

A Linear/Producer/Consumer Model of Classical Linear Logic

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Abstract

This paper defines a new proof- and category-theoretic framework for *classical linear logic* that separates reasoning into one linear regime and two persistent regimes corresponding to $!$ and $?$. The resulting linear/producer/consumer (LPC) logic puts the three classes of propositions on the same semantic footing, following Benton’s linear/non-linear formulation of intuitionistic linear logic. Semantically, LPC corresponds to a system of three categories connected by adjunctions reflecting the linear/producer/consumer structure. The paper’s metatheoretic results include admissibility theorems for the cut and duality rules, and a translation of the LPC logic into category theory. The work also presents several concrete instances of the LPC model.

1 Introduction

Since its introduction by Girard in 1987, linear logic has been found to have a range of applications in logic, proof theory, and programming languages. Its power stems from its ability to carefully manage resource usage: it makes a crucial distinction between *linear* (used exactly once) and *persistent* (unrestricted use) hypotheses, internalizing the latter via the $!$ connective. From a semantic point of view, the literature has converged (following Benton [3]) on an interpretation of $!$ as a comonad given by $! = F \circ G$ where $F \dashv G$ is a symmetric monoidal adjunction between categories $\mathcal{L}$ and $\mathcal{P}$ arranged as shown in Figure 1.

Here, $\mathcal{L}$ (for “linear”) is a symmetric monoidal closed category and $\mathcal{P}$ (for “persistent”) is a cartesian category. This is, by now, a standard way of interpreting *intuitionistic* linear logic (for details, see the discussion in Melliès’ article [13]). If, in addition, the category $\mathcal{L}$ is $*$ -autonomous, then the structure above is sufficient to interpret *classical* linear logic, where the monad $?$ is determined by $? = (F^{op} (G^{op} (-^\perp)))^\perp$. While sound, this situation unnecessarily commits to a particular implementation of $?$ in term of $\mathcal{P}^{op}$. The LPC framework absolves this commitment by opening up a new range of semantic models, discussed in Section 4.

With that motivation, this paper defines a proof- and category-theoretic framework for full *classical linear logic* that uses *two* persistent categories: one corresponding to $!$ and one to $?$. The resulting categorical structure is shown in Figure 2, where $\mathcal{P}$ now takes the place of the “producing” category, in duality with $\mathcal{C}$ as the “consuming” category. This terminology comes from the observations that:

$$!A \vdash 1_L \quad !A \vdash A \quad !A \vdash !A \otimes !A \quad \perp_L \vdash ?A \quad A \vdash ?A \quad ?A \wp ?A \vdash ?A$$

Intuitively, the left group means that $!A$ is sufficient to *produce* any number of copies of A and, dually, the right group says that $?A$ can *consume* any number of copies of A .

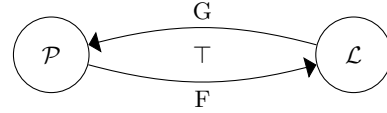


Figure 1: Categorical model of ILL with linear and persistent categories.

2 LPC Logic

The syntax of the LPC logic is made up of three syntactic forms for propositions: linear propositions A , producer propositions P , and consumer propositions C .

$A ::=$	$\top$	$A_1 \& A_2$	0	$A_1 \oplus A_2$
	1_L	$A_1 \otimes A_2$	$\perp_L$	$A_1 \wp A_2$
	$F_! P$	$F_? C$		
$P ::=$	1_P	$P_1 \otimes P_2$	$\lceil A \rceil$	
$C ::=$	$\perp_C$	$C_1 \wp C_2$	$\lfloor A \rfloor$	

The syntactic form of a proposition is called its *mode*—linear L, producing P or consuming C. The meta-variable X ranges over propositions of any mode, and the tagged meta-variable X^m ranges over propositions of mode m. The term *persistent* refers to either producer or consumer propositions.

