

Ludics without Designs I: Triads

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Abstract

In this paper, we introduce the concept of triad. Using this notion, we study, revisit, discover and rediscover some basic properties of ludics from a very general point of view.

1 Introduction

An orthodox introduction of a paper on ludics should begin as follows. First, the authors say what ludics is commonly intended to be: typically, they could say that it is a kind of game semantics which is close to the more popular categorical game models for linear logic and *PCF* introduced in the last twenty years. Having set up the context, then they could informally describe ludics as an untyped and monistic framework which provides a semantics for proofs of a linear (i.e., without exponentials) fragment of linear logic. The authors should also stress that ludics is a semantics which is based on interaction. Finally, — trying not to frighten too much the casual reader — the authors should give an intuitive account of some of the basic constituents of ludics: design, orthogonality, behaviour, etc., putting more emphasis on the concepts which are more related to the contribution of the paper.

Of course, there is *nothing* wrong (or bad) in starting an article on ludics in the “orthodox way” described above. However, for this paper we find more instructive to take another approach. Namely, we give from the very beginning the most important definition of our work.

Definition 1.1 (Triad). A **triad** is an ordered triple $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$ where:

- $\mathcal{P}_A = \{ p, q, r, \dots \}$ is a set. Its elements are said to be **positive terms**.
- $\mathcal{N}_A = \{ n, m, \ell, \dots \}$ is a set. Its elements are said to be **negative terms**.
- The sets $\mathcal{P}_A$ and $\mathcal{N}_A$ are disjoint.
- $\perp_A$ is a function

$$\begin{array}{ccc} \perp_A & : & \mathcal{P}_A \times \mathcal{N}_A \longrightarrow \{ \mathbf{t}, \mathbf{f} \} \\ \cup & & \cup \\ (p, n) & \longmapsto & \perp_A(p, n) \end{array}$$

called **orthogonality**. Here $\{ \mathbf{t}, \mathbf{f} \}$ denotes the standard two–elements set of the **booleans**, where $\mathbf{t}$ stands for *true* and $\mathbf{f}$ stands for *false*. $\triangle$

Example 1.2. Let $I = (\mathcal{P}_I, \mathcal{N}_I, \perp_I)$, where $\mathcal{P}_I \stackrel{\text{DEF}}{=} \{ 0, 1 \}$, $\mathcal{N}_I \stackrel{\text{DEF}}{=} \{ 2, 3 \}$, and $\perp_I$ is given by: $\perp_I(p, n) \stackrel{\text{DEF}}{=} \mathbf{t}$ if $p + n$ is even, and $\perp_I(p, n) \stackrel{\text{DEF}}{=} \mathbf{f}$ if $p + n$ is odd. The triple I is a triad in our sense. $\triangle$

The notion of triad is not at all a new mathematical concept. In *proof–theory* (especially in the proof–theory related to *linear logic*), our notion is very similar to the concept of (boolean–valued) *game* [5] or *Chu space* [7], as we discuss in Section 4. How is this notion related to ludics?

As the title of this work should suggest (“ludics without designs” is indeed quite provocative; it sounds like “proof–theory without proofs”), here we do not consider a design — the very central concept of ludics — as the *well–specified* and concrete proof–like object as defined in [4] (see also [2, 8] for

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other more or less equivalent definitions of design). Rather, a design is seen as special case of what we are calling *term*: just an *unspecified* and primitive element of the set $\mathcal{P}_A \cup \mathcal{N}_A$ of a given triad A .

Similarly, in ludics there is a *well-specified* orthogonality function [4]: the one which sends a pair (p, n) to the value t if and only if the procedure of normalization between the designs p and n successfully terminates. Here, we consider a more general situation: we are interested in all the possible orthogonality functions. Given two disjoint sets $\mathcal{P}_A$ and $\mathcal{N}_A$, any function from $\mathcal{P}_A \times \mathcal{N}_A$ to $\{t, f\}$ is an orthogonality in our sense. In particular, we do not need to introduce any kind of procedure of normalization.

What can we do with triads? Let $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$ be an arbitrary triad. In Section 2:

- We use the orthogonality function $\perp_A$ to define *closed sets* (in the sense of [3]) in $\mathcal{P}_A$ and $\mathcal{N}_A$, and we study some basic properties. Closed sets correspond to the *behaviours* of ludics [4], the semantical counter-part of the syntactical notion of formula in logic.
- We define the *specialization order* (in the sense of [3]) on $\mathcal{P}_A$ and $\mathcal{N}_A$, and again, we study some basic properties. The specialization order is exactly the *precedence relation* between designs [4].

Our methodology is the following: except for Section 4, we always work with *arbitrary triads*. This (obviously) means that the results we are going to show hold in any triad, but more importantly, that these results hold in ludics without explicitly introducing the notion of design nor the specific orthogonality function based on the (successful) termination of the procedure of normalization. The reader should be able to understand our results without any previous knowledge of ludics: all we need to know is in Definition 1.1. This should explain the “oddity” in the title of this paper.

