

# Near semi-rings and lambda calculus

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**Abstract.** A connection between lambda calculus and the algebra of near semi-rings is discussed. Among the results is the following completeness theorem.

A first-order equation in the language of binary associative distributive algebras is true in all such algebras if and only if the interpretations of the first order terms as lambda terms beta-eta convert to one another.

A similar result holds for equations containing free variables.

## 1 Near semirings and bad algebra

A “near semi-ring” consists of a monoid  $(S, +, 0)$  together with a semigroup  $(S, *)$  satisfying the right distributive law  $(x + y) * z = x * z + y * z$  and the left annihilator law  $0 * x = 0$  [11] (for motivation see [13]). Examples are the non-negative integers (natural numbers), which is a near ring, and the ordinal numbers. An interesting case is when  $(S, *)$  possesses an identity 1, and the monoid  $(S, *, 1)$  replaces  $(S, *)$ . Now if  $(S, *, 1)$  is any monoid then Cayley’s representation sends the point  $s : S$  to the function  $f_s : S \rightarrow S$  defined by  $f_s(t) = s * t$ , with  $f_1 =$  the identity function. This is sometimes called the regular representation, [12]. Now if  $+$  is defined pointwise  $(f_r + f_s)(t) = f_r(t) + f_s(t)$  from  $(S, +, 0)$ , then the Cayley representation satisfies

$$\begin{aligned} f_0 &= \text{the function identically } 0 \\ f_r + f_s &= f_{r+s}. \end{aligned}$$

This is the representation theorem of Hoogewijs [7]. Thus if we begin with a pair of monoids we can use the Cayley representation of  $(S, +, 0)$ , say  $(S', +', 0')$ , where  $+'$  is composition of functions  $f : S' \rightarrow S'$ , with  $*$  lifted to some operation  $*'$  on  $S' \rightarrow S' \times S' \rightarrow S'$ . Now we can apply the above Cayley representation of the result embedding it in  $(S' \rightarrow S') \rightarrow (S' \rightarrow S')$  so that the operations become

final  $+$  = pointwise composition  
final  $*$  = composition.

There are, however, interesting examples with more than two binary operations. One example is the polynomial ring over a field together with substitution. A second is finite cardinal numbers together with  $\inf$ ,  $\sup$ ,  $+$  and  $*$ . In

each case “lower” operations distribute over “higher” ones. In the latter case, although each semigroup is a monoid the law of left annihilation fails. This suggests the following definition. A “binary associative distributive (bad) algebra of height  $n$ ” consists of a set  $S$  together with  $n + 1$  binary operations  $*_0, \dots, *_n$  such that for each  $i = 0, \dots, n$ ,  $(S, *_i)$  is a semigroup and the distributive laws  $(r *_i s) *_j t = (r *_j t) *_i (s *_j t)$  hold when  $j < i$ . We say that the bad algebra is “monoidal” if there exist  $n + 1$  elements  $I_0, \dots, I_n : S$ , such that for each  $i = 0, \dots, n$ ,  $(S, *_i, I_i)$  is a monoid and when  $j > i$   $I_j *_i x = I_j$ .

For a given  $S$ , a type structure is a map  $S(0) = S$ ,  $S(m + 1)$  is non- empty, and  $S(m + 1) \subset S(m) \rightarrow S(m)$ . We can generalize Hoogewijs’ representation theorem as follows.

**Theorem 1.** *Every bad algebra can be embedded into a type structure.*

*Proof.* The construction embeds a bad algebra of height  $n$  into  $S(n + 1)$  recursively by starting with  $*_{n-1}$  and  $*_n$  and working down the  $*_i$  while up the  $S(n - i + 1)$ . At the stage for  $*_i$  the  $*_j$  for  $j > i$  are embedded pointwise and the  $*_j$  for  $j < i$  are embedded by definition. The construction clearly works for the monoidal case. End of proof.

## 2 Bad algebra and lambda calculus

Some notation will be useful. We adopt for the most part the notation and terminology of [1].

$B := \lambda xyz. x(yz)$   
 $K := \lambda xy. x$   
 $I := \lambda x. x$   
 $1 := \lambda xy. xy.$   
 $\sim :=$  beta – eta conversion  
 $\rightarrow_0 :=$  beta – eta reduction.

For the most part we shall use the colon “:” for membership and as an abbreviation for “has type”, except when it is used in “:=” for syntactic identity and definitional equality.

Both Church and Curry observed that the combinators form a monoid under multiplication  $B$ , identity  $I$ , left zero  $KI$ , and beta-eta conversion. The same is true for addition  $\lambda xyuv. xu(yuv)$ , identity  $KI$  (the Church numeral 0), left zero  $K$ , and beta-eta conversion  $\sim$ . Since these satisfy the right distributive law

$$\begin{aligned}
 & (\lambda xyz. x(yz))((\lambda xyuv. xu(yuv))ab) c \\
 & \sim (\lambda xyuv. xu(yuv)) ((\lambda xyz. x(yz))ac) ((\lambda xyz. x(yz))bc)
 \end{aligned}$$

they form a near semi-ring.

