

Modular Systems for Intuitionistic Modal Logic in Nested Sequents

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In this talk we prove the modularity of a nested sequent system for intuitionistic modal logic; for each of the modal axioms d , t , b , 4 and 5 , we add an equivalent set of structural and logical rules to the basic system for intuitionistic modal logic $\mathbf{IK}$ such that the resulting system admits cut elimination. This result is similar to the one presented at “Gentzen Systems and Beyond 2011” about classical modal logics. A full paper containing both results has been accepted for publication at “AiML 2014” [MS14].

We will use a variant of the nested sequent system $\mathbf{NIK}$ introduced in [Str13] for intuitionistic modal logic (see Fig. 1). We recall that in an intuitionistic nested sequent, formulas carry a polarity—there are two polarities, *input polarity* (marked with a black dot $\bullet$) and *output polarity* (marked with a white dot $\circ$)—such that exactly one formula in the whole sequent has the output polarity.

For any combination of the modal axioms $X \subseteq \{d, t, b, 4, 5\}$, we can add to $\mathbf{NIK}$ the corresponding set $X^\circ \cup X^\bullet$ of logical $\Diamond^\circ$ - and $\Box^\bullet$ -rules. To get a complete cut-free system, we will see that we also need to make use of the corresponding set $X^\square$ of structural rules (all shown in Fig. 2).

In the course of the proofs, we will use some additional structural rules:

$$\text{w} \frac{\Gamma\{\emptyset\}}{\Gamma\{A\}} \quad \text{cut} \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$$

Weakening turns out to be admissible for any system $\mathbf{NIK} \cup X^\bullet \cup X^\circ \cup X^\square$ but not all combinations of modal rules lead to complete cut-free systems. Therefore, to get a cut-elimination proof, the notion of 45-closure needs to be introduced.

$$\begin{array}{ll} \perp^\bullet \frac{}{\Gamma\{\perp^\bullet\}} & \text{c} \frac{\Gamma\{A^\bullet, A^\bullet\}}{\Gamma\{A^\bullet\}} \quad \text{id} \frac{}{\Gamma\{a^\bullet, a^\circ\}} \\ \wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} & \wedge^\circ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \\ \vee^\bullet \frac{\Gamma\{A^\bullet\} \quad \Gamma\{A^\bullet\}}{\Gamma\{A \vee B^\bullet\}} & \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}} \\ \supset^\bullet \frac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} & \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}} \\ \Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} & \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}} \\ \Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} & \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} \end{array}$$

Figure 1: System $\mathbf{NIK}$

The 45-closure of X is defined as in [Brü09]:

$$\hat{X} = \begin{cases} X \cup \{4\} & \text{if } \{b, 5\} \subseteq X \text{ or if } \{t, 5\} \subseteq X \\ X \cup \{5\} & \text{if } \{b, 4\} \subseteq X \\ X & \text{otherwise} \end{cases}$$

This allows us then to state the main result of [Str13]:

Theorem 1 ([Str13]) *Let $X \subseteq \{d, t, b, 4, 5\}$, and let*

$$Z = \begin{cases} \mathbf{NIK} \cup \hat{X}^\bullet \cup \hat{X}^\circ & \text{if } d \notin X \\ \mathbf{NIK} \cup \hat{X}^\bullet \cup \hat{X}^\circ \cup \{d^\square\} & \text{if } d \in X \end{cases}$$

Then a formula A is a theorem of $\mathbf{IK} + X$ if and only if it is derivable in Z .

The aim of the talk is to show that we can drop the 45-closure condition that appears in Theorem 1 if

$$\begin{array}{lll}
d^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} & d^\bullet \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} & d^\square \frac{\Gamma\{\emptyset\}}{\Gamma\{\emptyset\}} \\
t^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} & t^\bullet \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} & t^\square \frac{\Gamma\{\Delta\}}{\Gamma\{\Delta\}} \\
b^\circ \frac{\Gamma\{\Delta, A^\circ\}}{\Gamma\{\Delta, \Diamond A^\circ\}} & b^\bullet \frac{\Gamma\{\Delta, A^\bullet\}}{\Gamma\{\Delta, \Box A^\bullet\}} & b^\square \frac{\Gamma\{\Sigma, [\Delta]\}}{\Gamma\{\Sigma, \Delta\}} \\
4^\circ \frac{\Gamma\{\Diamond A^\circ, \Delta\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}} & 4^\bullet \frac{\Gamma\{\Box A^\bullet, \Delta\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} & 4^\square \frac{\Gamma\{\Delta, [\Sigma]\}}{\Gamma\{[\Delta], \Sigma\}} \\
5^\circ \frac{\Gamma\{\emptyset\}\{\Diamond A^\circ\}}{\Gamma\{\Diamond A^\circ\}\{\emptyset\}} & 5^\bullet \frac{\Gamma\{\emptyset\}\{\Box A^\bullet\}}{\Gamma\{\Box A^\bullet\}\{\emptyset\}} & 5^\square \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}
\end{array}$$