Next, in Section 3, we introduce the concept of *functional for a triad* A . Roughly speaking, as a program P induces a function $\hat{P} : \text{Inputs} \rightarrow \text{Outputs}$, a functional for A is an object f with induces a function $\hat{f} : \mathcal{P}_A \cup \mathcal{N}_A \rightarrow \mathcal{P}_A \cup \mathcal{N}_A$. There are two properties that a functional f for A has to satisfy:

- *Preservation of polarity*: for every $p \in \mathcal{P}_A$, and every $n \in \mathcal{N}_A$, $\hat{f}(p) \in \mathcal{P}_A$ and $\hat{f}(n) \in \mathcal{N}_A$.
- *Linearity*: for every $p \in \mathcal{P}_A$, and every $n \in \mathcal{N}_A$, $\perp_A(\hat{f}(p), n) = \perp_A(p, \hat{f}(n))$.

In ludics, functionals have been introduced in [1] as a particular class of designs. They are particularly important because, categorically speaking, they constitute the “morphisms of ludics” (see [1]). One of the aims of this paper is to study the some properties of functionals at a more general level. We show that in any triad A : every functional for A is *continuous* (w.r.t. the notion of continuity for closure spaces [3]) and *preserves the specialization order*. Up to the knowledge of the present author, these results are still unnoticed when applied to ludics.

In Section 4, we give a couple of examples of triads and functionals. We show that designs and functionals as given in [1] meet our conditions. In Section 5, we conclude. This introduction, together with the final two sections, are the only points of the paper which require some familiarity with ludics.

To conclude this section, we would like point out that the results we show in this paper are neither completely new nor particularly surprising. The main purpose (and contribution) of this paper is to *present* and *give complete proofs* of these results in a sufficiently general and abstract setting.

2 Closed Sets and the Specialization Order

To begin with, we fix some notation and terminology.

Notation 2.1. Let $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$ be a triad.

- (1) We call an arbitrary element of $\mathcal{P}_A \cup \mathcal{N}_A$ a **term**. We also use $d, e, c, \dots$ to range over terms.
- (2) We use the letter $\mathcal{O}_A$ as a variable ranging over $\{\mathcal{P}_A, \mathcal{N}_A\}$. Furthermore, if $\mathcal{O}_A = \mathcal{P}_A$, then we write $\overline{\mathcal{O}_A}$ for $\mathcal{N}_A$. Similarly, if $\mathcal{O}_A = \mathcal{N}_A$, then we write $\overline{\mathcal{O}_A}$ for $\mathcal{P}_A$. Note that $\overline{\overline{\mathcal{O}_A}} = \mathcal{O}_A$.

- (3) By an abuse of notation, for $d \in \mathcal{O}_A$ and $e \in \overline{\mathcal{O}_A}$ we write $\perp_A(d, e)$, or equivalently $\perp_A(e, d)$, for $\perp_A(\text{ the positive term in } \{d, e\}, \text{ the negative term in } \{d, e\})$.

This notation makes sense precisely because the sets $\mathcal{O}_A$ and $\overline{\mathcal{O}_A}$ are supposed to be disjoint.

- (4) Finally, we write $\emptyset_{\mathcal{O}_A}$ to mean that the empty-set $\emptyset$ has to be intended as a subset of $\mathcal{O}_A$. $\triangle$

From now on, up to the end of this section, we fix an arbitrary triad $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$. To ease notation, we always omit the subscript A when referring to some constituent of A i.e., we write $\mathcal{P}$ for $\mathcal{P}_A$, $\mathcal{N}$ for $\mathcal{N}_A$ and so on.

In this section, we use the orthogonality function $\perp$ to equip the sets $\mathcal{P}$ and $\mathcal{N}$ with some “topological structure.” Namely, we introduce the concept of *closed set* in our setting. Here, closed sets are not to be intended as “closed sets in a *topological space*” but as “closed sets in a *closure space* [3]”, a slightly more general topological notion. Closed sets are important to us because they correspond to *behaviours* [4, 2, 8], the “semantical” notion of formula in ludics. Closed sets induced by orthogonality are not uncommon in the literature of theoretical computer science, see e.g., [9, 6].

Definition 2.2 (Closed sets). Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$, and let $X \subseteq \mathcal{O}$. We define $X^\perp \subseteq \overline{\mathcal{O}}$ as

$$X^\perp \stackrel{\text{DEF}}{=} \{e \in \overline{\mathcal{O}} \mid \perp(d, e) = \text{t}, \text{ for every } d \in X\}.$$

A **closed set in $\mathcal{O}$** is a set $X \subseteq \mathcal{O}$ such that $X = X^{\perp\perp}$. $\triangle$

Note that the definition of closed sets in $\mathcal{O}$ strongly depends on the function $\perp$. It is likely that two triads which only differ on their orthogonality functions have different collections of closed sets.

The following theorem justifies the terminology given in Definition 2.2.

Theorem 2.3 (Closure). Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$. Let $X \subseteq \mathcal{O}$ and $Y \subseteq \mathcal{O}$. Then,

- (1) $X \subseteq X^{\perp\perp}$ (2) $X \subseteq Y$ implies $Y^\perp \subseteq X^\perp$ (3) $X \subseteq Y$ implies $X^{\perp\perp} \subseteq Y^{\perp\perp}$
- (4) $X^\perp = X^{\perp\perp\perp}$ (5) $X^{\perp\perp} = X^{\perp\perp\perp\perp}$.