Many years ago, see [6], I noticed a generalization of this near semi-ring structure to a hierarchy of monoids. Define

$$A_n := \lambda xy u_1 \dots u_n v. x u_1 \dots u_n (y u_1 \dots u_n v)$$

so  $A_0 := B$  and  $A_1$  is Church's addition. Then combinators form a monoid with multiplication  $A_n$ , identity  $K^n I$ , left zero  $K^{n+1}$  and beta-eta conversion. There are level  $m + 1$  integers  $n \sim$

$$A_{m+1}(K^m I)(\dots (A_{m+1}(K^m I)(K^m I)) \dots) \sim \lambda u_1 \dots u_m xy. x(\dots (xy) \dots),$$

which for  $m = 1$  coincide with Church's finite ordinals. Again the right distributive law holds. More precisely we have

(associativity)  $A_m(A_n xy) \sim A_0(A_n x)(A_n y)$  if  $m = n$ , (distributivity)  $A_m(A_n xy) \sim A_{n+1}(A_m x)(A_m y)$  if  $m < n$ ,

and in addition,

$$\begin{aligned} (i) \quad & x \sim A_m x(K^m I) \\ (ii) \quad & A_m x \sim A_{m+1}(Kx)I \\ (iii) \quad & K(A_m xy) \sim A_{m+1}(Kx)(Ky) \\ (iv) \quad & K^n x \sim A_m(K^n x) \quad \text{if } m < n. \end{aligned}$$

Thus we have a monoidal bad algebra of height  $n$ .

The entire hierarchy of near semi-rings admits a very simple combinatory theory. If we endow the combinators  $I, K$ , and  $A_n$  for  $n = 0, \dots$  with their weak beta reduction rules

$$\begin{aligned} I x & \rightarrow_1 x \\ K xy & \rightarrow_1 x \\ A_n xy u_1 \dots u_n v & \rightarrow_1 x u_1 \dots u_n (y u_1 \dots u_n v) \end{aligned}$$

then termination can be proved. Indeed, one way to prove termination is to interpret the re-write arrow " $\rightarrow_1$ " as strict descent in a well-ordered structure with a strictly monotone function. The following was observed in [9].

Let  $O$  be the set of countable ordinals and  $V : O \times O \rightarrow O$  Veblen's function defined by

$$\begin{aligned} V(0, b) &= \omega^b \\ V(a, b) &= \text{the } b\text{th common fixed point of the functions } \lambda y V(x, y) \text{ for } x < a. \end{aligned}$$

Let  $W = (O, V, 1)$ .  $W$  is a structure, in the logician's sense, for any language with a single binary function symbol interpreted as  $V$ , and any number of individual constants interpreted as the (finite) ordinal 1. A proper combinator  $P$  with reduction rule

$$P x_1 \dots x_n \rightarrow_1 X$$

is said to terminate in  $W$  if for all  $x_1, \dots, x_n$  in  $O$  we have

$$X < Px_1 \dots x_n$$

**Theorem 2.** [9]; *The proper combinators which terminate in  $W$  are precisely those which convert to applicative combinations of  $I, K$  and the  $A_n$ .*

For our purposes we need only the weaker result that if the simpler structure  $(\omega, \lambda xy. 2^x + y, 1)$  is used, then  $I, K$ , and all the  $A_n$  terminate. But we can add to this the reductions

$$\begin{aligned} \text{(associativity)} \quad & A_m (A_m xy)z \rightarrow_2 A_m x(A_m yz) \\ \text{(monoidality)} \quad & A_m (K^m I)x \rightarrow_2 x \\ & A_m x(K^m I) \rightarrow_2 x \\ \text{(annihilation)} \quad & A_m (K^n I)x \rightarrow_2 K^n I \quad \text{if } n > m \end{aligned}$$

which will be useful below. Our use of the symbol  $\rightarrow_2$  is intended to also include  $\rightarrow_1$  as our use of the symbol  $\rightarrow_3$  below is intended to include  $\rightarrow_2$ . Since  $\rightarrow_2$  obviously satisfies the weak diamond property, each  $\rightarrow_2$  reduction terminates with a unique normal form. If we add

$$\text{(distributivity)} \quad A_m(A_n xy)z \rightarrow_3 A_{n+1}(A_m xz)(A_m yz) \text{ if } m < n$$

then the weak diamond property holds but it is not so straightforward to verify Church-Rosser as before. We shall do this in the needed context by a roundabout argument below. The congruence generated by associativity, monoidality, annihilation and distributivity will be referred to as “bad conversion”.

Interestingly, when viewed as lambda terms there are non normalizable combinations; for example,

$$A_1 IK \rightarrow_0 \lambda xy. xx$$

and

$$(\lambda xy. xx)(\lambda xy. xx) \rightarrow_0 \lambda z. (\lambda xy. xx)(\lambda xy. xx \rightarrow_0 \dots)$$

We believe that this can only happen if  $K$  is not in function position. We shall prove that this can only happen with  $K$  or an  $A_m$  not in function position.

### 3 Functional completeness of bad algebra for lambda calculus

Simple types are built up from type variables  $p, q, r, \dots$  by  $\rightarrow$ . We shall adopt the Church typing discipline for terms requiring sub-terms to be explicitly typed. Nevertheless, it will be convenient to use Curry style notation

$$X : T$$

for  $X$  has type  $T$ . Fix a type variable  $s$ . We define the numerical types

$\text{Num}(m)$  by  
 $\text{Num}(0) = s$   
 $\text{Num}(m+1) = \text{Num}(m) \rightarrow \text{Num}(m)$ .