LPC replaces the usual constructors $!$ and $?$ with two pairs of connectives: $F_!$ and $\lceil - \rceil$ for $!$ and $F_?$ and $\lfloor - \rfloor$ for $?$. If A is a linear proposition, $\lceil A \rceil$ is a producer and $\lfloor A \rfloor$ is a consumer. On the other hand, a producer proposition P may be “frozen” into a linear proposition $F_! P$, effectively discarding its persistent characteristics. Similarly for a consumer C , $F_? C$ is linear. The linear propositions $!A$ and $?A$ are encoded in this system as $F_!(\lceil A \rceil)$ and $F_?(\lfloor A \rfloor)$ respectively.

The inference rules of the logic are shown in Figures 3 and 4. There are two judgments: the linear sequent $\Gamma \vdash \Delta$ and the persistent sequent $\Gamma \Vdash \Delta$. In the linear sequent, the (unordered) contexts Γ and Δ may be made up of propositions of any mode; in the persistent sequent, the contexts may contain only persistent propositions. The meta-variable Γ^P refers to contexts made up entirely of producer propositions, and Δ^C refers to contexts of consumer propositions.

The linear inference rules in Figures 3 and 4a encompass rules for the units and the linear operators $\oplus$, $\&$, $\otimes$ and $\wp$. It is worth noting that the multiplicative product $\otimes$ is defined only on linear and producer propositions, while the multiplicative sum $\wp$ is defined only on linear and consumer propositions.¹

Weakening and contraction can be applied for producers on the left-hand-side and consumers on the right-hand-side of both the linear and persistent sequents. For producers, that is:

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta} \text{W}^{\vdash}\text{-L} \quad \frac{\Gamma \Vdash \Delta}{\Gamma, P \Vdash \Delta} \text{W}^{\Vdash}\text{-L} \quad \frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta} \text{C}^{\vdash}\text{-L} \quad \frac{\Gamma, P, P \Vdash \Delta}{\Gamma, P \Vdash \Delta} \text{C}^{\Vdash}\text{-L}$$

The rules for the operators $F_!$, $F_?$, $\lceil - \rceil$ and $\lfloor - \rfloor$ are given in Figure 4b. These rules encode dereliction and promotion for $!$ and $?$ by passing through the adjunction. For example:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \Rightarrow \frac{\frac{\Gamma, A \vdash \Delta}{\Gamma, \lceil A \rceil \vdash \Delta}}{\Gamma, F_! \lceil A \rceil \vdash \Delta} \quad \frac{\Gamma^! \vdash \Delta^?, A}{\Gamma^! \vdash \Delta^?, !A} \Rightarrow \frac{\frac{\Gamma^P \vdash \Delta^C, A}{\Gamma^P \Vdash \Delta^C, \lceil A \rceil}}{\Gamma^P \vdash \Delta^C, F_! \lceil A \rceil}$$

¹The persistent operators in this paper are necessary for LPC, but in general are not restricted to the sum and product. Other operators, like $\rightarrow$ or $\vee$, could be incorporated so long as every producer operator has a dual for consumers.

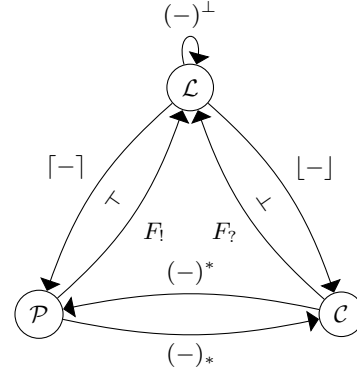


Figure 2: Categorical model of classical linear logic with linear, producing and consuming categories.

