \bar{e}_i}{q, \bar{v} + 2\bar{e}_i} \text{ (expansion)}$$

Note that expansions could be simulated by unary rules $q \xrightarrow{-\bar{e}_i} q_i \xrightarrow{\bar{e}_i} q'_i \xrightarrow{\bar{e}_i} q$ for all q in Q and $0 < i \leq d$; we prefer to see them as new deduction rules.

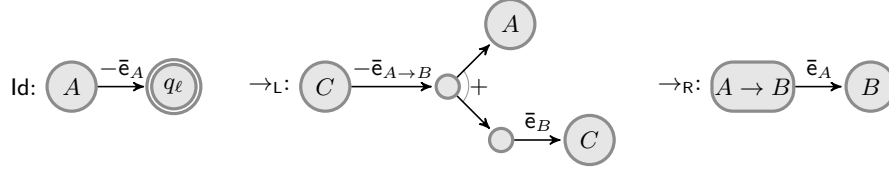
This yields a new rule for root judgements, which we denote using ‘ $\triangleright_e$ ’ to emphasise that we allow expansion rules:

$$\frac{\mathcal{B}, T, q_\ell \triangleright_e q, \bar{v} + 2\bar{e}_i}{\mathcal{B}, T, q_\ell \triangleright_e q, \bar{v} + \bar{e}_i}$$

The *expansive reachability* problem then asks, given an expansive BVASS $\mathcal{B}$ and two states q_r and q_ℓ , whether $\mathcal{B}, q_\ell \triangleright_e q_r, \bar{0}$.

4.2 From $LR_{\rightarrow}$ to Expansive Reachability

We prove here the following reduction:

Fig. 1: The rules and intermediate states of $\mathcal{B}_F$.

Proposition 3. *There is a logarithmic space reduction from provability in $\mathbf{R}_{\rightarrow}$ to expansive reachability in ordinary BVASSs.*

Let us consider an instance $\langle F \rangle$ of the provability problem for $\mathbf{R}_{\rightarrow}$ for an implicational formula F . The instance is positive if and only if we can find a proof for $\vdash F$ in $\mathbf{LR}_{\rightarrow}$. Thanks to the subformula property, we know that in such a proof, all the sequents $\Gamma \vdash A$ must use subformulae of F . That is, if we denote by S the set of subformulae of F , then Γ is in $\mathbb{N}^S$ and A in S .

We construct from F an expansive BVASS $\mathcal{B}_F$ that implements proof search in $\mathbf{LR}_{\rightarrow}$ restricted to subformulae of F . We define for this $\mathcal{B}_F \stackrel{\text{def}}{=} \langle Q_F, |S|, T_u, T_s \rangle$ where the state space Q_F includes S and a distinguished leaf state q_ℓ . It also includes some intermediate states as introduced in the translations of the rules of $\mathbf{LR}_{\rightarrow}$ into rules in $T_u \cup T_s$ depicted in Fig. 1. Note that (C) has no associated rule; it relies instead on expansions in $\mathcal{B}_F$. The full state space Q_F of $\mathcal{B}_F$, including intermediate states, is thus of size $O(|F|^2)$.

Let us write $\bar{v}_\Gamma$ for the vector in $\mathbb{N}^{|S|}$ associated with a multiset Γ in $\mathbb{N}^S$. The proof of Prop. 3 is a consequence of the following claim instantiated with $A = F$ and $\Gamma = \emptyset$:

Claim 3.1. For all Γ in $\mathbb{N}^S$ and A in S , $\Gamma \vdash A$ if and only if $\mathcal{B}_F, q_\ell \triangleright_e A, \bar{v}_\Gamma$.

This results from a straightforward induction on the structure of proofs in $\mathbf{LR}_{\rightarrow}$ and expansive root judgements for $\mathcal{B}_F$; see the full paper for details.

4.3 From Expansive Reachability to Coverability

The second step of our proof that $\mathbf{R}_{\rightarrow}$ provability is in 2-EXPTIME is then to reduce expansive reachability to coverability in BVASS. Our reduction incurs an exponential blow-up in the number of states, but thanks to Cor. 2, this still results in a 2-EXPTIME algorithm:

Proposition 4. *There is a polynomial space reduction from BVASS expansive reachability to BVASS coverability.*

Topmost Increments. Consider an instance $\langle \mathcal{B}, q_r, q_\ell \rangle$ of the expansive reachability problem with $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$. Because the root vector of an expansive reachability witness must be $\bar{0}$, we can identify along each branch of the witness and for each coordinate $0 < i \leq d$ the topmost (i.e. closest to the root)

application of an (incr) rule—possibly no such increment ever occurs on some branches.