Our combinators have most general (principal) types

$$\begin{aligned}
 A_m : (r_1 \rightarrow (\dots(r_m \rightarrow (p \rightarrow q))\dots)) \\
 \rightarrow ((r_1 \rightarrow (\dots(r_m \rightarrow (r \rightarrow p))\dots)) \\
 \rightarrow (r_1 \rightarrow (\dots(r_m \rightarrow (r \rightarrow q))\dots)))
 \end{aligned}$$

$$\begin{aligned}
 K : p \rightarrow (q \rightarrow p) \\
 I : p \rightarrow p.
 \end{aligned}$$

Now substituting  $\text{Num}(n)$  for  $p, q, r$  and  $\text{Num}(m-i+n+1)$  for  $r_i$  we obtain

$$\begin{aligned}
 A_m : \text{Num}(m+n+1) \rightarrow \text{Num}(m+n+2) \\
 K : \text{Num}(n) \rightarrow \text{Num}(n+1) \\
 I : \text{Num}(n+1).
 \end{aligned}$$

If we suppose that  $M$  is an applicative combination of the  $A_m, K$  and  $I$  such that every occurrence of an  $A_m$  or  $K$  is in function position then  $M$  has a numerical type and is strongly beta-eta normalizable. Indeed, we may assume that  $M$  is in combinatory normal form, and by conversion (ii), that every occurrence of  $A_m$  has two arguments, and by conversion (iii), that every occurrence of  $K$  is in a sub-term of the form  $K^m I$ . Thus  $M$  is the interpretation of a term in the language of monoidal bad algebras under the generalized Hoogewijs representation. A term, not assumed to be in combinatory normal form, where every  $A_m$  has at least two arguments and each  $K$  occurs in a sub-term  $K^m I$  is said to be “equable”. A conversion generated by the reduction relation  $\rightarrow_3$  is said to be equable if every term in the conversion is equable. It is easily seen that every term in an equable conversion can be simply typed with a numerical type using the typings above for  $A_m, K$ , and  $I$ .

If  $M$  is equable let  $M!$  result from  $M$  by replacing each sub-term  $A_n XY$  by  $\lambda u_1 \dots u_n v. Xu_1 \dots u_n (Yu_1 \dots u_n v)$ , and each maximal  $K^n I$  by  $\lambda u_1 \dots u_n v. v$ .  $M!$  is a beta reduct of  $M$  and  $M!$  has the property

(modesty) If a free variable  $z$  occurs in a sub-term  $Z$  then  $Z$  has numerical type not larger than the type of  $z$ .

Modesty is preserved under beta reduction and eta expansion so it is shared by the long beta-eta normal form of  $M!$ . We now proceed to show the converse.

We say that  $X$  is “openly equable” if  $X$  is a combination of  $A^m, K, I$  and the variables  $x_k, x_{k+1}, \dots, x_n$ , for  $k < \text{or} = n$ , such that  $X : \text{Num}(k)$ ,  $x_i : \text{Num}(i)$  for  $i = k, \dots, n$ , each occurrence of  $A_m$  has at least two arguments, and each occurrence of  $K$  is in a sub-term  $K^m I$ . If  $X$  is openly equable and  $X$  is in combinatory normal form then  $X!$  is modest. If  $X$  is openly equable we inductively

construct a term  $[x_k]X$  satisfying the same properties as  $X$  with  $k$  replaced by  $k+1$  such that  $([x_k]X)x_k \rightarrow_1 X$ . This is “bracket abstraction” for our context.

The basis case is  $X := x_k$  in which case we let  $[x_k]X := I$ . We can also include in this step  $X := K^m I$  in which case we set  $[x_k]X := K^{m+1}I$ . For the induction step we distinguish three cases with  $l > 0$ .

**Case 1:**  $X := x_j X_1 \dots X_l$ . Then  $x_j : \text{Num}(j)$ ,  $X_1 : \text{Num}(j-1) \dots$ ,  $X(l) : \text{Num}(j-l) = \text{Num}(k)$ , so by modesty  $x_k$  occurs only in  $X_l$ . In this case we can put  $[x_k]X := A_0(x_j X_1 \dots X_{l-1})([x_k]X(l))$ .

**Case 2:**  $X := A_m X_1 \dots X_l$ . This case is similar to Case 1 if  $l > 2$ . If  $l = 2$  then  $X_i : \text{Num}(k)$  for  $i = 1, 2$  and we can put  $[x_k]X := A_{m+1}([x_k]X_1)([x_k]X_2)$ .

**Case 3:**  $X := (K_m I) X_1 \dots X_l [x_k]X := A_0((K_m I) X_1 \dots X_{l-1})([x_k]X_l)$ .

**Theorem 3.** (*functional completeness*) *If  $M$  is a modest simply typed term of numerical type then  $M$  beta-eta converts to an equable applicative combination of  $A_m, K$ , and  $I$ .*

*Proof.* By a previous remark it suffices to consider normal  $M$ . For modest normal  $X$  with variables  $x_k, x_{k+1}, \dots, x_n$ , for  $k < \text{or} = n$ , such that  $X : \text{Num}(k)$  and  $x_i : \text{Num}(i)$  for  $i = k, \dots, n$ , we construct  $\$X$ , a combinatory normal applicative combination of  $A_m, K, I$ , and the variables  $x_k, x_{k+1}, \dots, x_n$  such that  $X \rightarrow_1 \$X$ , each occurrence of  $A_m$  has two arguments, and each occurrence of  $K$  is in a sub-term  $K^m I$ . We proceed by induction and we can write  $X := \lambda x_{k-1} \dots x_{k-l}. x_i X_1 \dots X_{i-k+l}$  where by assumption  $x_j : \text{Num}(j)$ , and  $X_j : \text{Num}(i-j)$ . If  $l > 0$  the induction hypothesis applies to  $\lambda x_{k-2} \dots x_{k-l}. x_i X_1 \dots X_{i-k+l}$  and we can apply  $[x_{k-1}]$ . Otherwise,  $x_k$  can only occur in  $X_{i-k}$  by modesty. Thus  $X := YZ$  and the induction hypothesis applies to  $Y$  and  $Z$ . End of proof.