$$\begin{array}{c}
\overline{X \vdash X} \quad \text{Ax}^\vdash \qquad \overline{\Gamma \vdash \Delta, \top} \quad \top^\vdash\text{-R} \qquad \overline{\Gamma, 0 \vdash \Delta} \quad 0^\vdash\text{-L} \\
\\
\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&^\vdash\text{-L1} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \&^\vdash\text{-L2} \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B} \oplus^\vdash\text{-R1} \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B} \oplus^\vdash\text{-R2} \\
\\
\frac{\Gamma \vdash \Delta, A_1 \quad \Gamma \vdash \Delta, A_2}{\Gamma \vdash \Delta, A_1 \& A_2} \&^\vdash\text{-R} \qquad \frac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \oplus A_2 \vdash \Delta} \oplus^\vdash\text{-L} \\
\\
\frac{\Gamma, X_1^m, X_2^m \vdash \Delta \quad m \in \{L, P\}}{\Gamma, (X_1 \otimes X_2)^m \vdash \Delta} \otimes^\vdash\text{-L} \qquad \frac{\Gamma_1 \vdash \Delta_1, X_1^m \quad \Gamma_2 \vdash \Delta_2, X_2^m \quad m \in \{L, P\}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, (X_1 \otimes X_2)^m} \otimes^\vdash\text{-R} \\
\\
\frac{\Gamma \vdash \Delta \quad m \in \{L, P\}}{\Gamma, 1_m \vdash \Delta} 1^\vdash\text{-L} \qquad \frac{m \in \{L, P\}}{\cdot \vdash 1_m} 1^\vdash\text{-R} \\
\\
\frac{\Gamma_1, X_1^m \vdash \Delta_1 \quad \Gamma_2, X_2^m \vdash \Delta_2 \quad m \in \{L, C\}}{\Gamma_1, \Gamma_2, (X_1 \wp X_2)^m \vdash \Delta_1, \Delta_2} \wp^\vdash\text{-L} \qquad \frac{\Gamma \vdash \Delta, X_1^m, X_2^m \quad m \in \{L, C\}}{\Gamma \vdash \Delta, (X_1 \wp X_2)^m} \wp^\vdash\text{-R} \\
\\
\frac{m \in \{L, C\}}{\perp_m \vdash \cdot} \perp^\vdash\text{-L} \qquad \frac{\Gamma \vdash \Delta \quad m \in \{L, C\}}{\Gamma \vdash \Delta, \perp_m} \perp^\vdash\text{-R}
\end{array}$$

Figure 3: Inference Rules for Linear Sequent

Displacement. The commas on the left-hand-side of both the linear and persistent sequents intuitively correspond to the $\otimes$ operator, and the commas on the right correspond to $\wp$. This correspondence motivates the context restriction in the rules that move between the linear and persistent regimes. The restriction ensures that almost all of the propositions have the “natural” mode—producers on the left and consumers on the right. We say “almost” because the principal formula in each rule defies this classification. We call such propositions *displaced*.

Definition 1. In a derivation of $\Gamma \Vdash \Delta$, a producer P is displaced if it appears in Δ . A consumer C is displaced if it appears in Γ .

Proposition 2 (Displacement). A derivation of $\Gamma \Vdash \Delta$ has exactly one displaced proposition.

Cut. The cut rules are presented in Figure 4c. Whenever a persistent cut term is displaced in a subderivation, that derivation must be persistent and satisfy the restrictions of Proposition 2. Lacking this restriction, simpler formulations of the cut rules disallow cut admissibility. For further discussion of the cut formulation and the full proof of Theorem 4, see the accompanying technical report [14].

To show admissibility of the CUT rules, it is sufficient to show admissibility of an equivalent set of rules called CUT+. The versions differ in their treatment of cut terms. The CUT+ formulation uses the observation that when a persistent proposition is *not* displaced in a sequent, it can be replicated any number of times. That is, the following are admissible in LPC (and similarly for the persistent sequent):

$$\frac{\Gamma, (P)_n \vdash \Delta}{\Gamma, P \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, (C)_n}{\Gamma \vdash \Delta, C}$$

Thus the CUT+ rules for persistent cut terms are as follows:

$$\frac{\Gamma_1^P \Vdash \Delta_1^C, P \quad (P)_n, \Gamma_2 \vdash \Delta_2}{\Gamma_1^P, \Gamma_2 \vdash \Delta_1^C, \Delta_2} \text{CUT+}_P^{\vdash} \qquad \frac{\Gamma_1^P \Vdash \Delta_1^C, P \quad (P)_n, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^P, \Gamma_2 \Vdash \Delta_1^C, \Delta_2} \text{CUT+}_P^{\Vdash}$$

