

# Simpler Proof Net Quantifiers: Unification Nets (Work in Progress)

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Girard formulated an approach to proof net quantifiers over a series of three papers. Since the approach uses explicit witnesses for existential quantifiers, it suffers from two problems: limited proof identification and non-local cut elimination. This paper introduces a more abstract approach, *unification nets*, which solves these problems by leaving existential witnesses *implicit*: a unification net is merely an axiom linking.

## 1 Introduction

Girard developed an approach to proof net quantifiers over a series of three papers [Gir87, Gir88, Gir91] using explicit existential witnesses. Consider the infinite collection of proofs  $\Pi_t$  below-left<sup>1</sup>, one per term  $t = x, y, f(z), g(a, f(y)), \dots$

$$\frac{\frac{\overline{\vdash \bar{P}t}, Pt}{\vdash \exists x \bar{P}x, Pt} \exists}{\vdash \exists x \bar{P}x, \exists y Py} \exists$$

Infinitely many  
proofs  $\Pi_t$  of  
 $\vdash \exists x \bar{P}x, \exists y Py$

$$\frac{\overline{\bar{P}t} \quad Pt}{\exists x \bar{P}x \quad \exists y Py}$$

Infinitely many cut-free  
Girard proof nets  $G_t$  on  
 $\vdash \exists x \bar{P}x, \exists y Py$

$$\frac{}{\exists x \bar{P}x \quad \exists y Py}$$

The unique cut-free  
unification net  $\mathcal{N}$  on  
 $\vdash \exists x \bar{P}x, \exists y Py$

Each proof  $\Pi_t$  has a distinct Girard proof net  $G_t$ , above-center. The subformulas  $\bar{P}t$  and  $Pt$  in  $G_t$ , retained from the proof  $\Pi_t$ , contain an explicit witness  $t$  for the existential quantifiers  $\exists x$  and  $\exists y$ . Thus there are infinitely many cut-free Girard proof nets  $G_t$  on the sequent  $\vdash \exists x \bar{P}x, \exists y Py$  (one per term  $t$ ). This paper introduces a more abstract approach, *unification nets*, identifying more proofs: every proof  $\Pi_t$  translates to the same unification net,  $\mathcal{N}$ , displayed above-right. Unification nets leave existential witnesses implicit: remaining faithful to propositional proof nets [Gir87, DR89], a unification net is just an axiom linking. (As in [Gir91], we focus on the multiplicative case.) The correctness of a unification net can be verified in linear time.

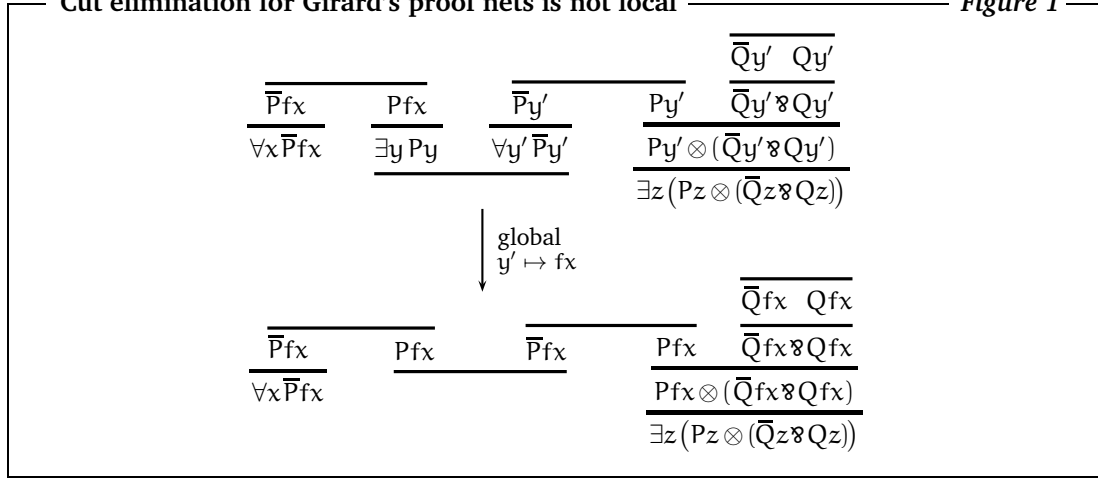
Cut elimination for Girard's proof nets [Gir91] fails to be local: Figure 1 reduces a cut between  $\exists y Py$  and  $\forall y' \bar{P}y'$ , via a global substitution of  $fx$  for  $y'$  throughout the net. In contrast, cut elimination for unification nets is local: Figure 2 shows the corresponding reduction step. A cut between  $\forall y A$  and  $\exists y \bar{A}$  reduces to a cut between  $A$  and  $\bar{A}$ , simply erasing the quantifiers  $\forall y$  and  $\exists y$ . As in the propositional case, all cuts can be eliminated in linear time.

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<sup>1</sup> $\bar{P}$  denotes the dual of the predicate symbol  $P$ . Thus  $\bar{P}t$  is the negation of the atomic formula  $Pt$ .

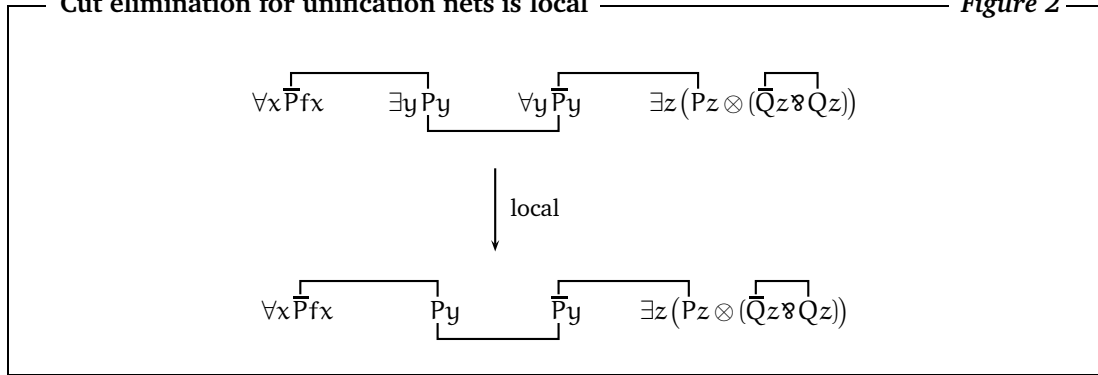
Cut elimination for Girard's proof nets is not local

Figure 1



Cut elimination for unification nets is local

Figure 2



## References

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