## 4 Logical completeness of lambda calculus for bad algebra

**Lemma 1.** *If  $X$  is openly equable,  $X!$  is modest and  $X \rightarrow_1 Y$  then*

$$[x_k]X \rightarrow_3 [x_k]Y.$$

*Proof.* By induction on the length of  $X$ . The result follows immediately from the induction hypothesis unless the contracted redex is the head redex. If the contracted redex is the head redex we distinguish two cases

**Case 1:**  $X = K^m I X_1 \dots X_l$ .

Subcase 1:  $m > 0$ . If  $l > 1$  then  $Y = K^{m-1} I X_2 \dots X_l$  and  $[x_k]X \rightarrow_1 [x_k]Y$ . If  $l = 1$  then  $[x_k]X \rightarrow_2 [x_k]Y$

Subcase 2:  $m = 0$ . In case  $l = 1$ ,  $[x_k]X \rightarrow_2 [x_k]Y$ . In case  $l > 1$ , by modesty  $x_k$  occurs only in  $X_l$  so  $[x_k]X = A_0(IX_1 \dots X_{l-1})([x_k]X_l) \rightarrow_1 [x_k]Y$ , since  $X$  is (openly) equable.

**Case 2:**  $X = A_m X_1 X_2 X_3 \dots X_l$ . This case is clear if  $l > 3$  since  $X$  is modest. If  $l = 3$  then

Subcase 1:  $m = 0$ . So  $Y = X_1(X_2 X_3)$ ,  $[x_k]X = A_0(A_0 X_1 X_2)([x_k]X_3) \rightarrow_2$  by associativity  $A_0 X_1(A_0 X_2([x_k]X_3)) = [x_k]Y$

Subcase 2:  $m > 0$ . Then  $Y = A_{m-1}(X_1 X_3)(X_2 X_3)$ . As before  $[x_k]X = A_0(A_m X_1 X_2)([x_k]X_3) \rightarrow_3$   
by distributivity  $= A_m(A_0 X_1[x_k]X_3)(A_0 X_2[x_k]X_3) = [x_k]Y$ . End of proof.

**Lemma 2.** *If  $X$  is openly equable,  $X!$  is modest and  $X \rightarrow_3 Y$  then  $[x_k]X \rightarrow_3 [x_k]Y$ .*

*Proof.* We have  $[x_k](A_m(A_{m'}XY)Z)$  = by modesty  $(A_m(A_{m'}[x_k]X[x_k]Y)[x_k]Z \rightarrow_3$   
by distributivity  $(A_{m'}(A_m[x_k]X[x_k]Z)(A_m[x_k]Y[x_k]Z) = [x_k](A_{m'}(A_m XZ)(A_m YZ))$   
when  $m' > m$ . Also when  $m' = m$ ,  $[x_k](A_m(A_m XY)Z)$  = by modesty  
 $(A_m(A_m[x_k]X[x_k]Y)[x_k]Z \rightarrow_2$  by associativity  $(A_m([x_k]X(A_m[x_k]Y[x_k]Z)) =$   
 $[x_k](A_m X(A_m YZ))$ . The lemma now follows from Lemma 1. End of proof.

**Lemma 3.** *If  $X$  is openly equable,  $X!$  is modest, and  $Y$  is the beta normal form of  $X$  then  $X \rightarrow_3 \$Y$  where  $\$Y$  is as in the proof of Theorem 3.*

*Proof.* There is a standard reduction of  $X$  to  $Y$ . The proof is by induction on the length of the standard reduction with a subsidiary induction on the length of  $X$ . Here we note that if  $X$  is an application then either  $X$  begins with a variable or  $X$  begins with a combinatory redex and contracting the redex commutes with the  $!$  operation, or  $X$  is combinatory head normal and  $X!$  begins with  $\lambda$ . We consider these cases.

**Case 1:**  $X$  begins with a variable. Then the induction hypothesis applies to the arguments of the variable.

**Case 2:**  $X$  has a combinatory head redex. Then the induction hypothesis applies to the result of contracting the head redex.

**Case 3**  $X$  is in combinatory head normal form, say  $X := A_m UV$ . Thus  $X! = \lambda u_1 \dots u_n v. U u_1 \dots u_n (V u_1 \dots u_n v)$ . Now  $Y = \lambda u_1 \dots u_n v. Z$  and the induction hypothesis applies to  $U u_1 \dots u_n (V u_1 \dots u_n v)$  and  $Z$ . Thus  $U u_1 \dots u_n (V u_1 \dots u_n v) \rightarrow_3 \$Z$ , so by Lemma 2

$$[u_1, \dots, u_n, v]U u_1 \dots u_n (V u_1 \dots u_n v) \rightarrow_3 [u_1, \dots, u_n, v]\$Z \text{ hence}$$

$$X = [u_1, \dots, u_n, v]U u_1 \dots u_n (V u_1 \dots u_n v) \\ \rightarrow_3 [u_1, \dots, u_n, v]\$Z = \$Y. \text{ End of proof.}$$

Lemma 3 clearly extends to beta-eta normal forms since  $[x](Xx) = A_0 X I$ .

