

On Affine Logic and Łukasiewicz Logic

Rob Arthan & Paulo Oliva

Queen Mary University of London

We report on a study [1] of intuitionistic fragments of the classical Łukasiewicz Logic, $\mathbf{LL}_c$. Of particular interest are intuitionistic affine logic¹, $\mathbf{AL}_i$, and, what we call intuitionistic Łukasiewicz Logic, $\mathbf{LL}_i$, which is $\mathbf{AL}_i$ extended with the axiom:

$$A \otimes (A \multimap B) \vdash B \otimes (B \multimap A) \quad [\text{CWC}]$$

often referred to as the axiom of *divisibility* in the Fuzzy Logic literature. $A \otimes (A \multimap B)$ can be thought of as a weak form of conjunction, so that the axiom of divisibility asserts the commutativity of this weak conjunction, hence our acronym [CWC]. From another point of view, $\mathbf{LL}_i$ is Hajek's Basic Logic [6] without the intuitionistically unacceptable axiom of *arrow prelinearity*:

$$(A \multimap B) \multimap C, (B \multimap A) \multimap C \vdash C \quad [\text{PREL}]$$

In algebraic terms, $\mathbf{LL}_i$ is the logic corresponding to the algebraic structures known as hoops [3, 4]. $\mathbf{LL}_i$ has been studied before, under several different names. For instance, Blok and Ferreirim refer to it as $\mathbf{S}_{\mathcal{HO}}$.

Our investigation of double negation translations for $\mathbf{LL}_c$ into $\mathbf{LL}_i$ led us to conjecture a homomorphism property for double negation: is $(A \otimes B)^{\perp\perp}$ equivalent to $A^{\perp\perp} \otimes B^{\perp\perp}$ over $\mathbf{LL}_i$? We were able to prove this conjecture and an analogue for $\multimap$ with the assistance of the Prover9 automated theorem prover.

Theorem 1 ([2]). *Write $A \simeq B$ to denote that $\mathbf{LL}_i$ proves $A \vdash B$ and $B \vdash A$. Then:*

$$\begin{aligned} (A \otimes B)^{\perp\perp} &\simeq A^{\perp\perp} \otimes B^{\perp\perp} \\ (A \multimap B)^{\perp\perp} &\simeq A^{\perp\perp} \multimap B^{\perp\perp}. \end{aligned}$$

The proofs, however, involve numerous intricate uses of [CWC]. Our analysis of Prover9's output, in an attempt to make the proofs comprehensible to a human reader, led us to consider four other *derived* connectives. These are,

- the weak conjunction: $A \wedge B \equiv A \otimes (A \multimap B)$
- the strong disjunction: $A \vee B \equiv (B \multimap A) \multimap A$
- the strong implication: $A \Rightarrow B \equiv A \multimap A \otimes B$, and

¹ $\mathbf{AL}_i$ is the fragment of linear logic [5] containing implication ($\multimap$), the multiplicative conjunction ($\otimes$), and falsehood (1), extended with the weakening rule but without double negation elimination. Negation is defined as $A^\perp = A \multimap 1$.

- a form of NOR: $A \downarrow B \equiv A^\perp \otimes (B \multimap A)$

The derived conjunction $A \wedge B$ and disjunction $A \vee B$ are well known from the literature and have been widely studied. It is noteworthy that abstractions that have proved useful in human reasoning appear frequently in the machine-generated proofs.

In the process of trying to understand and present the proof of Theorem 1, we have discovered several natural properties of both the primitive and the derived connectives, which allow us to give a structured account of the proof.

Theorem 2 ([1]). *Again with $\simeq$ denoting equiprovability in $\mathbf{LL}_i$, we have:*

$$\begin{array}{ll}
(A \wedge B)^\perp \simeq A \Rightarrow B^\perp & B \downarrow A \simeq A \downarrow B \\
A \downarrow B \simeq (A \vee B)^\perp & (A^{\perp\perp} \multimap A)^\perp \simeq 1 \\
(A \otimes B)^\perp \simeq A \multimap B^\perp & (A \multimap B)^\perp \simeq A^{\perp\perp} \otimes B^\perp \\
(A \wedge B)^\perp \simeq A \Rightarrow B^\perp & (A \downarrow B)^\perp \simeq A^\perp \Rightarrow B^{\perp\perp} \\
(A \Rightarrow B)^\perp \simeq A^{\perp\perp} \wedge B^\perp & (A \vee B)^\perp \simeq A^\perp \wedge B^\perp
\end{array}$$

Finally, it is easy to check that both the Kolmogorov and Gödel double negation translations are correct in the sense of Troelstra [7] when viewed as translations from classical affine logic $\mathbf{AL}_c$ into $\mathbf{AL}_i$. Neither the Gentzen nor the Glivenko translation is a correct double negation translation for $\mathbf{AL}_c$ into $\mathbf{AL}_i$, as shown in [2], where we use model-theoretic methods to prove that certain formulas are unprovable. The situation for $\mathbf{LL}_i$ is much more satisfactory. The following theorem is an almost immediate consequence of Theorem 1, since, given these homomorphism properties for double negation, all four double negation translations are equivalent.

Theorem 3 ([1]). *Both the Gentzen and the Glivenko translations are correct double negation translations for $\mathbf{LL}_i$.*

References

- [1] Rob Arthan and Paulo Oliva. On affine logic and Łukasiewicz logic. <http://arXiv.org/abs/1404.0570>, 2014.
- [2] Rob Arthan and Paulo Oliva. On pocrim and hoops. <http://arXiv.org/abs/1404.0816>, 2014.
- [3] W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *Algebra Universalis*, 43(2-3):233–257, 2000.
- [4] Nikolaos Galatos and Peter Jipsen. A survey of generalized basic logic algebras. In *Witnessed years—essays in honour of Petr Hájek*, volume 10 of *Tributes*, pages 303–327. Coll. Publ., London, 2009.
- [5] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.
- [6] Petr Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.
- [7] A. S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*. Springer, Berlin, 1973.