$$\begin{array}{c}
\overline{P \Vdash P} \text{ Ax}_P^\perp \qquad \overline{C \Vdash C} \text{ Ax}_C^\perp \\
\\
\frac{\Gamma_1, C_1 \Vdash \Delta_1 \quad \wp_C^\perp\text{-L}}{\Gamma_1, \Gamma_2, C_1 \wp C_2 \Vdash \Delta_1, \Delta_2} \quad \frac{\Gamma_1 \Vdash \Delta_1, P_1 \quad \otimes_P^\perp\text{-R}}{\Gamma_1, \Gamma_2 \Vdash \Delta_1, \Delta_2, P_1 \otimes P_2} \quad \frac{\Gamma, P \vdash \Delta}{\Gamma, F_! P \vdash \Delta} F_!\text{-L} \quad \frac{\Gamma^P \Vdash \Delta^C, P}{\Gamma^P \vdash \Delta^C, F_! P} F_!\text{-R} \\
\frac{\Gamma, P_1, P_2 \Vdash \Delta}{\Gamma, P_1 \otimes P_2 \Vdash \Delta} \otimes_P^\perp\text{-L} \quad \frac{\Gamma \Vdash \Delta, C_1, C_2}{\Gamma \Vdash \Delta, C_1 \wp C_2} \wp_C^\perp\text{-R} \quad \frac{\Gamma^P, C \Vdash \Delta^C}{\Gamma^P, F_? C \vdash \Delta^C} F_?\text{-L} \quad \frac{\Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, F_? C} F_?\text{-R} \\
\overline{\perp_C \Vdash \cdot} \perp_C^\perp\text{-L} \quad \overline{\cdot \Vdash 1_P} 1_P^\perp\text{-R} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, [A] \vdash \Delta} [-]\text{-L} \quad \frac{\Gamma^P \vdash \Delta^C, A}{\Gamma^P \Vdash \Delta^C, [A]} [-]\text{-R} \\
\frac{\Gamma \Vdash \Delta}{\Gamma, 1_P \Vdash \Delta} 1_P^\perp\text{-L} \quad \frac{\Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \perp_C} \perp_C^\perp\text{-R} \quad \frac{\Gamma^P, A \vdash \Delta^C}{\Gamma^P, [A] \Vdash \Delta^C} [-]\text{-L} \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, [A]} [-]\text{-R} \\
\text{(a) Inference Rules for Persistent Sequent} \qquad \text{(b) Adjunction Inference Rules} \\
\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{CUT}_L^\perp \\
\\
\frac{\Gamma_1^P \Vdash \Delta_1^C, P \quad P, \Gamma_2 \vdash \Delta_2}{\Gamma_1^P, \Gamma_2 \vdash \Delta_1^C, \Delta_2} \text{CUT}_P^\perp \quad \frac{\Gamma_1^P \Vdash \Delta_1^C, P \quad P, \Gamma_2 \Vdash \Delta_2}{\Gamma_1^P, \Gamma_2 \Vdash \Delta_1^C, \Delta_2} \text{CUT}_P^\perp \\
\frac{\Gamma_1 \vdash \Delta_1, C \quad C, \Gamma_2^P \Vdash \Delta_2^C}{\Gamma_1, \Gamma_2^P \vdash \Delta_1, \Delta_2^C} \text{CUT}_C^\perp \quad \frac{\Gamma_1 \Vdash \Delta_1, C \quad C, \Gamma_2^P \Vdash \Delta_2^C}{\Gamma_1, \Gamma_2^P \Vdash \Delta_1, \Delta_2^C} \text{CUT}_C^\perp \\
\text{(c) CUT Inference Rules}
\end{array}$$

Figure 4: Persistent and Auxiliary Inference Rules

Lemma 3 (CUT+ Admissibility). *The CUT+ rules are admissible in LPC.*

Theorem 4 (CUT Admissibility). *The CUT rules in Figure 4c are admissible in LPC.*

Duality. Unlike standard presentations of classical linear logic, LPC does not contain an explicit duality operator $(-)^{\perp}$, nor a linear implication $\multimap$ with which to encode duality. Instead, we define $(-)^{\perp}$ to be a meta-operation on propositions and prove that the following duality rules are admissible in LPC:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta} (-)^{\perp}\text{-L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}} (-)^{\perp}\text{-R}$$