**Proposition 1.** (*Church-Rosser theorem for  $\rightarrow_3$* ) *If  $M$  and  $N$  are equable and are linked by an equable conversion then there exists an equable  $P$  such that  $M \rightarrow_3 P_3 \leftarrow N$ . Moreover  $P$  can be taken to be in combinatory normal form.*

*Proof.* If  $M$  and  $N$  are linked by an equable conversion then they beta-eta convert. Thus there exists beta-eta normal  $Q$  such that  $M$  and  $N$  both beta- eta reduce to  $Q$ . By Lemma 3  $M \rightarrow_3 Q_3 \leftarrow N$ . End of proof.

**Theorem 4.** (*logical completeness*); *If  $M$  and  $N$  are normal equable applicative combinations of  $A_m, K$  and  $I$  then  $M$  bad converts to  $N$  if and only if  $M$  beta-eta converts to  $N$ .*

We can state a corollary to Theorem 4 in informal terms as follows. A first-order equation in the language of monoidal bad algebras is true in all monoidal bad algebras if, and only if, the corresponding lambda terms beta-eta convert. A similar result holds for equations containing free variables. For this it suffices to extend our typing to intersection types. The first order free variables appear as parameters with types which are intersections of numerical types. We shall have a less trivial application of intersection types below.

## 5 The $B, I$ monoid

The lowest level, the monoid generated by  $B$  and  $I$  alone, turns out to be of independent interest. It is the “positive part” of Richard Thompson’s group  $F$ , and it generates this group. In 1965, Thompson discovered this group in connection to his theorem characterizing the groups with solvable word problems. The group  $F$  was rediscovered by Peter Freyd and Alex Heller in 1993 in connection with homotopy retracts  $x * x \equiv x$  which do not split  $y * z \equiv I$  and  $x \equiv z * y$ . It will be convenient to write  $Bxy$  in infix “monoid” notation  $x * y$ . The following beta-eta conversions hold

$$\begin{array}{ll}
 Nx & \sim x * \dots * x * 1 \text{ if } N \text{ is the Church numeral for } n \\
 & \text{and there are } n \text{ occurrences of } x \\
 B(x * y) & \sim Bx * By \\
 B * N & \sim N * B \\
 B(Bx) * B & \sim B * Bx \\
 K(x * y) & \sim Bx * Ky \\
 Kx & \sim Bx * 0 \\
 B(Bx) * K & \sim K * x.
 \end{array}$$

Note that the 3rd and 4th equation have  $B$  not in function position in contrast to the previous paragraphs. “Monomials” have the form

$$B(\dots(BA)\dots) \quad d \text{ occurrences of } B$$

where  $A$  is one of the atoms  $B, K$  or a Church numeral  $N$ .  $d$  is the “degree” of the monomial and  $A$  is its “sort”. A product of monomials is a “multinomial”. We have the

**Lemma 4.** *Every applicative combination of  $B, K, I$  and Church numerals converts to a multinomial.*

*Proof.* Observe that if  $P$  is normal w.r.t. weak beta reductions and associativity then  $P$  can be reduced a product of monomials by the reductions

$$\begin{aligned} BI &\rightarrow_4 1 \\ Nx &\rightarrow_4 x * \dots * x * 1 \text{ if } N \text{ is the Church numeral for } n \\ &\quad \text{and there are } n \text{ occurrences of } x \\ B(x * y) &\rightarrow_4 Bx * By \\ K(x * y) &\rightarrow_4 Bx * Ky \\ Kx &\rightarrow_4 Bx * 0. \end{aligned}$$

In a multinomial  $M$ , if for any two consecutive monomials  $B(\dots(BA')\dots) * B(\dots(BA'')\dots)$ , of degrees  $d'$  and  $d''$ , we have  $|d'' - d'| < 2$ ,  $M$  is said to be “semi-regular”. A maximal consecutive sub-product of monomials of the same degree is called a “block” of  $M$ . If  $M$  is semi-regular and all of its blocks have the same number of monomials we say that  $M$  is “regular”.

**Proposition 2.** *Every multinomial converts to one which is regular.*

This can be seen by inserting factors  $B(\dots(B1)\dots)$ .

Church observed that the Church numeral for  $n$  applied to the Church numeral for  $m$  beta reduces to the Church numeral for  $m^n$ . Consequently, every applicative combination of Church numerals beta-eta strongly normalizes. We have already proved that every applicative combination of  $B, K, I$ , and Church numerals, where every occurrence of  $B$  and  $K$  is in function position, is strongly normalizable. We shall now show that the condition on  $K$  can be relaxed. Given a regular multinomial  $M$  we may write  $M = M(m) * \dots * M(1)$  where each  $M(i)$  is a block. Put  $s(i) =$  the number of monomials in  $M(i)$  whose sort is  $B$  or  $K$ . We call the sequence  $s(1)\dots s(m)$  the “statistics of  $M$ ”.