Proof. (1) Let $d \in X$. By definition of $X^\perp$, for every $e \in X^\perp$ we must have $\perp(d, e) = \text{t}$. Hence, $d \in X^{\perp\perp}$.
(2) Let $d \in Y^\perp$. Then, $\perp(d, e) = \text{t}$ for all $e \in Y$. As $X \subseteq Y$, we have $\perp(d, e) = \text{t}$ for all $e \in X$. So, $d \in X^\perp$.
(3) By applying (2) two times.
(4) By (1), we have $X^\perp \subseteq (X^\perp)^{\perp\perp} = X^{\perp\perp\perp}$. By (1) again, $X \subseteq X^{\perp\perp}$. By (2), $X^{\perp\perp\perp} = (X^{\perp\perp})^\perp \subseteq X^\perp$.
(5) By (4), $X^\perp = X^{\perp\perp\perp}$. Hence, $X^{\perp\perp} = (X^\perp)^\perp = (X^{\perp\perp\perp})^\perp = X^{\perp\perp\perp\perp}$. $\square$

Properties (1), (3) and (5) above say that $^{\perp\perp}$ is a closure operator on the sets $\mathcal{P}$ and $\mathcal{N}$, i.e., $(\mathcal{P}, ^{\perp\perp})$ and $(\mathcal{N}, ^{\perp\perp})$ form two *closure spaces*, in the sense of [3]. Furthermore, for $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$:

- (i) By property (4), for every $X \subseteq \mathcal{O}$ the set $X^\perp$ is a closed set in $\overline{\mathcal{O}}$;
- (ii) By property (5), for every $X \subseteq \mathcal{O}$ the set $X^{\perp\perp}$ is a closed set in $\mathcal{O}$.
- (iii) By property (5) again, every closed set in $\mathcal{O}$ can be written as $X^{\perp\perp}$, for some $X \subseteq \mathcal{O}$.

In the sequel, we frequently use the properties (i), (ii) and (iii) above without any explicit reference.

Examples 2.4. Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$.

- (1) $\emptyset$ is a closed set in $\mathcal{O}$. To see this, note that $\emptyset = \{d \in \mathcal{O} \mid \perp(d, e) = \text{t}, \text{ for every } e \in \emptyset_{\overline{\mathcal{O}}}\} = (\emptyset_{\overline{\mathcal{O}}})^\perp$. Furthermore, for every $X \subseteq \mathcal{O}$, we obviously have $X^{\perp\perp} \subseteq \mathcal{O}$. So, $\emptyset$ is the greatest closed set in $\mathcal{O}$.
- (2) $(\emptyset_{\overline{\mathcal{O}}})^{\perp\perp}$ is a closed set in $\mathcal{O}$. Furthermore, $(\emptyset_{\overline{\mathcal{O}}})^{\perp\perp}$ is the least closed set in $\mathcal{O}$. To see the last point, let $X \subseteq \mathcal{O}$. Since $\emptyset_{\overline{\mathcal{O}}} \subseteq X^{\perp\perp}$, by Theorem 2.3(3) and (5), we have $(\emptyset_{\overline{\mathcal{O}}})^{\perp\perp} \subseteq X^{\perp\perp\perp\perp} = X^{\perp\perp}$.
- (3) For every $d \in \mathcal{O}$, the set $\{d\}^{\perp\perp}$ is the least closed set in $\mathcal{O}$ which contains d . For, let $X \subseteq \mathcal{O}$ and assume $d \in X^{\perp\perp}$, i.e., $\{d\} \subseteq X^{\perp\perp}$. By Theorem 2.3(3) and (5), we have $\{d\}^{\perp\perp} \subseteq X^{\perp\perp\perp\perp} = X^{\perp\perp}$.

- (4) The closed set $(\emptyset_{\mathcal{O}})^{\perp\perp}$ in $\mathcal{O}$ is quite special: for every $d \in (\emptyset_{\mathcal{O}})^{\perp\perp}$ we have by (2) and (3) above that $\{d\}^{\perp\perp} = (\emptyset_{\mathcal{O}})^{\perp\perp}$. See also Proposition 2.8.

The next proposition tell us that closed sets are closed under taking arbitrary intersections.