Assume without loss of generality that q_ℓ has no outgoing transition in $\mathcal{B}$. We construct a new BVASS $\mathcal{B}^\dagger = \langle Q^\dagger, d, T_u^\dagger, T_s^\dagger \rangle$ with additional states q_ℓ^i and unary rules $q_\ell \xrightarrow{\bar{e}_i} q_\ell^i \xrightarrow{-\bar{e}_i} q_\ell$ for every $0 < i \leq d$. Then $\mathcal{B}, q_\ell \triangleright_e q_r, \bar{0}$ if and only if $\mathcal{B}^\dagger, q_\ell \triangleright_e q_r, \bar{0}$ (observe in particular that no expansion in q_ℓ^i can occur in an expansive reachability witness). Additionally, the new rules allow us to assume that there *is* a topmost increment for each branch and every coordinate of an expansive reachability witness of $\mathcal{B}^\dagger$.

Let $[d] \stackrel{\text{def}}{=} \{1, \dots, d\}$. The root judgement relation can be refined as ' $\triangleright_e^s$ ' with a set $s \subseteq [d]$ of coordinates. The intended semantics for $i \in s$ is that there is at least one increment on coordinate i earlier on the path from the root in the expansive reachability witness. Formally, at the leaves

$$\overline{\mathcal{B}^\dagger, \emptyset, q_\ell \triangleright_e^{[d]} q_\ell, \bar{0}}$$

since by assumption every coordinate must see an increase. Then, an increment is either topmost or not:

$$\frac{\mathcal{B}^\dagger, T, q_\ell \triangleright_e^{s \uplus \{i\}} q_1, \bar{w} + \bar{e}_i}{\mathcal{B}^\dagger, T \cup \{q \xrightarrow{\bar{e}_i} q_1\}, q_\ell \triangleright_e^s q, \bar{w}} \quad \frac{\mathcal{B}^\dagger, T, q_\ell \triangleright_e^{s \cup \{i\}} q_1, \bar{v} + \bar{e}_i}{\mathcal{B}^\dagger, T \cup \{q \xrightarrow{\bar{e}_i} q_1\}, q_\ell \triangleright_e^{s \cup \{i\}} q, \bar{v}}$$

where $\bar{w}(i) = 0$ and ' $\uplus$ ' denotes disjoint union. Decrements and expansions are necessarily dominated by the topmost increment:

$$\frac{\mathcal{B}^\dagger, T, q_\ell \triangleright_e^{s \cup \{i\}} q_1, \bar{v}}{\mathcal{B}^\dagger, T \cup \{q \xrightarrow{-\bar{e}_i} q_1\}, q_\ell \triangleright_e^{s \cup \{i\}} q, \bar{v} + \bar{e}_i} \quad \frac{\mathcal{B}^\dagger, T, q_\ell \triangleright_e^{s \cup \{i\}} q, \bar{v} + 2\bar{e}_i}{\mathcal{B}^\dagger, T, q_\ell \triangleright_e^{s \cup \{i\}} q, \bar{v} + \bar{e}_i}$$

Finally, the same topmost increments have been seen on both branches of a split:

$$\frac{\mathcal{B}^\dagger, T_1, q_\ell \triangleright_e^s q_1, \bar{v}_1 \quad \mathcal{B}^\dagger, T_2, q_\ell \triangleright_e^s q_2, \bar{v}_2}{\mathcal{B}^\dagger, T_1 \cup T_2 \cup \{q \rightarrow q_1 + q_2\}, q_\ell \triangleright_e^s q, \bar{v}_1 + \bar{v}_2}$$

The refined root judgements verify

$$\mathcal{B}, q_\ell \triangleright_e q_r, \bar{0} \text{ implies } \mathcal{B}^\dagger, q_\ell \triangleright_e^\emptyset q_r, \bar{0}, \quad (1)$$

the converse implication being immediate by removing the ' s ' annotations.

Reduction to Coverability. We construct a BVASS $\mathcal{B}^\ddagger = \langle Q^\ddagger \times 2^{[d]}, d, T_u^\ddagger, T_s^\ddagger \rangle$ and build a coverability instance $\langle \mathcal{B}^\ddagger, (q_r, \emptyset), (q_\ell, [d]) \rangle$. The idea is to maintain a set $s \subseteq [d]$ as in the refined judgements $\triangleright_e^s$; however since we cannot test to zero we will rely instead on nondeterminism. Let

$$\begin{aligned} T_u^\ddagger &\stackrel{\text{def}}{=} \{(q, s) \xrightarrow{\bar{e}_i} (q_1, s \cup \{i\}) \mid q \xrightarrow{\bar{e}_i} q_1 \in T_u^\dagger, s \subseteq [d]\} & (\text{incr}^\ddagger) \\ &\cup \{(q, s \cup \{i\}) \xrightarrow{-\bar{e}_i} (q_1, s \cup \{i\}) \mid q \xrightarrow{-\bar{e}_i} q_1 \in T_u^\dagger, s \subseteq [d]\}, & (\text{decr}^\ddagger) \\ T_s^\ddagger &\stackrel{\text{def}}{=} \{(q, s) \rightarrow (q_1, s) + (q_2, s) \mid q \rightarrow q_1 + q_2 \in T_s^\dagger, s \subseteq [d]\}. & (\text{split}^\ddagger) \end{aligned}$$

For $s \subseteq [d]$ and $\bar{v}$ in $\mathbb{N}^d$, we define $s \cdot \bar{v}$ for each $0 < i \leq d$ by

$$(s \cdot \bar{v})(i) \stackrel{\text{def}}{=} \begin{cases} \bar{v}(i) & \text{if } i \in s, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

We show the following claims in the full paper:

Claim 4.1. If $\mathcal{B}^\dagger, q_\ell \triangleright_e^s q, \bar{v}$, then there exists $\bar{v}' \geq \bar{v}$ such that $\mathcal{B}^\dagger, (q_\ell, [d]) \triangleright (q, s), \bar{v}'$.

