

A proof theoretical approach to standard completeness.

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1 Abstract

Hypersequents are a natural generalization of Gentzen sequents, whose basic objects of inference are “disjunctions” of sequents. Since their introduction in [1], hypersequent calculi have been largely used in nonclassical logics, above all in fuzzy logic, see e.g. [8, 6]. One of the most important applications of hypersequents in fuzzy logic is for showing proof-theoretically *standard completeness*, i.e. the completeness of logics with respect to algebras on the real interval $[0, 1]$. The proof-theoretic approach to standard completeness, first introduced in [7], is based on the idea that adding to a logic a special rule, called *density*, makes the logic *rational complete*, i.e. complete with respect to algebras over the rationals in $[0, 1]$. Thus, showing the admissibility or *elimination* of the density rule entails rational completeness for the original logic. More precisely, given a logic L , the proof-theoretical approach to standard completeness can be described as follows :

- (a) Define a hypersequent calculus HL for L extended with the density rule.
- (b) Prove that the density rule is eliminable in HL , i.e. that it does not enlarge the set of provable hypersequents. This shows rational completeness for L .
- (c) Standard completeness may then be obtained from rational completeness by means of the Dedekind-MacNeille completion.

Step (b) above (density-elimination) was established in [7] for various hypersequent calculi, in close analogy to Gentzen-style methods for cut-elimination. A more elegant approach to density elimination, *by substitutions*, has been then introduced in [4] and is inspired to normalization for natural deduction systems. In our contribution, we use key properties of hypersequent calculi, such as the *proof by cases* property, to extend the method by substitutions in [4] and prove density elimination for wider classes of hypersequent calculi. For instance, we will show density elimination, hence standard completeness, for axiomatic extensions of uninorm logic UL [2] and monoidal t-norm logic MTL [3].

References

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