Proposition 2.5 (Intersection). *Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$. For every subset $\mathcal{X}$ of the power-set of $\mathcal{O}$, we have $\bigcap_{X \in \mathcal{X}} X^{\perp} = (\bigcup_{X \in \mathcal{X}} X)^{\perp}$. In particular, $\bigcap_{X \in \mathcal{X}} X^{\perp}$ is a closed set in $\overline{\mathcal{O}}$.*

Proof. For every $X \in \mathcal{X}$, we have $X \subseteq \bigcup_{X \in \mathcal{X}} X$. By Theorem 2.3(2), we have $(\bigcup_{X \in \mathcal{X}} X)^{\perp} \subseteq X^{\perp}$. Since this holds for every $X \in \mathcal{X}$, we conclude $(\bigcup_{X \in \mathcal{X}} X)^{\perp} \subseteq \bigcap_{X \in \mathcal{X}} X^{\perp}$. As for the converse, let $e \in \bigcap_{X \in \mathcal{X}} X^{\perp}$. Then, $e \in Y$ for some $Y \in \mathcal{X}$. Let $c \in \bigcap_{X \in \mathcal{X}} X^{\perp}$. Then, $c \in X^{\perp}$ for every $X \in \mathcal{X}$. So, $\perp(c, e) = \mathbf{t}$, as $c \in Y^{\perp}$. Thus, $c \in (\bigcup_{X \in \mathcal{X}} X)^{\perp}$. Hence, $\bigcap_{X \in \mathcal{X}} X^{\perp} \subseteq (\bigcup_{X \in \mathcal{X}} X)^{\perp}$. $\square$

By the previous proposition, for every $X \subseteq \mathcal{O}$, the set $X^{\perp}$ is completely determined by the sets $\{d\}^{\perp}$, for $d \in X$. To see this, take $\mathcal{X} \stackrel{\text{DEF}}{=} \{\{d\} \mid d \in X\}$. Since $\mathcal{X}$ is a subset of the power-set of $\mathcal{O}$, we have $X^{\perp} = (\bigcup_{d \in X} \{d\})^{\perp} = (\bigcup_{\{d\} \in \mathcal{X}} \{d\})^{\perp} = \bigcap_{\{d\} \in \mathcal{X}} \{d\}^{\perp} = \bigcap_{d \in X} \{d\}^{\perp}$.

Unfortunately, the analogous property for unions $\bigcup_{X \in \mathcal{X}} X^{\perp} = (\bigcap_{X \in \mathcal{X}} X)^{\perp}$ fails in general. To see a counter-example, let $\mathcal{X}$ be the empty-set. Then, we would have $\emptyset_{\overline{\mathcal{O}}} = \bigcup_{X \in \mathcal{X}} X^{\perp} = (\bigcap_{X \in \mathcal{X}} X)^{\perp} = \mathcal{O}^{\perp}$. But in general, it is not true that $\emptyset_{\overline{\mathcal{O}}} = \mathcal{O}^{\perp}$, as the set $\mathcal{O}^{\perp}$ is determined by the orthogonality function¹. Closed sets can also be defined by means of Theorem 3.4: any set of the form X_f is a closed set in $\mathcal{O}$.

We now introduce the *specialization order* in our setting. Specialization is a standard ordering relation which is usually defined in topological spaces. This notion is commonly attributed to P. S. Alexandroff, and it has been extensively used in topology, order theory and domain theory. In [3], the specialization order is also defined for closure spaces, and here we follow a similar approach.

Definition 2.6 (Specialization order). Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$. We define the **specialization order** as the binary relation $\leq^{\mathcal{O}} \subseteq \mathcal{O} \times \mathcal{O}$ given by:

$$d \leq^{\mathcal{O}} e \stackrel{\text{DEF}}{\iff} \{d\}^{\perp} \subseteq \{e\}^{\perp} \quad (\text{for } d \in \mathcal{O} \text{ and } e \in \mathcal{O}). \quad \triangle$$

Note that $(\mathcal{O}, \leq^{\mathcal{O}})$ is a *preorder*, as the relation $\leq^{\mathcal{O}}$ is reflexive ($d \leq^{\mathcal{O}} d$, for every $d \in \mathcal{O}$) and transitive ($d \leq^{\mathcal{O}} e$ and $e \leq^{\mathcal{O}} c$ imply $d \leq^{\mathcal{O}} c$, for every $d \in \mathcal{O}$, $e \in \mathcal{O}$ and $c \in \mathcal{O}$).

We point out that many authors define the specialization order as the inverse relation of $\leq^{\mathcal{O}}$. For instance, in [3] the specialization order is defined as in Proposition 2.7(3), but with the role of d and e interchanged. Our definition of $\leq^{\mathcal{O}}$ follows the analogous one given in [4] (but there, the specialization order is called *precedence* relation, and in [2, 8] it is called *observational order*²).

Proposition 2.7. *Let $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$. Let $d \in \mathcal{O}$ and $e \in \mathcal{O}$. The following claims are equivalent:*

- (a) $d \leq^{\mathcal{O}} e$. (b) $e \in \{d\}^{\perp\perp}$. (c) For every $X \subseteq \mathcal{O}$, $d \in X^{\perp\perp}$ implies $e \in X^{\perp\perp}$.

In particular, as (a) and (b) are equivalent, we have $\{d\}^{\perp\perp} = \{e \in \mathcal{O} \mid d \leq^{\mathcal{O}} e\}$ for every $d \in \mathcal{O}$.