Claim 4.2. If $\mathcal{B}^\dagger, (q_\ell, [d]) \triangleright (q, s), \bar{v}$, then $\mathcal{B}^\dagger, q_\ell \triangleright_e q, s \cdot \bar{v}$.

Proof of Prop. 4. If $\mathcal{B}, q_\ell \triangleright_e q_r, \bar{0}$, then by (1), $\mathcal{B}^\dagger, q_\ell \triangleright_e^\emptyset q_r, \bar{0}$, thus by Claim 4.1, there exists $\bar{v}$ such that $\mathcal{B}^\dagger, (q_\ell, [d]) \triangleright (q_r, \emptyset), \bar{v}$, i.e. we can cover $(q_r, \emptyset)$ in $\mathcal{B}^\dagger$.

Conversely, if $\mathcal{B}^\dagger, (q_\ell, [d]) \triangleright (q_r, \emptyset), \bar{v}$, then by Claim 4.2, $\mathcal{B}^\dagger, q_\ell \triangleright_e q_r, \emptyset \cdot \bar{v}$ where $\emptyset \cdot \bar{v} = \bar{0}$. Therefore $\mathcal{B}, q_\ell \triangleright_e q_r, \bar{0}$ in the original BVASS $\mathcal{B}$. $\square$

Theorem 5. *Provability in $\mathbf{R}_{\rightarrow}$ is in 2-EXPTIME.*

Proof. By Prop. 3 and Prop. 4, from a provability instance $\langle F \rangle$, we can reduce to a coverability instance $\langle \mathcal{B}_F^\dagger, (q_r, \emptyset), (q_\ell, [|F|]) \rangle$ where $\mathcal{B}_F^\dagger$ has dimension $|F|$ and a number of states in $2^{p(|F|)}$ for a polynomial p . By Cor. 2, this coverability instance can be solved in double exponential time in $|F|$. Note that the coverability check can be performed on-the-fly from F to avoid the explicit construction of $\mathcal{B}_F^\dagger$. $\square$

5 Lower Bound

In this section, we exhibit a reduction from BVASS coverability to $\mathbf{R}_{\rightarrow}$ provability, thereby showing its 2-EXPTIME-hardness.

Previous reductions from counter machines to substructural logics in [12, 22, 11] actually reduce to provability in the logic extended with a *theory* encoding the rules of the system, which is then reduced to the basic logic. This last step relies in an essential way on the presence of exponential or additive connectives to ‘dispose’ of unused rules.

Having neither exponential nor additive connectives at our disposal, we introduce in Sec. 5.1 a *comprehensive* variant of the expansive reachability problem, where every rule should be employed at least once in the deduction. We further avoid the use of a theory and define in Sec. 5.2 a *focusing* calculus for $\mathbf{R}_{\rightarrow}$, from which the correctness of the reduction given in Sec. 5.3 will be facilitated.

5.1 Comprehensive Reachability

Given an expansive BVASS $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$ and two states q_r and q_ℓ , the *comprehensive reachability* problem asks whether there exists a deduction tree of $\mathcal{B}$ with root label $(q_r, \bar{0})$ and leaves label $(q_\ell, \bar{0})$, such that every rule in $T_u \cup T_s$ is used at least once. Termed differently, it asks whether $\mathcal{B}, T_u \cup T_s, q_\ell \triangleright_e q_r, \bar{0}$. We show that BVASS coverability can be reduced to comprehensive expansive reachability, hence by Fact 1:

Proposition 6. *Comprehensive reachability in expansive ordinary BVASS is 2-EXPTIME-hard.*

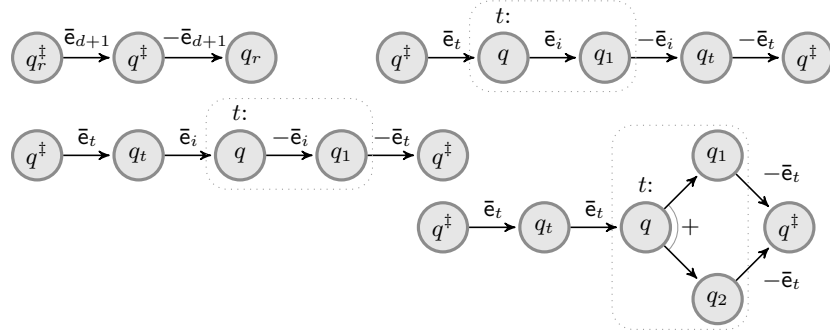


Fig. 2: The rules of $\mathcal{B}^\dagger$ in the proof of Prop. 6, where t ranges over $T_u^\dagger \cup T_s$.