**Theorem 5.** *An applicative combination of  $B, K$ , and Church numerals where each occurrence of  $B$  is in function position is strongly normalizable.*

*Proof.* First we will show that if a  $\rightarrow_1$  or  $\rightarrow_4$  reduct of a combination is strongly beta-eta normalizable then so is the combination. We show this by showing that intersection types [4] lift from contracta to redexes. Lifting is straightforward for combinatory redexes. As to  $\rightarrow_4$  consider the following beta-eta conv’s. Note that in each, eta is only used in the reductions from right to left. We observe that in the reduction from left to right, if a given r.h.s. has an intersection type then its expansion has the same type. In addition, in the reduction from right to left, if a term has an intersection type then its reduct has the same type by the subject reduction theorem. This includes eta reductions. These remarks cover all  $\rightarrow_4$  except the special case  $(1x) \rightarrow_4 x$ . For this case, observe that in application  $x$  will always be a closed term beginning with lambda, so  $(1x)$  has the desired type by the normal form for intersection types.

- (i)  $B(Bxy)z \rightarrow_0 \lambda v. (Bxy)(zv) \rightarrow_0 \lambda v. x(y(zv))_0 \leftarrow Bx(Byz)$
  - (ii)  $Kx \rightarrow_0 \lambda y. x \_0 \leftarrow \lambda yv. xv \_0 \leftarrow \lambda yv. x(KIyv) \_0 \leftarrow \lambda y. Bx(KIy) \_0 \leftarrow (Bx)(KI)$
  - (iii)  $K(Bxy) \rightarrow_0 \lambda u(Bxy) \rightarrow_0 \lambda uv. x(yv) \_0 \leftarrow \lambda uv. x(yv) \_0 \leftarrow \lambda uv. x(Kyu) \_0 \leftarrow \lambda u. (Bx)(Kyu)_0 \leftarrow B(Bx)(Ky)$
  - (iv)  $B(Bxy) \rightarrow_0 \lambda uv. (Bxy)(uv) \rightarrow_0 \lambda uv. x(y(uv)) \_0 \leftarrow \lambda uv. x(Byuv) \_0 \leftarrow \lambda u. (Bx)(Byu) \_0 \leftarrow B(Bx)(By)$
  - (v)  $Nx \rightarrow_0 \lambda y. x(\dots(xy)\dots) \_0 \leftarrow \dots \_0 \leftarrow \lambda y. x(\dots(Bxx)\dots)y) \_0 \leftarrow Bx(\dots(Bxx)\dots).$
- The cases for the remaining are straightforward.

Now suppose that  $M$  is a multinomial all of whose sorts are either  $K$  or Church numerals. We shall show that  $M$  has an intersection type. To simplify notation we shall assume that each Church numeral is 2. The cases of 0 and 1 are straightforward and the cases of  $n > 2$  are treated exactly like the case of 2, except for much notational complication. Suppose that  $M = M(m) * \dots * M(1)$  where each  $M(i)$  is a monomial of degree  $d(i)$  with sort 2 or  $K$ . We construct a rooted tree with the following properties.

- (0) The root is a line incident with only a single point; the “low point”.
- (1) The tree is binary branching; that is, each point has either 0 (a leaf), 1, or 2 ancestors. All points, except the low point, have a unique descendent.
- (2) Each path from the root to a leaf has length  $m + 1$ , as measured by the number of points on the path.
- (3) Each line of the tree is labeled with a simple type. This includes the root. The types associated with lines joining a point to its ancestors are “ancestor types”, and the type associated with the line joining a point to its descendent (or, in the case of the low point, the type associated with the root) is the “descendent type”.

The tree will be constructed and the types will be assigned in such a way that

- (i) if  $M(i)$  has a Church numeral sort then each point at depth  $i$  has exactly two ancestors and if  $T''$ ,  $T'$  are the types associated with these two ancestors and  $T$  is the descendent type then

$$M(i) : (T'' \wedge T') \rightarrow T$$

- (ii) If  $M(i)$  has  $K$  as its sort then each point at depth  $i$  has a unique ancestor and if  $T'$  is a type associated with this ancestor and  $T$  is the descendent type then

$$M(i) : T' \rightarrow T.$$

First we construct the tree without labels. This is completely determined by (i) and (ii). Next, we label each line with 1 or 2 simple types s.t.

- (a) A line incident with a leaf, and the root, are labeled with a single simple type.
- (b) All other lines are labeled with two simple types: +type and -type. The pair of types associated with a given line will have to be unified in order to achieve the final labelling. This preliminary labelling goes as follows.

We define the revised degree

$$d(i)' = \begin{cases} d(i) & \text{if the sort of } M(i) \text{ is } K \\ d(i) + 1 & \text{if the sort of } M(i) \text{ is } 2 \end{cases}$$

and let  $d = \max\{d(i)' | i = 1, \dots, m\}$ . Set  $e(i) = d - d(i) + s(1) + \dots + s(i - 1)$ .

