

Coherentisation of accessibility conditions in labelled sequent calculi*

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1 Introduction

One methodology for going beyond¹ the expressive power of traditional Gentzen systems for non-classical (e.g. intermediate and modal) logics is (instead of hypersequents or nested sequents) the use of labelled sequent calculi, with rules for analysing implications or modal formulae and a representation of conditions on the accessibility relation(s); this allows a wide range of such logics to be represented. Two ways have been studied of representing the accessibility conditions as inference rules: (1) converts the conditions into CNF, using Skolemisation to reduce quantifier alternation, then generating rules that act on both sides of the sequent [4, 6], and (2) is, where the accessibility conditions are coherent (aka geometric) implications, to represent them directly as inference rules (acting only on the antecedent) [7, 11]. These latter rules are natural and transparent, whereas clausification tends to render the conditions rather opaquely (see [4, Fig 5, p 582] and also [3]).

It has been suggested that the second approach is limited to just those accessibility conditions representable as coherent implications. We show this suggestion to be false, by proving that *every first-order theory has a coherent conservative extension*, i.e. one in which all the necessary reasoning can be carried out using the (coherent) rule-based approach of [7, 11]. This widens the range of logics that approach (2) can handle to all those based on a first-order Hintikka-Kripke semantics.

Such an extension is easily constructed by adding new predicate symbols (and no Skolemisation). Several approaches can be adopted: we consider them in turn, both from a historical and from a practical perspective.

2 Technical background

A “coherent implication” is a sentence of first-order logic that is the universal closure of an implication $D_1 \supset D_2$, where each D_i is a *positive formula*, i.e. a disjunction of existentially quantified conjunctions of atoms. Such a sentence can be recast [8, Prop. D.1.3.10] as (intuitionistically equivalent to) a conjunction of sentences that are universal closures of implications $C \supset D$, where C is a conjunction of atoms and D is a positive formula. We shall use *coherent implication* mainly in the latter more specialised sense. Note in particular that such a formula is always a sentence and is not necessarily an “implication”.

C can be the empty conjunction $\top$ and D can be the empty disjunction $\perp$: obvious notational simplifications can then be applied. A universal sentence $\forall \mathbf{x} A$ (with quantifier-free A) can be rewritten as a conjunction of coherent implications.

A coherent implication is also known as a “geometric axiom” [6, 10], a “coherent axiom”, a “basic geometric sequent” [15] or a “coherent formula” [3]; and a *coherent theory* (i.e. a theory axiomatised by a set of coherent implications) is also known as a “geometric theory” [9]. We follow the distinction drawn by some, and covered well in [8], between geometric formulae (in which infinitary disjunctions are allowed) and positive formulae (no such infinitary operations).

This notion is important for several reasons: see for example [3, 9, 12, 15]. That of interest here is that any coherent implication can be converted directly [10] to an inference rule in such a fashion that admissibility of the structural rules of the underlying calculus is unaffected.

3 Examples

Kreisel-Putnam logic **KP** is an intermediate logic with a characteristic frame condition that is not a coherent implication. This condition (see [5, p. 55]) is, with the symbol R for the accessibility relation replaced by $\leq$ and with some negative antecedent formulae converted to positive succedent formulae [7],

$$\forall xyz. (x \leq y \wedge x \leq z) \supset (y \leq z \vee z \leq y \vee \exists u. (x \leq u \wedge u \leq y \wedge u \leq z \wedge F(u, y, z))) \quad (1)$$

where $F(u, y, z)$ abbreviates $\forall v. u \leq v \supset \exists w. (v \leq w \wedge (y \leq w \vee z \leq w))$; the complexity of $F(u, y, z)$ upsets the coherence. However, instead of using it in an abbreviative definition, we take F as a new predicate symbol and add the axiom

$$\forall uv yz. (F(u, y, z) \wedge u \leq v) \supset (\exists w (v \leq w \wedge y \leq w) \vee \exists w (v \leq w \wedge z \leq w)). \quad (2)$$

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¹ Or, at least, extending it in a uniform fashion.

(1) and (2) are now coherent implications, adequate as frame conditions for **KP**. We omit their presentation as rules.