Increasing Reachability. Let us consider an instance $\langle \mathcal{B}, q_r, q_\ell \rangle$ of the coverability problem in an ordinary BVASS $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$. As a first step, we construct an ordinary BVASS $\mathcal{B}^\dagger \stackrel{\text{def}}{=} \langle Q, d, T_u^\dagger, T_s \rangle$ with additional *increases* $q \xrightarrow{\bar{e}_i} q$ for every q in Q and $0 < i \leq d$. We claim that coverability in $\mathcal{B}$ is equivalent to reachability in $\mathcal{B}^\dagger$:

Claim 6.1. There exists $\bar{v}$ in $\mathbb{N}^d$ such that $\mathcal{B}, q_\ell \triangleright q_r, \bar{v}$ iff $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{0}$.

Proof Sketch. Clearly, if $\mathcal{B}, q_\ell \triangleright q_r, \bar{v}$ for some $\bar{v}$ in $\mathbb{N}^d$, then $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{v}$, and using increases in q_r shows $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{0}$. Conversely, if there is a reachability witness for $\langle \mathcal{B}^\dagger, q_r, q_\ell \rangle$, then we can assume that increases $q \xrightarrow{\bar{e}_i} q$ occur as close to the root as possible. As increases occurring right below increments, decrements, or splits can be permuted locally to occur right above, such a reachability witness has all its increases at the root. The deduction tree below those increases is labelled $(q_r, \bar{v})$ for some $\bar{v}$ in $\mathbb{N}^d$ and is also a deduction tree of $\mathcal{B}$. $\square$

Comprehensive Root Rules. The second step of the reduction from BVASS coverability builds an ordinary BVASS $\mathcal{B}^\ddagger \stackrel{\text{def}}{=} \langle Q^\ddagger, d+1 + |T_u^\dagger \cup T_s|, T_u^\ddagger, T_s \rangle$ where $Q^\ddagger \stackrel{\text{def}}{=} Q \uplus \{q_r^\ddagger, q_t^\ddagger\} \uplus \{q_t \mid t \in T_u^\dagger \cup T_s\}$. It features an additional set of unary ‘root’ rules—depicted in Fig. 2—designed to allow any rule in $T_u^\dagger \cup T_s$ to be employed in a reachability witness.

The idea is to introduce a new state $q^\ddagger$ and a new coordinate for each rule t in $T_u^\dagger \cup T_s$. Starting from $q^\ddagger$, $\mathcal{B}^\ddagger$ can simulate any rule t from $T_u^\dagger \cup T_s$ by first incrementing by the corresponding unit vector $\bar{e}_t$, then applying the rule, and finally decrementing by $\bar{e}_t$ to return to $q^\ddagger$. This ensures that, if $\mathcal{B}^\dagger, T, q_\ell \triangleright_e q^\ddagger, \bar{v}$ for some $\bar{v}$ in $\mathbb{N}^d$, then $\mathcal{B}^\ddagger, T', q_\ell \triangleright_e q^\ddagger, \bar{v}$ where $T_u^\dagger \cup T_s \subseteq T'$. The additional states and rules from $q_r^\ddagger$ and to q_r then show that:

Claim 6.2. If $\mathcal{B}^\dagger, T, q_\ell \triangleright q_r, \bar{0}$ for some $T \subseteq T_u^\dagger \cup T_s$, then $\mathcal{B}^\ddagger, T_u^\ddagger \cup T_s, q_\ell \triangleright_e q_r^\ddagger, \bar{0}$.

Conversely, assume that $\mathcal{B}^\ddagger, q_\ell \triangleright_e q_r^\ddagger, \bar{0}$. This entails $\mathcal{B}^\dagger, q_\ell \triangleright_e q^\ddagger, \bar{e}_{d+1}$.

First assume that no decrement by $\bar{e}_t$ for any t in $T_u^\dagger \cup T_s$ is ever used in the corresponding expansive reachability witness. Then also no increment by $\bar{e}_t$

occurs, and the only rule applicable at this point is $q^\ddagger \xrightarrow{-\bar{e}_{d+1}} q_r$. This yields a node n labelled $(q_r, \bar{0})$, and a deduction subtree rooted by n where only rules of $\mathcal{B}^\dagger$ and expansions are applied. Because any expansion can be simulated in $\mathcal{B}^\dagger$ using an increase, this yields $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{0}$.