In fact, there are three versions of this duality operation: $(-)^{\perp}$ for linear propositions, $(-)^*$ for producers and $(-)_*$ for consumers. For a linear proposition A , $A^{\perp}$ is linear, but for a producer P , P^* is a consumer, and for a consumer C , C_* is a producer. We define these duality operations as follows:

$$\begin{array}{llll}
\top^{\perp} := 0 & 1_L^{\perp} := \perp_L & 1_P^* := \perp_C & (F_! P)^{\perp} := F_? P^* \\
0^{\perp} := \top & \perp_L^{\perp} := 1_L & \perp_C^* := 1_P & (F_? C)^{\perp} := F_! C_* \\
(A \& B)^{\perp} := A^{\perp} \oplus B^{\perp} & (A \otimes B)^{\perp} := A^{\perp} \wp B^{\perp} & (P \otimes Q)^* := P^* \wp Q^* & [A]^* := [A^{\perp}] \\
(A \oplus B)^{\perp} := A^{\perp} \& B^{\perp} & (A \wp B)^{\perp} := A^{\perp} \otimes B^{\perp} & (C \wp D)_* := C_* \otimes D_* & [A]_* := [A^{\perp}]
\end{array}$$

It is easy to show that the following axioms hold in LPC:

$$\overline{A, A^{\perp} \vdash \cdot} \quad \overline{\cdot \vdash A, A^{\perp}} \quad \overline{P, P^* \Vdash \cdot} \quad \overline{\cdot \Vdash P, P^*}$$

Using cut we can further prove $P, P^* \vdash \cdot$ and $\cdot \vdash P, P^*$. From these axioms it is possible to derive the following duality inference rules:

$$\frac{\Gamma \vdash \Delta, A \quad \overline{A, A^\perp \vdash \cdot}}{\Gamma, A^\perp \vdash \Delta} \text{CUT}_L^\perp \quad \frac{\Gamma^P \Vdash \Delta^C, P \quad \overline{P, P^* \Vdash \cdot}}{\Gamma^P, P^* \Vdash \Delta^C} \text{CUT}_P^\perp \quad \frac{\Gamma^P \Vdash \Delta^C, P \quad \overline{P, P^* \vdash \cdot}}{\Gamma^P, P^* \vdash \Delta^C} \text{CUT}_P^\perp$$

For linear duality, the above is a straightforward application of $\text{CUT}_L^\perp$. For duality of persistent propositions there are two sets of rules: one for the linear sequent and one for the persistent sequent. For the persistent sequent rule, the context restriction induced by the cut rule is actually redundant by the displacement proposition— P is the *only* displaced proposition in any derivation of $\Gamma \Vdash \Delta, P$. However the rule induced by $\text{CUT}_P^\perp$ is *not* the rule we expect for producer duality in the linear sequent—the hypothesis and conclusion are different kinds of sequents, and linear propositions are completely excluded from the contexts. Instead we can prove the more general form of the rule directly: For any derivation of $\Gamma \vdash \Delta, P$, there is a derivation of $\Gamma, P^* \vdash \Delta$.

Consistency. Define the negation of a linear proposition to be $\neg A := A^\perp \wp 0$.

Theorem 5 (Consistency). *There is no A such that A and $\neg A$ are both provable in LPC.*

3 Categorical Model

In this section we describe a categorical axiomatization of LPC based on Figure 2. Some definitions have been omitted for brevity; for details see the companion paper [14].

Traditionally the multiplicative fragment of linear logic is modeled by a *-autonomous category. For LPC, the multiplicative fragment is modeled by a symmetric linearly distributive category with negation, an equivalent notion that puts $\otimes$ and $\wp$ on equal footing [5].

Definition 6. Let $\mathcal{L}$ be a category with two symmetric monoidal structures: $\otimes$ and $\wp$, with a natural transformation $\delta_{A_1, A_2, A_3} : A_1 \otimes (A_2 \wp A_3) \rightarrow (A_1 \otimes A_2) \wp A_3$. Then $\mathcal{L}$ is symmetric linearly distributive if δ satisfies the coherence conditions described by Cockett and Seely [5].