Proof. (a) $\implies$ (c): Let $X \subseteq \mathcal{O}$ and assume $d \in X^{\perp\perp}$, i.e., $\{d\} \subseteq X^{\perp\perp}$. By Theorem 2.3(2) and (4), we have $X^{\perp} = X^{\perp\perp\perp} \subseteq \{d\}^{\perp}$. Since $d \leq^{\mathcal{O}} e$, i.e., $\{d\}^{\perp} \subseteq \{e\}^{\perp}$, we have $X^{\perp} \subseteq \{e\}^{\perp}$. By Theorem 2.3(2), we have $\{e\}^{\perp\perp} \subseteq X^{\perp\perp}$. Hence, $e \in \{e\}^{\perp\perp} \subseteq X^{\perp\perp}$, by Theorem 2.3(1).

(c) $\implies$ (b): Let $X = \{d\}$. By Theorem 2.3(1), $d \in \{d\} \subseteq \{d\}^{\perp\perp}$. So, $e \in \{d\}^{\perp\perp}$ by (c).

(b) $\implies$ (a): Assume $e \in \{d\}^{\perp\perp}$, i.e., $\{e\} \subseteq \{d\}^{\perp\perp}$. By Theorem 2.3(2) and (4), we have $\{d\}^{\perp} = \{d\}^{\perp\perp\perp} \subseteq \{e\}^{\perp}$. Hence, $\{d\}^{\perp} \subseteq \{e\}^{\perp}$, i.e., $d \leq^{\mathcal{O}} e$. $\square$

¹ Suppose that $\mathcal{P} = \{0\}$ and $\mathcal{N} = \{1\}$. Then, for each $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$, we have $\emptyset_{\overline{\mathcal{O}}} = \mathcal{O}^{\perp}$ if and only if $\perp(0, 1) = \mathbf{f}$.

² In ludics, we can read $d \leq^{\mathcal{O}} e$ as “ d is more defined than e ” or “ d is less *opportunistic* than e ” (see [4] for a discussion).

We now give a characterization of the terms which inhabit the closed set $(\emptyset_{\mathcal{O}})^{\perp\perp}$ in $\mathcal{O}$ by means of the specialization order.

Proposition 2.8. *Let $\mathcal{O} \in \{ \mathcal{P}, \mathcal{N} \}$, and let $d \in (\emptyset_{\mathcal{O}})^{\perp\perp}$. Then, we have $e \leq^{\mathcal{O}} d$ for every $e \in \mathcal{O}$.*

Proof. By Example 2.4(4), we have $\{d\}^{\perp\perp} = (\emptyset_{\mathcal{O}})^{\perp\perp}$. Hence, $\{d\}^{\perp} = (\emptyset_{\mathcal{O}})^{\perp} = \overline{\mathcal{O}}$, by using Theorem 2.3(4). For every $e \in \mathcal{O}$, we have $\{e\}^{\perp} \subseteq \overline{\mathcal{O}} = \{d\}^{\perp}$. Thus, $e \leq^{\mathcal{O}} d$. $\square$

3 Functionals

We now define the concept of set of functionals for a triad.

Definition 3.1 (Set of functionals). Let $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$ be a triad. A **set of functionals for A** is a pair $F = (\mathcal{F}_F, \text{eval}_F)$ where:

- $\mathcal{F}_F = \{f, g, h, \dots\}$ is a set. Its members are called **functionals**.
- eval_F is a function, that we call **evaluation**

$$\begin{array}{ccc} \text{eval}_F & : & \mathcal{F}_F \times (\mathcal{P}_A \cup \mathcal{N}_A) \longrightarrow \mathcal{P}_A \cup \mathcal{N}_A \\ & & \cup \qquad \qquad \qquad \cup \\ & & (f, d) \longmapsto \text{eval}_F(f, d) \quad (\text{for } f \in \mathcal{F}_F \text{ and } d \in \mathcal{P}_A \cup \mathcal{N}_A) \end{array}$$

which satisfies the following clauses (called **preservation of polarity** and **linearity** respectively):

(POL) For every $p \in \mathcal{P}_A$, $n \in \mathcal{N}_A$ and $f \in \mathcal{F}_F$, $\text{eval}_F(f, p) \in \mathcal{P}_A$ and $\text{eval}_F(f, n) \in \mathcal{N}_A$.

(LIN) For every $p \in \mathcal{P}_A$, $n \in \mathcal{N}_A$ and $f \in \mathcal{F}_F$, $\perp_A(\text{eval}_F(f, p), n) = \perp_A(p, \text{eval}_F(f, n))$. $\triangle$

Example 3.2. Let I be the triad given in Example 1.2. Let $W = (\mathcal{F}_W, \text{eval}_W)$ be the pair given by: $\mathcal{F}_W \stackrel{\text{DEF}}{=} \{*, \star\}$ (where $*$ and $\star$ are just two distinct symbols), and

$$\text{eval}_W(*, d) \stackrel{\text{DEF}}{=} d, \quad \text{eval}_W(\star, d) \stackrel{\text{DEF}}{=} \begin{cases} d-1 & \text{if } d=1 \text{ or } d=3 \\ d+1 & \text{if } d=0 \text{ or } d=2 \end{cases}, \quad (\text{for } d \in \mathcal{P}_I \cup \mathcal{N}_I = \{0, 1, 2, 3\}).$$

According to our definition, W is a set of functionals for I . $\triangle$

Let us now discuss Definition 3.1. Let $A = (\mathcal{P}_A, \mathcal{N}_A, \perp_A)$ be an arbitrary but fixed triad, and let $F = (\mathcal{F}_F, \text{eval}_F)$ be an arbitrary but fixed set of functionals for A . To ease notation, we omit the subscripts A and F in the rest of this section.