The second example, illustrating a technique² for handling negations, is the sentence $\forall x.\exists y.xRy \wedge \neg(yRx)$. Let S (for “strictly serial”) be a new binary predicate symbol, to express the negation of R ; the sentence can now be rewritten as the conjunction of $\forall x.\exists y.xRy \wedge ySx$ and $\forall xy.(xRy \wedge xSy) \supset \perp$; from these we obtain (by the method in [10]) the two rules

$$\frac{xRy, ySx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \alpha \quad \frac{}{xRy, xSy, \Gamma \Rightarrow \Delta} \beta$$

where, in the first rule, y is fresh and wlog x occurs in Γ, Δ . In general, x and y may be instantiated as terms.

[8, p 887] shows that this is not just a matter of finding an equivalent coherent axiomatisation but, by extending the language (and thus changing the morphisms and thus the category of models), can be the replacement of an incoherent theory $\mathcal{T}$ by a coherent one, albeit similar in the sense of being a conservative extension of $\mathcal{T}$.

4 Results and related work

Theorem. Any sentence A in a first-order language $\mathcal{L}$ can be converted to a finite set of coherent implications (in an extension $\mathcal{L}'$ of $\mathcal{L}$ with new predicate symbols) that has the same (so far as formulae in $\mathcal{L}$ are concerned) consequences as A , that is, the theory axiomatised by A has a coherent conservative extension.

Choice of new predicate symbols and associated coherent implications to express their meanings can be achieved by adapting the ideas of Skolem [16], who showed, by a systematic replacement of complex subformulae of a prenex normal form, that every first-order theory has a conservative extension axiomatised just by $\forall\exists$ -formulae.

Alternatively (and extendable to infinitary logic), Antonius [1, pp 34–36]³ showed that, for each formula A of a first-order language $\mathcal{L}$, one may construct a positive formula $\bar{A}$ in an extended language $\mathcal{L}'$, as follows: $\bar{P}(\mathbf{x}) = P(\mathbf{x})$ for atomic $P(\mathbf{x})$; $\bar{B} \wedge \bar{C} = \bar{B} \wedge \bar{C}$, $\bar{B} \vee \bar{C} = \bar{B} \vee \bar{C}$, $\bar{B} \supset \bar{C} = N_B(\mathbf{x}) \vee \bar{C}$, $\neg \bar{B} = N_B(\mathbf{x})$, $\exists x \bar{B} = \exists x \bar{B}$, and $\forall x \bar{B} = N_{\exists x \neg B}(\mathbf{x}')$, where N_B (resp. $N_{\exists x \neg B}$) is a new predicate symbol whose arity is the number of free variables $\mathbf{x}$ of B (resp. of $\mathbf{x}'$, i.e. those of $\exists x \neg B$). Let $\mathcal{T}'$ be the theory got by replacing each axiom A of a theory $\mathcal{T}$ by $\top \supset \bar{A}$, and $\bar{\mathcal{T}}$ the theory got by adding as axioms to $\mathcal{T}'$ all the coherent implications⁴ $\forall \mathbf{x}.\top \supset (N_{B(\mathbf{x})}(\mathbf{x}) \vee \bar{B}(\mathbf{x}))$ and $\forall \mathbf{x}.(N_{B(\mathbf{x})}(\mathbf{x}) \wedge \bar{B}(\mathbf{x})) \supset \perp$; these are just those obtained in cases where the subformula translated is either a negation or universally quantified. Her result then is that a theory $\mathcal{T}$ proves a sequent $B \Rightarrow B'$ (implicitly universally quantified) iff $\bar{\mathcal{T}}$ proves the sequent $\bar{B} \Rightarrow \bar{B}'$, where the latter form of proof is in a sequent calculus⁵ restricted to coherent sequents. Conservativity is not mentioned; we can strengthen the result as follows:

Theorem. Let A be a sentence, axiomatising the theory $\mathcal{T}$; then the coherent theory $\bar{\mathcal{T}}$, axiomatised by $\top \supset \bar{A}$ and the coherent implications generated from A , is a definitional (and thus conservative) extension of $\mathcal{T}$.

Johnstone [8] and Bezem & Coquand [3] also treat this topic; the former presents, in the language of categorical logic, a related result [Lemma D 1.5.13]; the latter is (rightly) more interested in the problems of automation, but does present an algorithm that turns a sentence A into a set of coherent implications that is consistent iff A is consistent—but this is a weaker result. Polonsky [13] further develops the automation theme.

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² An early example of this trick for handling negations is that in [14, p 127]; it also appears in [2].

³ We thank Wedad Antonius for her permission to quote from this unpublished work. Cf also Reyes [14].

⁴ Antonius’ presentation uses unquantified but schematic sequents, a terminology we have avoided in order to give a unified presentation.

⁵ Geared to interpretation in coherent categories.