$\mathcal{L}$ is said to have negation if there exists a map $(-)^{\perp}$ on objects of $\mathcal{L}$ and families of maps $\gamma_A^{\perp} : A^{\perp} \otimes A \rightarrow \perp_L$ and $\gamma_A^{\perp} : 1_L \rightarrow A \wp A^{\perp}$ commuting with δ in certain ways.

Theorem 7 (Cockett and Seely). *Symmetric linearly distributive categories with negation correspond to *-autonomous categories.*

Definition 8. A linear/producing/consuming (LPC) model consists of the following components:

1. A symmetric linearly distributive category $(\mathcal{L}, \otimes, \wp)$ with negation $(-)^{\perp}$, finite products $\otimes$ and finite coproducts $\oplus$.
2. Symmetric monoidal categories $(\mathcal{P}, \otimes)$ and $(\mathcal{C}, \wp)$ in duality by means of contravariant functors $(-)^* : \mathcal{P} \Rightarrow \mathcal{C}$ and $(-)_* : \mathcal{C} \Rightarrow \mathcal{P}$, where $(-)^*$ is monoidal and $(-)_*$ is comonoidal, with natural isomorphisms $\epsilon_{*C}^* : (C_*)^* \rightarrow C$ and $\eta_{*P}^* : P \rightarrow (P^*)^*$.
3. Monoidal natural transformations $e_P^{\otimes} : P \rightarrow 1_P$ and $d_P^{\otimes} : P \rightarrow P \otimes P$ in $\mathcal{P}$ and comonoidal natural transformations $e_C^{\wp} : \perp_C \rightarrow C$ and $d_C^{\wp} : C \wp C \rightarrow C$ in $\mathcal{C}$, interchanged under duality, such that: (a) for every P , $(P, d_P^{\otimes}, e_P^{\otimes})$ forms a commutative comonoid in $\mathcal{P}$; and (b) for every C , $(C, d_C^{\wp}, e_C^{\wp})$ forms a commutative monoid in $\mathcal{C}$.
4. Symmetric monoidal functors $[-] : \mathcal{L} \Rightarrow \mathcal{P}$ and $F_! : \mathcal{P} \Rightarrow \mathcal{L}$ and symmetric comonoidal functors $[-] : \mathcal{L} \Rightarrow \mathcal{C}$ and $F_? : \mathcal{C} \Rightarrow \mathcal{L}$, which respect the dualities in that $(F_! P)^{\perp} \simeq F_? (P^*)$ and $[A] \simeq [A^{\perp}]$, and that form monoidal/comonoidal adjunctions $[-] \dashv F_!$ and $F_? \dashv [-]$.

LPC and other linear logic models. As LPC is inspired by Benton’s linear/non-linear paradigm, this section formalizes the relationship between LPC, LNL, and single-category models of linear logic.

Definition 9 (Melliès [12]). *A linear/non-linear (LNL) model consists of: (1) a symmetric monoidal closed category $\mathcal{L}$; (2) a cartesian category $\mathcal{P}$; and (3) functors $G : \mathcal{L} \Rightarrow \mathcal{P}$ and $F : \mathcal{P} \Rightarrow \mathcal{L}$ that form a symmetric monoidal adjunction $F \dashv G$.²*

In LPC, because every object in $\mathcal{P}$ forms a commutative comonoid, $\mathcal{P}$ is cartesian [8]. Thus:

Proposition 10. *Every LPC model is an LNL model.*

In addition, a $*$ -autonomous category in a linear/non-linear model induces an LPC triple:

Proposition 11. *If the category $\mathcal{L}$ in an LNL model is $*$ -autonomous, then $(\mathcal{L}, \mathcal{P}, \mathcal{P}^{op})$ is a model of LPC.*

Next we prove that every LPC model contains a classical linear category [16], an extension of Benton et al’s linear category [2] to classical linear logic.