(a) *The set $\mathcal{F}$ and the function eval.* To begin with, note that $\mathcal{F}$ can be any set! Indeed, for any set Z , the pair (Z, eval_Z) , where $\text{eval}_Z(f, d) \stackrel{\text{DEF}}{=} d$, for all $f \in Z$ and all $d \in \mathcal{P} \cup \mathcal{N}$, is a set of functionals for A . In this paper, we decided to take the most liberal definition for the collection of functionals $\mathcal{F}$. We do not require that $\mathcal{F}$ has to contain some “identity functional”, and we do not require that $\mathcal{F}$ has to be closed under some notion of composition. So, we are not considering functionals because we want to form a category (this is done in [1]). Here we want to look at them from a quite general perspective.

Functionals have a function-like input–output behaviour which is completely determined by the function eval. To better understand this point, let $f \in \mathcal{F}$ and consider the set $\hat{f} \stackrel{\text{DEF}}{=} \{(d, \text{eval}(f, d)) \mid d \in \mathcal{P} \cup \mathcal{N}\}$. We claim that $\hat{f}$ is a function from $\mathcal{P} \cup \mathcal{N}$ to itself (in the standard set-theoretical sense). For, we have to prove that for every $d \in \mathcal{P} \cup \mathcal{N}$ there exists a unique $e \in \mathcal{P} \cup \mathcal{N}$ such that $(d, e) \in \hat{f}$. Let $d \in \mathcal{P} \cup \mathcal{N}$. As for existence, let e be $\text{eval}(f, d)$. Then, $\text{eval}(f, d) \in \mathcal{P} \cup \mathcal{N}$ and $(d, \text{eval}(f, d)) = (d, e) \in \hat{f}$, by definition of eval and $\hat{f}$. As for uniqueness, let $(d, e) \in \hat{f}$ and $(d, e') \in \hat{f}$. Then, by definition of $\hat{f}$, we have $e = \text{eval}(f, d)$ and $e' = \text{eval}(f, d)$. Since eval is a function, $e = e'$.

Hence, a functional $f \in \mathcal{F}$ determines, via eval, a *genuine* function $\hat{f} : \mathcal{P} \cup \mathcal{N} \longrightarrow \mathcal{P} \cup \mathcal{N}$. But in general, *functionals are not functions* (in Example 3.2, the symbols $*$ and $\star$ are certainly not *functions*).

Since for every $f \in \mathcal{F}$ and every $d \in \mathcal{P} \cup \mathcal{N}$ we have $\hat{f}(d) = \text{eval}(f, d)$ in the sequel we also write $\hat{f}(d)$ for $\text{eval}(f, d)$, as we did in the introduction.

(b) *Preservation of polarity.* The condition (POL) comes from ludics, where functionals (as defined in [1]) always satisfy this property (see also Example 4.4).

(c) *Linearity.* Again, (LIN) comes from ludics, where functionals always satisfy this property (see Example 4.4). The name “linearity” comes from [5], where maps between *games* are said to be *linear* if they satisfy the analogous condition of our (LIN) (see Example 4.3). In turn, this terminology comes from a property of linear maps in vector spaces (see [5]). In the standard terminology for Chu spaces, condition (LIN) is commonly called *adjunction* condition.

We now relate functionals to closed sets, continuity, and the specialization order. We recall from [3] that a *continuous function* from a closure space (X, Γ) to a closure space (Y, Δ) (here X, Y are sets and Γ, Δ are closure operators) is a function $F : X \rightarrow Y$ such that: for every closed set W in (Y, Δ) , the set $\{x \in X \mid F(x) \in W\}$ (the *preimage of W under F*) is a closed set in (X, Γ) . In our setting, we define the concept of continuity for functionals in a similar fashion.

Definition 3.3 (Continuous functional). Let $f \in \mathcal{F}$ be a functional. We say that f is **continuous** if for every $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$ and every $X \subseteq \mathcal{O}$ the following set X_f is a closed set in $\mathcal{O}$

$$X_f \stackrel{\text{DEF}}{=} \{d \in \mathcal{O} \mid \hat{f}(d) \in X^{\perp\perp}\} . \quad \triangle$$

We now show that in our setting every functional is continuous. Our proof only uses conditions (POL) and (LIN) of Definition 3.1. We also point out that analogous results have already been noticed in similar works which use orthogonality to generate topological structure (see e.g., Lemma 11 in [9]).

