

Sequent Systems for Classical Modal Logics

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This paper develops sequent calculi for several classical modal logics. Utilizing a polymodal translation of the standard modal language, we are able to establish a base system for the minimal classical modal logic from which we generate extensions in a modular, and uniform, manner. We also briefly suggest potential applications to epistemic logic.

Since their inception (brought about by Montague [9] and Scott [11], independently), neighborhood frames have become a standard semantic resource for logicians and philosophers wishing to work with modal logics weaker than those characterized by relational frames. However, while the domain of application of these so-called “classical” modal logics and their neighborhood semantics grows ever wider, and the model theory of neighborhood structures becomes better understood (e.g. [7]), there remains a noticeable absence in the development of the proof theory for these systems (exceptions to this include [6] and [5], both of which provide tableau systems for monotonic modal logics). This paper takes some first steps towards filling this void.

In particular, we introduce a labelled sequent system (such systems for normal modal logics can be found in [3] or [10], amongst others) for the minimal classical modal logic (often referred to as **E** [1]), as well as several extensions. Labelled sequent systems are variants of standard sequent calculi that encode semantic information into the syntax of the rules (in a manner parallel to the prefixes of some tableau systems [2]). These semantic markers allow us to bridge the gap more easily between the extant model theoretic results and the purely syntactic approaches familiar to proof theorists.

The systems we present enjoy the properties usually required of well-behaved sequent calculi. These include the admissibility of structural rules (weakening and contraction), the invertibility of logical rules, and cut elimination. In virtue of these admissibility results, the correspondence between

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our sequent systems and the standard Hilbert axiomatizations can be proved systematically in the sense that it is possible to derive all the Hilbert axioms and rules by applying a proof-search procedure. This easily provides an indirect proof of completeness for the corresponding logics. Moreover, the systems maintain sufficient semantic information enabling the efficient construction of counter models for invalid principles.

An attractive feature of Hilbert-style axiomatizations of modal logics is that they are *modular*: axioms can be added or subtracted in a component-wise fashion, generating a well-behaved lattice of logics with a common base. Maintaining this modularity is challenging when developing Gentzen systems for these logics. Importantly, our approach allows us to preserve this feature – single axioms are each associated with a set of rules, and various logics are then obtained via the addition of these sets to a common core governing the logical connectives and modal operators. That these rules can be added in a cumulative manner without jeopardizing cut elimination is a consequence of the way in which they are extracted directly from the frame conditions. The benefit of this is that several non-normal modal logics can be dealt with uniformly.

Because labelled systems have been developed primarily in the context of logics characterized by relational semantics, it is not immediately obvious how to adapt this method when one is working with structures described by a set-theoretical, rather than purely logical, language as it is unclear how one ought to represent this richer structural information in the labelled system. We address this problem by utilizing the fact that one can simulate a classical modal logic by way of a normal polymodal logic (for details, see [8] or [4]). This allows us straightforwardly to borrow much of the basic proof-theoretic apparatus already developed for normal modal logics. Furthermore, working in the translated language facilitates a clearer understanding of the set-theoretic frame conditions from a purely relational perspective. However, the translation only helps establish a starting point, and leaves open the important question of how best to achieve a uniform approach incorporating the intermediate logics between **E** and **K**. We demonstrate how to extend the modularity of labelled sequent systems for normal modal logics to the logics in this range.

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