Definition 12 (Schalk [16]). *A category $\mathcal{L}$ is a model for classical linear logic if and only if it: (1) is $*$ -autonomous; (2) has finite products & and thus finite coproducts $\oplus$; and (3) has a linear exponential comonad $!$ and thus a linear exponential monad $?$.*

Proposition 13. *The category $\mathcal{L}$ from the LPC model is a model for classical linear logic.*

Proposition 14. *Every model for classical linear logic forms an LPC category.*

The producer category can be derived from either the Kleisli or Eilenberg-Moore constructions, as demonstrated by Benton [3], with its opposite category for $\mathcal{C}$.

Interpretation of the Logic. We define an interpretation of the LPC logic that maps propositions to objects in the categories, and derivations to morphisms. For objects, the $\llbracket - \rrbracket_{\mathcal{L}}$ interpretation function is defined on all propositions, but $\llbracket - \rrbracket_{\mathcal{P}}$ and $\llbracket - \rrbracket_{\mathcal{C}}$ are defined only on persistent propositions. On the operators the interpretations are straightforward, and they extend to arbitrary persistent propositions as follows:

$$\llbracket P \rrbracket_{\mathcal{L}} = F! \llbracket P \rrbracket_{\mathcal{P}} \quad \llbracket C \rrbracket_{\mathcal{L}} = F? \llbracket C \rrbracket_{\mathcal{C}} \quad \llbracket C \rrbracket_{\mathcal{P}} = (\llbracket C \rrbracket_{\mathcal{C}})_* \quad \llbracket P \rrbracket_{\mathcal{C}} = (\llbracket P \rrbracket_{\mathcal{P}})^*$$

Contexts can be interpreted with the comma as either the tensor or cotensor in the linear category; we define $\llbracket \Gamma \rrbracket_{\mathcal{L}}^{\otimes}$ and $\llbracket \Delta \rrbracket_{\mathcal{L}}^{\wp}$ respectively. In the producer category there is no cotensor and vice versa for the consumer category, so $\llbracket \Gamma^P \rrbracket_{\mathcal{P}}$ interprets the comma as the tensor in the producer category, and $\llbracket \Gamma^C \rrbracket_{\mathcal{C}}$ interprets the comma as the cotensor in the consumer category.

A linear derivation $\mathcal{D}$ of the form $\Gamma \vdash \Delta$ is interpreted as a morphism $\llbracket \mathcal{D} \rrbracket_{\mathcal{L}} : \llbracket \Gamma \rrbracket_{\mathcal{L}}^{\otimes} \rightarrow \llbracket \Delta \rrbracket_{\mathcal{L}}^{\wp}$, but this will not suffice for persistent derivations $\Gamma \Vdash \Delta$ —when mapped into $\mathcal{P}$, the codomain cannot be interpreted as a $\wp$ -separated list. Proposition 2 states that every persistent derivation $\mathcal{D}$ contains exactly one displaced proposition. This means that $\mathcal{D}$ is either of the form $\Gamma^P \Vdash \Delta^C, P$ or $\Gamma^P, C \Vdash \Delta^C$. In the category $\mathcal{P}$, this derivation is interpreted as a morphism

$$\llbracket \mathcal{D} \rrbracket_{\mathcal{P}} : \llbracket \Gamma^P \rrbracket_{\mathcal{P}} \otimes \llbracket \Delta^C \rrbracket_{\mathcal{P}} \rightarrow \llbracket P \rrbracket_{\mathcal{P}} \quad \text{or} \quad \llbracket \mathcal{D} \rrbracket_{\mathcal{P}} : \llbracket \Gamma^P \rrbracket_{\mathcal{P}} \otimes \llbracket \Delta^C \rrbracket_{\mathcal{P}} \rightarrow \llbracket C \rrbracket_{\mathcal{P}},$$

respectively. Similarly for interpretation in $\mathcal{C}$. The full details can be found in [14].

²The LNL model given by Benton [3] has the added stipulation that the cartesian category be cartesian closed, but other works have since disregarded this condition.

4 Examples

This section provides some concrete instances of the LPC model. The following chart summarizes the three examples and their LPC categories.