Assume now the opposite: there is at least one occurrence of a decrement by $\bar{e}_t$ in the expansive reachability witness. Consider the bottommost such occurrence along some branch, necessarily yielding a node with $q^\ddagger$ as state label. Then in the same way, below this bottommost occurrence, no increment by $\bar{e}_t$ occurs, and the only rule applicable at this point is $q^\ddagger \xrightarrow{-\bar{e}_{d+1}} q_r$, which yields a node n labelled $(q_r, \bar{v})$ for some $\bar{v}$ in $\mathbb{N}^d$. The deduction subtree rooted by n only uses rules of $\mathcal{B}^\dagger$ and expansions, thus as in the previous case $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{v}$. Using increases in q_r then shows $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{0}$. Therefore, in all cases:

Claim 6.3. If $\mathcal{B}^\dagger, T_u^\ddagger \cup T_s, q_\ell \triangleright_e q_r^\ddagger, \bar{0}$, then $\mathcal{B}^\dagger, q_\ell \triangleright q_r, \bar{0}$.

By Claims 6.1, 6.2, and 6.3, $\mathcal{B}^\dagger, T_u^\ddagger \cup T_s, q_\ell \triangleright_e q_r^\ddagger, \bar{0}$ if and only if there exists $\bar{v}$ in $\mathbb{N}^d$ such that $\mathcal{B}, q_\ell \triangleright q_r, \bar{v}$, thereby showing the correctness of our reduction.

5.2 Focusing Proofs in $\mathbf{R}_{\rightarrow}$

We enforce a particular proof policy in our simulation of BVASSs in $\mathbf{R}_{\rightarrow}$, which is inspired by the *focusing proof* techniques [2] employed to reduce non-determinism during proof search in sequent calculi. With only implication at our disposal, we find ourselves in a ‘negative fragment’, where focusing proofs have a very simple calculus $F\mathbf{R}_{\rightarrow}$. This is equivalent to restricting oneself to long normal forms in the associated λ -calculus.

A *focusing sequent* is of one of the two forms ‘ $\Gamma, [A] \Vdash B$ ’ or ‘ $\Gamma \Vdash A$ ’ where ‘ $[A]$ ’ is called a *focused* formula. We let $\Gamma, \Delta, \dots$ denote as before multisets of implicational formulæ and A, B, C implicational formulæ. Here are the rules of the focusing calculus $F\mathbf{R}_{\rightarrow}$:

$$\begin{array}{c} \frac{}{[a] \Vdash a} \text{ (atomic)} \quad \frac{\Gamma, [A] \Vdash a}{\Gamma, A \Vdash a} \text{ (focus)} \quad \frac{\Gamma, A, A \Vdash a}{\Gamma, A \Vdash a} (\mathbf{C}^f) \\[10pt] \frac{\Gamma \Vdash A \quad \Delta, [B] \Vdash a}{\Gamma, \Delta, [A \rightarrow B] \Vdash a} (\rightarrow_L^f) \quad \frac{\Gamma, A \Vdash B}{\Gamma \Vdash A \rightarrow B} (\rightarrow_R^f) \end{array}$$

Note that our focusing calculus $F\mathbf{R}_{\rightarrow}$ gives the priority to right implications ($\rightarrow_R^f$) over the left implications ($\rightarrow_L^f$), focus (focus) and contractions ($\mathbf{C}^f$): the latter can only be applied to sequents with atomic succedents a in $\mathcal{A}$. A similar observation is that a focusing sequent $\Gamma, [a] \Vdash A$ is provable if and only if $A = a$ is atomic and $\Gamma = \emptyset$ is the empty multiset, since (atomic) is the only rule yielding a sequent with a focused atomic formula $[a]$.

Theorem 7 ($F\mathbf{R}_{\rightarrow}$ is sound and complete). *A sequent $\Gamma \vdash A$ is provable in $L\mathbf{R}_{\rightarrow}$ if and only if the focusing sequent $\Gamma \Vdash A$ is provable in $F\mathbf{R}_{\rightarrow}$.*

We prove Thm. 7 in the full paper, using the admissibility of a suitable cut rule in $F\mathbf{R}_{\rightarrow}$.

5.3 From Comprehensive Expansive Reachability to $F\mathbf{R}_{\rightarrow}$

Let us consider a comprehensive expansive reachability instance $\langle \mathcal{B}, q_r, q_\ell \rangle$ with $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$. We are going to construct an implicational formula F such that $\vdash F$ if and only if $\mathcal{B}, T_u \cup T_s, q_\ell \triangleright_e q_r, \bar{0}$.