Theorem 3.4 (Continuity of functionals). *Every functional $f \in \mathcal{F}$ is continuous.*

Proof. Let $f \in \mathcal{F}$ be a functional, and let $X \subseteq \mathcal{O}$. We have

$$\begin{aligned} d \in X_f &\iff \hat{f}(d) \in X^{\perp\perp} &\iff \perp(\hat{f}(d), e) = \mathbf{t} \text{ for every } e \in X^\perp \\ &\iff \perp(d, \hat{f}(e)) = \mathbf{t} \text{ for every } e \in X^\perp &\iff \perp(d, c) = \mathbf{t} \text{ for every } c \in \{\hat{f}(e) \mid e \in X^\perp\} \\ &\iff d \in \{\hat{f}(e) \mid e \in X^\perp\}^{\perp\perp} . \end{aligned}$$

By Theorem 2.3(4), the set $\{\hat{f}(e) \mid e \in X^\perp\}^{\perp\perp}$ is a closed set in $\mathcal{O}$. This concludes the proof. $\square$

Definition 3.5 (Preservation of the specialization order). Let $f \in \mathcal{F}$ be a functional. We say that f **preserves the specialization order** if for every $\mathcal{O} \in \{\mathcal{P}, \mathcal{N}\}$, every $d \in \mathcal{O}$ and every $e \in \mathcal{O}$

$$d \leq^\mathcal{O} e \text{ implies } \hat{f}(d) \leq^\mathcal{O} \hat{f}(e) . \quad \triangle$$

We now prove that in our setting every functional preserves the specialization order. As before, our proof only uses conditions (POL) and (LIN) of Definition 3.1.

Theorem 3.6. *Every functional $f \in \mathcal{F}$ preserves the specialization order.*

Proof. Assume $d \leq^\mathcal{O} e$, i.e., $\{d\}^\perp \subseteq \{e\}^\perp$. We have to show that $\hat{f}(d) \leq^\mathcal{O} \hat{f}(e)$ i.e., $\{\hat{f}(d)\}^\perp \subseteq \{\hat{f}(e)\}^\perp$. Let $c \in \{\hat{f}(d)\}^\perp$. Then, $\perp(\hat{f}(d), c) = \mathbf{t} = \perp(d, \hat{f}(c))$. Hence, $\hat{f}(c) \in \{d\}^\perp$. Since $\{d\}^\perp \subseteq \{e\}^\perp$, we have $\hat{f}(c) \in \{e\}^\perp$. Since $\perp(\hat{f}(c), e) = \mathbf{t} = \perp(c, \hat{f}(e))$, we conclude $c \in \{\hat{f}(e)\}^\perp$. $\square$

4 Examples: Boolean-Valued Games and Ludics

We now give two examples of triads.

Example 4.1 (Boolean-valued games). A *boolean-valued game* [5] (or *boolean-valued Chu space* [7]) is a triple $z = (P, O, r)$, where P (“strategies”, “points”) and O (“co-strategies”, “open sets”) are sets, and $r : P \times O \rightarrow \{t, f\}$. Every boolean-valued game with P and O disjoint is a triad in our sense. $\triangle$

Example 4.2 (Ludics). We now show that ludics fits into our general setting. For convenience, we consider ludics as formulated in [8]. In Subsection 2.1 of [1] the reader can find all the notions required to understand this example. Let $\mathcal{A}$ be a signature in the sense of [8, 1]. Consider the triple $\mathbb{A} = (\mathcal{P}_{\mathbb{A}}, \mathcal{N}_{\mathbb{A}}, \perp_{\mathbb{A}})$ where:

- $\mathcal{P}_{\mathbb{A}}$ is the set of linear, cut-free and positive designs with at most x_0 as free variable (i.e., the set of atomic designs *augmented* by Ω , or “partial³ atomic designs”, according to [1]) over the signature $\mathcal{A}$;
- $\mathcal{N}_{\mathbb{A}}$ is the set of linear, cut-free and negative designs without free variables (i.e., the set of negative atomic designs of [1]) over the signature $\mathcal{A}$;
- $\perp_{\mathbb{A}}(p, n) \stackrel{\text{DEF}}{=} t$ if $\llbracket p[n/x_0] \rrbracket = \mathbf{\Psi}$, and $\perp_{\mathbb{A}}(p, n) \stackrel{\text{DEF}}{=} f$ if $\llbracket p[n/x_0] \rrbracket = \Omega$, where $p \in \mathcal{P}_{\mathbb{A}}$ and $n \in \mathcal{N}_{\mathbb{A}}$.

The triple $\mathbb{A}$ is a triad because in ludics, the sets $\mathcal{P}_{\mathbb{A}}$ and $\mathcal{N}_{\mathbb{A}}$ are disjoint and because, for $p \in \mathcal{P}_{\mathbb{A}}$ and $n \in \mathcal{N}_{\mathbb{A}}$, the operation $(p, n) \mapsto \llbracket p[n/x_0] \rrbracket$ is a function from $\mathcal{P}_{\mathbb{A}} \times \mathcal{N}_{\mathbb{A}}$ to $\{\mathbf{\Psi}, \Omega\}$. $\triangle$

We now give examples of sets of functionals for the triads given in Example 4.1 and Example 4.2.