If the sort of  $M(i)$  is 2 then we consider any point at depth  $i$  and select new type variables  $r_1 \dots r_{d(i)}, p_1, \dots, p_{e(i)}, p, q_1, \dots, q_{e(i)}, q, r$ . The left ancestor of this point gets -type

$$r_1 \rightarrow (\dots (r_{d(i)} \rightarrow ((q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots))) \dots))$$

and the right ancestor gets -type

$$r_1 \rightarrow (\dots (r_{d(i)} \rightarrow ((q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots)) \rightarrow (p_1 \rightarrow (\dots (p_{e(i)} \rightarrow r) \dots)))) \dots)$$

and the descendent of this point gets +type

$$r_1 \rightarrow (\dots (r_{d(i)} \rightarrow (p \rightarrow (p_1 \rightarrow (\dots (p_{e(i)} \rightarrow r) \dots)))) \dots).$$

If the sort of this point is  $K$  then we consider any point at depth  $i$  and select new type variables  $r_1 \dots r_{d(i)}, p, q_1, \dots, q_{e(i)}, q$ . The ancestor point of this point gets -type

$$r_1 \rightarrow (\dots (r_{d(i)} \rightarrow (q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots))) \dots)$$

and the descendent of this point gets +type

$$r_1 \rightarrow (\dots (r_{d(i)} \rightarrow (p \rightarrow (q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots)))) \dots).$$

Now if  $M(i)$  has sort 2 then

$$\begin{aligned} r_1 &\rightarrow (\dots (r_{d(i)} \rightarrow (p \rightarrow (q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots)))) \dots) \wedge \\ r_1 &\rightarrow (\dots (r_{d(i)} \rightarrow ((q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots)) \rightarrow (p_1 \rightarrow (\dots (p_{e(i)} \rightarrow r) \dots)))) \dots) \end{aligned}$$

which is Coppo and Dezani [4] type equivalent to

$$\begin{aligned} r_1 &\rightarrow (\dots (r_{d(i)} \rightarrow ((p \rightarrow (q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots))) \wedge \\ &((q_1 \rightarrow (\dots (q_{e(i)} \rightarrow q) \dots)) \rightarrow (p_1 \rightarrow (\dots (p_{e(i)} \rightarrow r) \dots)))) \dots) \end{aligned}$$

so (i) and (ii) are clearly satisfied.

Finally there is a single substitution which simultaneously unifies all the pairs of +type and -type. This is defined directly by recursion on  $m$ — the depth of the point noting that any such pair have the same number of components.

Now if we put the same variable  $x$  at each leaf with type Coppo and Dezani equivalent to the intersection of all the types at the leaves we see that  $Mx$  has and is strongly normalizable. End of proof.

## 6 Church's Theorem

If we endow the set of natural numbers with the discrete topology and the set of functions from natural numbers to natural numbers with the product topology then a term  $M$  could compute a uniformly continuous functional from functions to numbers if there exists a  $k$  such that for any sequence of Church numerals  $N(1) \dots N(k)$

$$MN(1) \dots N(k)$$

converts to a Church numeral.

A consequence of our simple type assignment is that if  $M$  is an applicative combination of the  $A_m, K$  and  $I$  such that every occurrence of an  $A_m$  or  $K$  is in function position then  $M$  computes a uniformly continuous functional. The reason is that  $M$  has a numerical type, and so we can pick the types of  $N(1), \dots, N(k)$  so that the type of

$$MN(1) \dots N(k)$$

is  $\text{Num}(2)$ , and the only long beta-eta normal forms of type  $\text{Num}(2)$  are Church numerals. Similarly, if  $M$  is an applicative combination of  $B, K$  and Church numerals such that every Church numeral is in function position then  $M$  is strongly normalizable and  $M$  computes a uniformly continuous functional. This can be seen as follows. We have the following beta-eta conversions

$$\begin{aligned} B(x * y) &\sim Bx * By \\ K(x * y) &\sim Bx * Ky \\ K(Bx) &\sim B * Kx \\ B(Kx) &\sim K(Kx) \\ Kx * y &\sim Kx \\ K * Bx &\sim B(Bx) * K \\ K * B &\sim BB * K \\ K * K^m I &\sim K^m I \quad \text{if } m > 0. \end{aligned}$$

The first four permit a multinomial form where each monomial has the form

- (a)  $B^m B$
- (b)  $B^m(KI)$
- (c)  $K$

The fifth conversion insures that if a monomial of the form (b) with  $m = 0$  occurs then it is the rightmost monomial in the multinomial. Finally, the conversions six, seven, and eight, together with the first, insure that all  $K$ 's occur rightmost left of such a monomial, and all  $B$ 's rightmost left of them.

It follows that every monomial is beta-eta convertible to one built up from  $I$  by the operation

$$X \rightarrow_5 BX * B \dots B * K \dots K$$

possibly with  $KI$  appended. Now it is clear that

$$B \dots B * K \dots K * KI$$

has a simple type of the form  $\text{Num}(n_2) \rightarrow \text{Num}(n_1)$ . Thus by induction if  $X$  has a simple type

$$\text{Num}(n_k) \rightarrow (\dots (\text{Num}(n_2) \rightarrow \text{Num}(n_1) \dots))$$

then

$$BX * B \dots B * K \dots K * KI$$

has simple type

$$\text{Num}(n'_{k+1}) \rightarrow (\dots (\text{Num}(n'_2) \rightarrow \text{Num}(n'_1) \dots).$$

Now we can proceed as before. The multinomial  $M$  is said to be non-decreasing (non-increasing) if the degrees of the blocks of  $M$  are non-decreasing (non-increasing) from right to left.