We work for this on the set of atomic formulæ $Q \uplus \{e_i \mid 0 < i \leq d\}$, and associate to a root judgement $\mathcal{B}, T, q_\ell \triangleright_e q, \bar{v}$ a focusing sequent

$$q_\ell, \Delta_T, \Gamma_{\bar{v}} \Vdash q \quad (3)$$

where Δ_T encodes the rules in $T \subseteq T_u \cup T_s$ and $\Gamma_{\bar{v}}$ encodes $\bar{v}$: let $T = \{t_1, \dots, t_k\}$ and $\bar{v} = c_1 \bar{e}_1 + \dots + c_d \bar{e}_d$, then

$$\Delta_T \stackrel{\text{def}}{=} \ulcorner t_1 \urcorner, \dots, \ulcorner t_k \urcorner, \quad (4)$$

$$\Gamma_{\bar{v}} \stackrel{\text{def}}{=} e_1^{c_1}, \dots, e_d^{c_d}, \quad (5)$$

where ‘ $\ulcorner t \urcorner$ ’ is the individual encoding of rule t and ‘ A^c ’ stands for c repetitions of the formula A . We use the following individual rule encodings:

$$\ulcorner q \xrightarrow{\bar{e}_i} q_1 \urcorner \stackrel{\text{def}}{=} (e_i \rightarrow q_1) \rightarrow q, \quad (6)$$

$$\ulcorner q \xrightarrow{-\bar{e}_i} q_1 \urcorner \stackrel{\text{def}}{=} q_1 \rightarrow (e_i \rightarrow q), \quad (7)$$

$$\ulcorner q \rightarrow q_1 + q_2 \urcorner \stackrel{\text{def}}{=} q_1 \rightarrow (q_2 \rightarrow q). \quad (8)$$

Then proof search in $F\mathbf{R}_{\rightarrow}$ is easily seen to implement deductions in $\mathcal{B}$:

Claim 8.1 (Completeness). If $\mathcal{B}, T, q_\ell \triangleright_e q, \bar{v}$, then $q_\ell, \Delta_T, \Gamma_{\bar{v}} \Vdash q$.

Proof. We proceed by induction on the structure of the root judgement. For the base case, i.e. for $\mathcal{B}, \emptyset, q_\ell \triangleright_e q_\ell, \bar{0}$, we have

$$\frac{\overline{[q_\ell] \Vdash q_\ell} \text{ (atomic)}}{q_\ell \Vdash q_\ell} \text{ (focus)}$$

as desired.

For the induction step, if the last applied rule is an increment $t = q \xrightarrow{\bar{e}_i} q_1$ on a judgement $\mathcal{B}, T, q_\ell \triangleright_e q_1, \bar{v} + \bar{e}_i$, then

$$\begin{array}{c} \text{i.h.} \\ \frac{\frac{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i \Vdash q_1}{q_\ell, \Delta_T, \Gamma_{\bar{v}} \Vdash e_i \rightarrow q_1} \text{ } (\rightarrow_R^f) \quad \frac{}{[q] \Vdash q} \text{ (atomic)}}{\frac{q_\ell, \Delta_T, \Gamma_{\bar{v}}, [(e_1 \rightarrow q_1) \rightarrow q] \Vdash q}{q_\ell, \Delta_T, \Gamma_{\bar{v}}, (e_1 \rightarrow q_1) \rightarrow q \Vdash q} \text{ } (\rightarrow_L^f)} \text{ (focus)} \end{array}$$

and an additional contraction (C^f) if $t \in T$ shows $q_\ell, \Delta_{T \cup \{t\}}, \Gamma_{\bar{v}} \Vdash q$ as desired.

If the last applied rule is a decrement $t = q \xrightarrow{-\bar{e}_i} q_1$ on a judgement $\mathcal{B}, T, q_\ell \triangleright_e q_1, \bar{v}$, then

$$\begin{array}{c}
\text{i.h.} \quad \frac{q_\ell, \Delta_T, \Gamma_{\bar{v}} \Vdash q_1}{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i, [q_1 \rightarrow (e_i \rightarrow q)] \Vdash q} \quad \frac{\frac{\frac{[e_i] \Vdash e_i}{e_i \Vdash e_i} \text{ (atomic)}}{e_i, [e_i \rightarrow q] \Vdash q} \text{ (focus)}}{\frac{[q] \Vdash q}{e_i, [e_i \rightarrow q] \Vdash q} \text{ (atomic)}} \text{ (}\rightarrow^f_{\perp}\text{)} \\
\frac{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i, [q_1 \rightarrow (e_i \rightarrow q)] \Vdash q}{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i, q_1 \rightarrow (e_i \rightarrow q) \Vdash q} \text{ (focus)}
\end{array}$$

and a contraction (C^f) if $t \in T$ shows $q_\ell, \Delta_{T \cup \{t\}}, \Gamma_{\bar{v} + \bar{e}_i} \Vdash q$ as desired.

If the last applied rule is an expansion on a judgement $\mathcal{B}, T, q_\ell \triangleright_e q, \bar{v} + 2\bar{e}_i$, then

$$\begin{array}{c}
\text{i.h.} \\
\frac{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i, e_i \Vdash q}{q_\ell, \Delta_T, \Gamma_{\bar{v}}, e_i \Vdash q} \text{ (}C^f\text{)}
\end{array}$$

as desired.