Example 4.3 (Linear maps). Given two boolean-valued games, $z = (P, O, r)$ and $z' = (P', O', r')$, a *linear map from z to z'* is a pair $\langle f, g \rangle$ of functions $f : P \rightarrow P'$ and $g : O' \rightarrow O$ such that $r'(f(p), o') = r(p, g(o'))$, for every $p \in P$ and every $o' \in O'$. Let $z = (P, r, O)$ be a boolean-valued game such that P and O are disjoint. Consider the pair $F = (\mathcal{F}_F, \text{eval}_F)$, where $\mathcal{F}_F$ is the set of all linear maps from z to itself, and eval_F is given by: $\text{eval}_F(\langle f, g \rangle, p) \stackrel{\text{DEF}}{=} f(p)$ and $\text{eval}_F(\langle f, g \rangle, o) \stackrel{\text{DEF}}{=} g(o)$, for every $\langle f, g \rangle \in \mathcal{F}_F$, every $p \in P$ and every $o \in O$. We claim that F is a set of functionals for z . To prove this claim, we need to show that the conditions (POL) and (LIN) are satisfied. Let $\langle f, g \rangle \in \mathcal{F}_F$, $p \in P$ and $o \in O$.

(POL) Since $\langle f, g \rangle \in \mathcal{F}_F$, we have $f : P \rightarrow P$ and $g : O \rightarrow O$. Hence, $\text{eval}_F(\langle f, g \rangle, p) = f(p) \in P$ and $\text{eval}_F(\langle f, g \rangle, o) = g(o) \in O$.

(LIN) Since $\langle f, g \rangle \in \mathcal{F}_F$, we have $r(f(p), o) = r(p, g(o))$. Thus, $r(\text{eval}_F(\langle f, g \rangle, p), o) = r(f(p), o) = r(p, g(o)) = r(p, \text{eval}_F(\langle f, g \rangle, o))$. $\triangle$

Example 4.4 (Functionals in ludics). Let $\mathbb{A}$ be the triad defined in Example 4.2. Consider the pair $H = (\mathcal{F}_H, \text{eval}_H)$, where $\mathcal{F}_H$ is the set of all linear, cut-free and negative designs with at most x_0 as free variable (i.e., the set of *functionals*, in the sense of Subsection 2.2 of [1]) over the signature $\mathcal{A}$, and eval_H is given as follows: $\text{eval}_H(f, p) \stackrel{\text{DEF}}{=} \llbracket p[f/x_0] \rrbracket$ and $\text{eval}_H(f, n) \stackrel{\text{DEF}}{=} \llbracket f[n/x_0] \rrbracket$, for every $f \in \mathcal{F}_H$, every $p \in \mathcal{P}_{\mathbb{A}}$ and every $n \in \mathcal{N}_{\mathbb{A}}$. To prove that H is a set of functionals for $\mathbb{A}$, we now show that the conditions (POL) and (LIN) hold. Let $f \in \mathcal{F}_H$, $p \in \mathcal{P}_{\mathbb{A}}$, $n \in \mathcal{N}_{\mathbb{A}}$ and $d \in \mathcal{P}_{\mathbb{A}} \cup \mathcal{N}_{\mathbb{A}}$.

(POL) Since $\mathcal{P}_{\mathbb{A}}$ and $\mathcal{N}_{\mathbb{A}}$ are disjoint, the operation

$$(f, d) \mapsto \begin{cases} \llbracket d[f/x_0] \rrbracket & \text{if } d \in \mathcal{P}_{\mathbb{A}} \\ \llbracket f[d/x_0] \rrbracket & \text{if } d \in \mathcal{N}_{\mathbb{A}} \end{cases}$$

is a function⁴ from $\mathcal{F}_H \times (\mathcal{P}_{\mathbb{A}} \cup \mathcal{N}_{\mathbb{A}})$ to $\mathcal{P}_{\mathbb{A}} \cup \mathcal{N}_{\mathbb{A}}$. So, eval_H is well-defined. Moreover, $\text{eval}_H(f, p) \in \mathcal{P}_{\mathbb{A}}$ and $\text{eval}_H(f, n) \in \mathcal{N}_{\mathbb{A}}$ follow from the definition of normalization.

³In this example, Ω is a member of $\mathcal{P}_{\mathbb{A}}$. In ludics, this situation is usually not allowed. The only differences — w.r.t. more traditional presentations of ludics — are the following: (i) the set $\emptyset_{\mathcal{N}_{\mathbb{A}}}$ is a closed set in $\mathcal{N}_{\mathbb{A}}$ (because $\emptyset_{\mathcal{N}_{\mathbb{A}}} = \{\Omega\}^{\perp_{\mathbb{A}}}$); (ii) the set $\mathcal{P}_{\mathbb{A}}$ is the unique closed set in $\mathcal{P}_{\mathbb{A}}$ which contains Ω (because $\{\Omega\}^{\perp_{\mathbb{A}} \perp_{\mathbb{A}}} = (\emptyset_{\mathcal{N}_{\mathbb{A}}})^{\perp_{\mathbb{A}}} = \mathcal{P}_{\mathbb{A}}$).

⁴It would be only a *partial* function if $\Omega \notin \mathcal{P}_{\mathbb{A}}$, as there are many p (different from Ω) and f such that $\llbracket p[f/x_0] \rrbracket = \Omega$. Exactly for this reason, we included Ω in $\mathcal{P}_{\mathbb{A}}$.