Examples:

- (i) An applicative combination of  $B, K$  and Church numerals where each positive Church numeral is in function position converts to a non-decreasing multinomial.
- (ii) An applicative combination of  $B, K$ , and Church numerals where each occurrence of  $B$ , and  $K$  is in function position converts to a multinomial all of whose sorts are Church numerals (statistics all 0's).
- (iii) A regular, non-decreasing multinomial with no sort a positive integer, whose statistics  $s(1) \dots s(m)$  satisfies

$$i > s(i) + \dots + s(1)$$

converts to a non-increasing multinomial.

A binary product  $M'' * M'$  of regular monomials with the same number of blocks and statistics  $s(1)'' \dots s(m)''$ ,  $s(1)' \dots s(m)'$  respectively is called an “oscillator” if  $M''$  is non-increasing,  $M'$  is non-decreasing, and  $s(i)'' = s(m + 1 - i)'$ . A multinomial  $M$  is said to be “harmonic” if it is a product of oscillators.

Examples(cont'd): the following kinds of multinomials convert to harmonic ones

- (iv) Monomials where every sort is an integer.

- (v)  $M * M$ , where  $M$  is regular, non-decreasing, no sort is a positive integer and the statistics  $s(1) \dots s(m)$  satisfy  $i > s(i) + \dots + s(1)$ .

We have the following generalization of Church's theorem.

**Theorem 6.** *If  $M$  is harmonic then  $M$  is strongly normalizable and for sufficiently large  $k$ , for any sequence of Church numerals  $N(1) \dots N(k)$*

$$MN(1) \dots N(k)$$

*converts to a Church numeral.*

*Proof.* Omitted for lack of space.

## 7 The Game of Alonzo

We would like to know Theorem 5 for all applicative combinations of  $B$ ,  $K$ , and Church numerals. We would like to know that every applicative combination of  $B$ ,  $K$ , and Church numerals is strongly normalizable. Indeed, we would like to know the truth of the following conjecture.

Conjecture: If  $M$  is a product of monomials each of whose sorts is either  $B$  or the Church numeral 2, then for sufficiently large  $k$ ,

$$\begin{array}{c} M \underbrace{1 \dots 1}_{k} \text{ conv. } 1 \\ | \\ k \end{array} .$$

Now  $M 1$  can be calculated as follows:

$$\begin{array}{ll} BP(M 1) & \sim P * (M 1) \\ B(P * Q) & \sim BP * BQ \\ B 1 & \sim 1 \\ 2 P & \sim P * P \end{array}$$

so we can formulate an equivalent conjecture about a game, which we call the "game of Alonzo". The game of Alonzo consists of  $n$  players and a dealer. The players and the dealer are seated around a circular table. Play takes place counter-clockwise around the table (right to left). Each player has a stack of 1,000\$ chips and is dealt a single card, which is either an ace (2) or a duce ( $B$ ), by the dealer. As play rotates counter-clockwise to a given player three rules apply; the "ante", the "cashout", and the "split".

- (1) If the player has at least one chip then he ante's up by putting one 1,000\$ chip in the center of the table. This is the "ante".
- (2) If the player has no chips and his card is a duce then he must pay for and give one 1,000\$ chip to each player to his left and to the right of the dealer. These chips are bought and payed for at a bank separate from the game. This is "cashing out". The player then leaves the game, unless he is the last player.

- (3) If a player has no chips and his card is an ace then he gets to play a copy of each of the hands of the players to his left and to the right of the dealer. Each copy gets the corresponding number of chips and all can be played independently. This is “splitting”. If there are no such players to his left he leaves the game unless he is the last player. Play continues to rotate counter-clockwise through the dealer and no new cards are dealt. Play ends when there is only one player left, and he wins the pot.

Conjecture: Every game ends.

## References

- [1] Barendregt, H. P., *The Lambda Calculus*, North Holland 1984
- [2] Birget, J.C., Monoid generalizations of the Richard Thompson groups, *Journal of Pure and Applied Algebra* **213**, (2009), pps 264-278.
- [3] Church, A., The Calculi of Lambda Conversion, *PUP* (1941).
- [4] Coppo, M., and Dezani, M., A new type assignment for lambda terms, *Archiv fur Math. Logik*, **19**, (1978)
- [5] Curry, H.B., *Combinatory Logic*, Vol 1, North Holland 1958.
- [6] Freyd, P. and Heller, A., Splitting homotopy idempotents II, *Journal of Pure and Applied Algebra*, **89**, (1993) pps 93-195.
- [7] Hoogewijs, J., Semi-nearring embeddings, *Med. Konink. Vlaamse Acad. Wetensch. Lett. Schone Kunst. Belgie Kl. Wetensch* **32**(2), (1970) pps 3-11.
- [8] Pottinger, G., *A type assignment for strongly normalizable lambda terms*, Curry Festschrift, (1980).
- [9] Statman, R., Freyd’s hierarchy of combinator monoids, *LICS*, 1991.
- [10] Thompson, R., Embeddings into finitely generated groups which preserve the word problem in Adian, Boone and Higman eds., *The Word Problem*, Studies in Logic and the Foundation of Mathematics, North Holland 95, (1980), pps 401-441.
- [11] Van Hoorn, W., and Van Rootselaar, B., Fundamental notions in the theory of semi-near rings, *Comp. Math.* **18**, (1967), pp. 65-78.
- [12] Artin, M., “Algebra”, Prentice Hall, 2011.
- [13] Internet: <http://www.algebra.uni-linz.ac.at/Nearrings>.