Finally, if the last applied rule is a split $t = q \rightarrow q_1 + q_2$ on two judgements $\mathcal{B}, T_1, q_\ell \triangleright_e q_1, \bar{v}_1$ and $\mathcal{B}, T_2, q_\ell \triangleright_e q_2, \bar{v}_2$, then

$$\begin{array}{c}
\text{i.h.} \\
\frac{\text{i.h.} \quad \frac{q_\ell, \Delta_{T_1}, \Gamma_{\bar{v}_1} \Vdash q_1}{q_\ell, q_\ell, \Delta_{T_1}, \Delta_{T_2}, \Gamma_{\bar{v}_1}, \Gamma_{\bar{v}_2}, [q_1 \rightarrow (q_2 \rightarrow q)] \Vdash q} \quad \frac{\frac{q_\ell, \Delta_{T_2}, \Gamma_{\bar{v}_2} \Vdash q_2}{q_\ell, \Delta_{T_2}, \Gamma_{\bar{v}_2}, [q_2 \rightarrow q] \Vdash q} \quad \frac{[q] \Vdash q}{[q] \Vdash q} \text{ (atomic)}}{q_\ell, \Delta_{T_2}, \Gamma_{\bar{v}_2}, [q_2 \rightarrow q] \Vdash q} \text{ (}\rightarrow^f_{\perp}\text{)} \\
\frac{q_\ell, q_\ell, \Delta_{T_1}, \Delta_{T_2}, \Gamma_{\bar{v}_1}, \Gamma_{\bar{v}_2}, [q_1 \rightarrow (q_2 \rightarrow q)] \Vdash q}{q_\ell, \Delta_{T_1 \cup T_2 \cup \{t\}}, \Gamma_{\bar{v}_1 + \bar{v}_2} \Vdash q} \text{ (}C^f\text{)}
\end{array}$$

as desired. $\square$

The interest of the focusing calculus $F\mathbf{R}_{\rightarrow}$ is that, starting from a sequent $q_\ell, \Delta, \Gamma, [\ulcorner t \urcorner] \Vdash q$ where Δ is in $\mathbb{N}^{\Delta_{T_u} \cup T_s}$ and Γ in $\mathbb{N}^{\{e_i \mid 0 < i \leq d\}}$ and the focus is on the encoding of a rule t , there is *no choice* but to follow the proof trees shown in the proof of Claim 8.1. Given a multiset m in $\mathbb{N}^E$ for some set E , we write

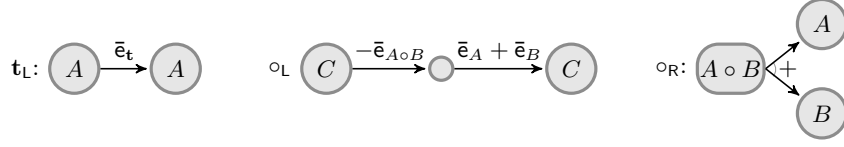
$$\sigma(m) \stackrel{\text{def}}{=} \{e \in E \mid m(e) > 0\} \quad (9)$$

for the *support* of m .

Claim 8.2 (Soundness). Let Δ be in $\mathbb{N}^{\Delta_{T_u} \cup T_s}$, Γ in $\mathbb{N}^{\{e_i \mid 0 < i \leq d\}}$, q in Q , and $n > 0$. Then $q_\ell^n, \Delta, \Gamma \Vdash q$ implies $\mathcal{B}, \sigma(\Delta), q_\ell \triangleright_e q, \bar{v}_\Gamma$.

Proof. Note that $n = 0$ would yield an unprovable sequent. We proceed by induction on the structure of a proof tree for the focusing sequent. The only applicable rules in a proof search from $q_\ell^n, \Delta, \Gamma \Vdash q$ are (**focus**) and (C^f). In the latter case, we distinguish three cases depending on the contracted formula A :

- If $A = q_\ell$, i.e. if $q_\ell^{n+1}, \Delta, \Gamma \Vdash q$, then by induction hypothesis $\mathcal{B}, \sigma(\Delta), q_\ell \triangleright_e q, \bar{v}_\Gamma$ as desired.
- If $A = e_i$ in Γ , then by induction hypothesis $\mathcal{B}, \sigma(\Delta), q_\ell \triangleright_e q, \bar{v}_\Gamma + \bar{e}_i$, and an expansion yields $\mathcal{B}, \sigma(\Delta), q_\ell \triangleright_e q, \bar{v}_\Gamma$ as desired.
- If $A = \ulcorner t \urcorner$ for some rule t in Δ , then the support $\sigma(\Delta)$ is not changed and by induction hypothesis $\mathcal{B}, \sigma(\Delta), q_\ell \triangleright_e q, \bar{v}_\Gamma$ as desired.


 Fig. 3: The additional BVASS rules for $LR_{\rightarrow, \circ}^t$.