Figure 2: Modal $\Diamond^\circ$ - and $\Box^\bullet$ -rules and structural modal rules for axioms $d, t, b, 4, 5$; with the proviso $depth(\Gamma\{\cdot\}\{\emptyset\}) \geq 1$ for $5^\circ, 5^\bullet$ and $5^\square$.

we use both the logical rules and the structural rules, leading to:

Theorem 2 ([MS14]) *Let $X \subseteq \{d, t, b, 4, 5\}$. A formula A is a theorem of $IK + X$ if and only if it is derivable in $NIK \cup X^\bullet \cup X^\circ \cup X^\square$.*

Indeed, all rules in $NIK \cup X^\bullet \cup X^\circ \cup X^\square$ are sound. This has already been shown (directly or indirectly) in [Brü09, BS09, Str13] and can easily be verified. Full details are in the appendix of [MS14]. Thus, any formula that is derivable in $NIK \cup X^\bullet \cup X^\circ \cup X^\square$ is also a theorem of $IK + X$.

Conversely, if A is a theorem of $IK + X$, then by Theorem 1 we have a proof of A in $NIK \cup \hat{X}^\bullet \cup \hat{X}^\circ \cup X^\square$. If $\hat{X} = X$, then we are done. Otherwise, we must have one of the following three cases:

- If $\{t, 5\} \subseteq X$ then $\hat{X} = X \cup \{4\}$. We show that $4^\bullet$ and 4° are admissible for $NIK \cup X^\bullet \cup X^\circ \cup X^\square$.
- If $\{b, 5\} \subseteq X$ then $\hat{X} = X \cup \{4\}$. We show that $4^\bullet$ and 4° are admissible for $NIK \cup X^\bullet \cup X^\circ \cup X^\square$.
- If $\{b, 4\} \subseteq X$ then $\hat{X} = X \cup \{5\}$. We can replace the $5^\bullet$ -rule (resp. 5° -rule) with three simpler variants $5_1^\bullet, 5_2^\bullet, 5_3^\bullet$ (resp. $5_1^\circ, 5_2^\circ, 5_3^\circ$), shown in Fig. 3, and then show that those rules are admissible for $NIK \cup X^\bullet \cup X^\circ \cup X^\square$.

$$\begin{array}{lll}
5_1^\circ \frac{\Gamma\{[\Delta], \Diamond A^\circ\}}{\Gamma\{[\Delta], \Diamond A^\circ\}} & 5_2^\circ \frac{\Gamma\{[\Delta], [\Diamond A^\circ, \Sigma]\}}{\Gamma\{[\Delta], \Diamond A^\circ, [\Sigma]\}} & 5_3^\circ \frac{\Gamma\{[\Delta], [\Diamond A^\circ, \Sigma]\}}{\Gamma\{[\Delta], \Diamond A^\circ, [\Sigma]\}} \\
5_1^\bullet \frac{\Gamma\{[\Delta], \Box A^\bullet\}}{\Gamma\{[\Delta], \Box A^\bullet\}} & 5_2^\bullet \frac{\Gamma\{[\Delta], [\Box A^\bullet, \Sigma]\}}{\Gamma\{[\Delta], \Box A^\bullet, [\Sigma]\}} & 5_3^\bullet \frac{\Gamma\{[\Delta], [\Box A^\bullet, \Sigma]\}}{\Gamma\{[\Delta], \Box A^\bullet, [\Sigma]\}}
\end{array}$$

Figure 3: Variants of the rules $5^\bullet$ and 5°

Therefore, in any case, A is derivable in $NIK \cup X^\bullet \cup X^\circ \cup X^\square$.

To obtain a modular cut-free system for all logics in the intuitionistic modal S5-cube, we have used here a combination of logical and structural rules. However, for some axioms only the structural or/and only the logical rules would be sufficient, depending on the system, i.e., depending on which other axioms are present. In order to better understand this phenomenon, it would be necessary to find a general pattern for translating axioms into structural and/or logical rules. In particular, it is an important question for future research, for which type of axioms such a translation is possible.

References

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