	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{C}$
Vectors	FINVECT	FINSET	FINSET ^{op}
Relations	REL	SET	SET ^{op}
Bool. Alg.	FINBOOLALG	FINPOSET	FINLAT

Vector Spaces. When considering an LPC model of linear algebra, we take $\mathcal{L}$ to be the category of finite-dimensional vector spaces over a finite field, $\mathcal{P}$ to be finite sets and functions, and $\mathcal{C}$ to be FINSET^{op}. The $\otimes$, $(-)^{\perp}$ and $\&$ of linear logic are easily interpreted as tensor product, duality and direct product in $\mathcal{L}$. The functors $[-]$ and $[-]$ are the forgetful functor from vector spaces to sets, and $F_!$ and $F_?$ take a set X to the free vector space generated by X .

Other related interpretations are found in the literature. Ehrhard [7] presents finiteness spaces, where the objects are spaces of vectors with finite support. In his model, the $!$ operator sends a space A to the space supported by finite multisets over A ; it takes some effort to show that this comonad respects the finiteness conditions. Pratt [15] proves that finite dimensional vector spaces over a field of characteristic 2 is a Chu space and thus a model of linear logic.

Relations. The category REL of sets and relations, along with the categories SET and SET^{op}, forms an LPC model. The functor $F_!$ ($F_?$) is the forgetful functor from functions (inverse functions) to relations, and $[-]$ ($[-]$) takes a set to its powerset.

Melliés [12] discusses a non-model of linear logic based on REL, where the exponential takes a set X to the finite subsets of X . That “model” fails because the comonad unit $\epsilon_A : !A \rightarrow A$ is not natural. In the LPC formulation, ϵ is derived from the adjunction, ensuring naturality.

Boolean Algebras. Let $\mathcal{L}$ be the category of finite boolean algebras, $\mathcal{P}$ be FINPOSET of finite partially ordered sets, and $\mathcal{C}$ be FINLAT of finite lattices. This trio forms an LPC category where the duality is given by Birkhoff’s representation theorem [4]. The functors $[-]$ and $[-]$ are forgetful functors, and $F_!$ and $F_?$ take the base set X of their respective structures to the boolean lattice $\mathcal{P}(X)$. Unlike the examples where $\mathcal{C} = \mathcal{P}^{op}$, this example utilizes a non-trivial duality between the persistent categories. Although FINLAT is both cartesian and cocartesian, FINPOSET is only cartesian.

5 Related work

Girard [9] first introduced linear logic to mix the constructivity of intuitionistic propositional logic with the duality of classical logic. Partly because of this constructivity, there has been great interest in the semantics of linear logic in both the classical and intuitionistic fragments. Consequently, there exist several categorical frameworks for its semantic models.

One influential framework is Benton *et al.*’s *linear category* [2], consisting of a symmetric monoidal closed category with products and a linear exponential comonad $!$. Schalk [16] adapted linear categories to the classical case by requiring that the symmetric monoidal closed category be $*$ -autonomous. The coproduct $\wp$ and coexponential $?$ are then induced from the duality.

Cockett and Seely [5], seeking to study $\otimes$ and $\wp$ as independent structures unobscured by duality, introduced linearly distributive categories, which make up the linear category in the LPC model. The authors extended this motivation to the exponentials by modeling $!$ and $?$ as

linear functors [6], meaning that $?$ is *not* derived from $!$ and $(-)^{\perp}$. The LPC model reflects that work by allowing $!$ and $?$ to have different adjoint decompositions.

Other variations of classical linear logic, notably Girard’s Logic of Unity [10], distinguish linear propositions from persistent ones. In the intuitionistic case, Benton [3] developed the linear/non-linear logic and categorical model described in Section 3. Barber used this model as the semantics for a term calculus called DILL [1]. A Lafont category [11] is a canonical instance of an LNL model where $!A$ is the free commutative comonoid generated by A . This construction automatically admits an adjunction between a linear category $\mathcal{L}$ and the category of commutative comonoids over $\mathcal{L}$. However, the LNL and LPC models have an advantage over Lafont categories by allowing a much greater range of interpretations for the exponential. Lafont’s construction excludes traditional models of linear logic like coherence spaces and the category REL.

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