In the former case, we also distinguish three cases depending on which formula A receives the focus:

- If $A = q_\ell$, necessarily Δ and Γ are empty and $q = q_\ell$, and indeed $\mathcal{B}, \emptyset, q_\ell \triangleright_e q_\ell, \bar{0}$.
- If $A = e_i$ in Γ , then proof search fails since $q \neq e_i$.
- If $A = \ulcorner t \urcorner$ in Δ , then proof search needs to follow the proof trees used in the proof of Claim 8.1, and applying the induction hypothesis on the open leaves of these trees allows to conclude in each case. $\square$

Theorem 8. *Provability in $\mathbf{R}_{\rightarrow}$ is 2-EXPTIME-hard.*

Proof. We reduce from the comprehensive expansive reachability problem, which is 2-EXPTIME-hard by Prop.6. From an instance $\langle \mathcal{B}, q_r, q_\ell \rangle$ where $\mathcal{B} = \langle Q, d, T_u, T_s \rangle$, we construct a formula $F \stackrel{\text{def}}{=} q_\ell \rightarrow \varphi(T_u \cup T_s, q_r)$ defined by

$$\varphi(\emptyset, q_r) \stackrel{\text{def}}{=} q_r, \quad \varphi(T \uplus \{t\}, q_r) \stackrel{\text{def}}{=} \ulcorner t \urcorner \rightarrow \varphi(T, q_r). \quad (10)$$

By Thm.7, $\vdash F$ if and only if $\Vdash F$. The latter holds if and only if $q_\ell, \Delta_{T_u \cup T_s} \Vdash q_r$ since we can only apply $(\rightarrow_R^f)$. Then this occurs if and only if $\mathcal{B}, T_u \cup T_s, q_\ell \triangleright_e q_r, \bar{0}$ by Claim 8.1 and Claim 8.2. $\square$

6 Extensions

Adding Multiplicatives. The sequent system $LR_{\rightarrow}$ for $\mathbf{R}_{\rightarrow}$ can be extended to accommodate further multiplicative connectives: the *fusion* connective $\circ$ (aka. ‘co-tenability’ in [1]) and the sentential constant $\mathbf{t}$:

$$\frac{\Gamma \vdash A}{\Gamma, \mathbf{t} \vdash A} (\mathbf{t}_L) \quad \frac{}{\vdash \mathbf{t}} (\mathbf{t}_R)$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \circ B \vdash C} (\circ_L) \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \circ B} (\circ_R)$$

Let us call $LR_{\rightarrow, \circ}^t$ the resulting sequent system. The BVASS $\mathcal{B}_F$ presented in Sec.4.2 can be extended in a straightforward manner with the rules of Fig.3 and by identifying q_ℓ with $\mathbf{t}$. Thanks to Prop.4 and Cor.2, this shows that provability in $LR_{\rightarrow, \circ}^t$ is in 2-EXPTIME.

Note that the sequent system $LR_{\rightarrow, \circ}^t$ is the same as that of intuitionistic multiplicative contractive linear logic (**IMLLC**), where $\rightarrow$, $\circ$, and $\mathbf{t}$ are usually noted respectively $\multimap$, $\otimes$, and $\mathbf{1}$.

Adding Exponentials. In fact, essentially the same reduction from sequent calculus to expansive BVASS reachability can be carried over for the more general *multiplicative exponential* fragment of intuitionistic contractive linear logic with bottom **IMELZC**, see [11, Prop. 9]. The main differences are that:

1. The exponential connectives incur an exponential blow-up in the number of states of the constructed BVASS $\mathcal{B}_F$: its state space now contains $S \times 2^{S_!}$ where $S_!$ is the set of exponential subformulae of F . The subsequent reduction to coverability in Sec. 4.3 then performs a product with $2^{S \setminus S_!}$ (contractions in exponential subformulae being already handled), hence the resulting state space remains of size $2^{p(|F|)}$ for some polynomial p .
2. The exponential connectives also require an additional operation of *full zero test*: as shown in [11, Lem. 3], this operation can be eliminated at no cost in complexity.

By Thm. 8, Prop. 4 and Cor. 2, we conclude:

Theorem 9. *Provability in any logic between $\mathbf{R}_{\rightarrow}$ and **IMELZC** is 2-EXPTIME-complete.*

This applies in particular to $\mathbf{R}_{\rightarrow}^t$, **IMLLC**, and **IMELLC**. It seems likely that the non-intuitionistic variants **MLLC** and **MELLC**, i.e. multiplicative and multiplicative exponential contractive linear logic, are also 2-EXPTIME-complete: the upper bound follows from the bound on **IMELZC**.

7 Concluding Remarks

Besides closing a longstanding open problem, the proof that $\mathbf{R}_{\rightarrow}$ is 2-EXPTIME-complete paves the way for new investigations:

- In spite of the high worst-case complexity of BVASS coverability, Majumdar and Wang [13] have recently presented a practical algorithm with encouraging initial results. The reduction in Sec. 4 allows to transfer their techniques to $\mathbf{R}_{\rightarrow}$ provability, but might incur a worst-case exponential blow-up.
- Provability in the related implicational fragment $\mathbf{T}_{\rightarrow}$ of ticket entailment has recently been proven decidable independently by Padovani [16] and Bimbó and Dunn [3]. Although the complexity of this problem is currently unknown, the latter proof relies on provability in $\mathbf{LR}_{\rightarrow}^t$, which we prove to be 2-EXPTIME-complete in Sec. 6.

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