

On Herbrand theorems for classical and non-classical logics

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Abstract: The talk is devoted to Herbrand-type theorems for an wide enough spectrum of first-order logics for both the classical and intuitionistic cases as well as for their modal extensions. A way for the construction of Herbrand theorems for first-order logics containing the ineliminable cut rule is discussed.

1 Introduction

The purpose of this talk is to present a general way for the construction of Herbrand theorems for an wide enough spectrum of first-order logics having the form of sequent calculi [1]. The research is based on some of the author's results relating to the construction of computer-oriented first-order sequent calculi not requiring performing preliminary skolemization before the starting of deduction.

As in the case of the original Herbrand theorem [2], any Herbrand theorem under consideration presents itself a (reduction) theorem reducing the problem of the deducibility of an initial sequent S of the form $\rightarrow F$ (where F is a closed formula) in a first-order sequent calculus to the problem of the deducibility of a certain quantifier-free sequent (being constructed on the basis of S) in its quantifier-free-rule part.

Herbrand theorems are divided into two classes: one contains the theorems for first-order classical (modal) logics and the other contains theorems for their intuitionistic modifications. Initially wording for logics without equality, all such results then are extended to logics with equality.

2 Preliminaries

The *signature* of any (sequent) logic contains: a finite (possibly, empty) set of function symbols; a finite non-empty set of predicate symbols containing, perhaps, the equality; all the logical connectives including the universal and existential quantifiers; and modal operators. A specificity of our formalism is that it uses *multisets* of formulas instead of sequences of formulas. This leads to that all the axioms of any logic have the form $\Gamma, A \rightarrow A, \Delta$, where Γ and Δ are multisets of formulas and A is an atomic formula including $t = t$ for any term t not containing so-called dummies [3] (i.e. variables admitting their replacement by any terms) in the case of equality.

Besides of the axioms, any sequent calculus contains the *usual inference rules* for all the logical connectives and, perhaps, the *cut rule*, which may be ineliminable in it or its expansion. Any logic *with equality* contains only the Kanger-type equality rules [3]. All the *modal logics* being considered in the paper are constructed from classical or intuitionistic ones (with or without equality) by adding specific inference rules for modal connectives. Any inference is supposed to have the form of an *inference tree* called a

proof tree, if all its leaves are axioms.

For obtaining Herbrand theorems in the form of reduction, the usual notion of an *Herbrand universe* H_F for F (consisting of terms) is modified in a certain way and the notion of an *Herbrand expansion* of F being the result of (multiple) “doubling” of certain subformulas of F is introduced. The result of a Herbrand reduction for $\rightarrow F$ can be presented in the form $\rightarrow M \cdot \sigma$, where M is the result of the removing of all the quantifiers from an expansion F' of F , σ is a substitution of terms from $H_{F'}$ for all the dummies from M , and $M \cdot \sigma$ is the result of the applying of σ to M . Namely, the existence of a proof tree (satisfying certain restrictions) for $\rightarrow M \cdot \sigma$ in the propositional part of a calculus under consideration guarantees the deducibility of $\rightarrow F$ in the calculus.

It is well known that in classical logic the deducibility (validity) of any formula is invariant w.r.t. the skolemization operation. In this connection, most of Herbrand-type theorems for classical logic are formulated in the form of reduction theorems using skolemization. But already in the case of (usual) intuitionistic logic, deducibility (validity) is not invariant w.r.t. skolemization in general, which makes us look for ways to get Herbrand theorems for the intuitionistic logic and different modal extensions without performing preliminary skolemization. Attempts to move in this direction were made in some of the author's papers relating to inference search in non-classical (and classical) logics without equality, in which the original notions of the *admissibility* of a substitution (of terms for variables) for a formula (sequent) and the *compatibility* of an inference tree with a substitution were used. Namely, both these notions and analogues of a usual Herbrand universe and Herbrand expansion lead to the opportunity to obtain Herbrand theorems for all the logics under consideration.

To obtain Herbrand theorems for the logics, in which the cut rule cannot be eliminated, it is suggested to introduce the notion of a *tautological expansion* of an initial formula F , according to which formulas of the form $G \supset G$, where the structure of G is connected with F , are allowed to use for the construction of a Herbrand expansion of F .

3 Herbrand theorems for logics without equality

For modal intuitionistic logics without equality rule, we have the following results.

Theorem 1. Let $\mathbf{SC}$ be a first-order intuitionistic modal sequent logic with maybe the cut. For a closed formula F , the sequent $\rightarrow F$ is deducible in $\mathbf{SC}$ if and only if there are a sequence of tautological and Herbrand expansions of F leading to the construction of a formula F' and a substitution σ of terms from a Herbrand universe $H_{F'}$ for all the dummies of M being the result of removing all the quantifiers from F' such that

- (i) there exists a proof tree Tr for $M \cdot \sigma$ in the propositional part of $\mathbf{SC}$,
- (ii) σ is an admissible substitution for F' , and
- (iii) the tree Tr is compatible with the substitution σ .

For other classes of logics without equality, we have:

- (1) In the case of intuitionistic (modal) logics containing the eliminable cut, reminder about tautological expansion can be removed from Theorem 1.
- (2) In the case of classical modal logics containing the ineliminable cut, the item (iii) can be removed Theorem 1.
- (3) In the case of classical modal logics (or simply classical logic) containing the eliminable cut, both the reminder about tautological expansion and the item (iii) can be removed from Theorem 1.
- (4) In the case of logics containing the ineliminable cut and admitting skolemization, the items (ii) and (iii) can be removed from Theorem 1, if skolemization is applied.
- (5) In the case of logics containing the eliminable cut and admitting skolemization, both the reminder about tautological expansion and the items (ii) and (iii) can be removed from Theorem 1, if skolemization is applied.

Applying Theorem 1 to Gentzen's calculi $\mathbf{LK}$ and $\mathbf{LJ}$ without equality [1], we obtain the Herbrand theorems stating that the deducibility of sequents in $\mathbf{LK}$ and $\mathbf{LJ}$ is equivalent to the deducibility of certain quantifier-free sequents in their propositional parts.

4 Herbrand theorems for logics with equality

In the case of the logics with equality, the Herbrand universe H_F for a formula F is partitioned into disjoint classes of terms in accordance with the set of all the so-called negative “equality atoms” of F and a substitution σ selected for replacement of all the dummies of F by terms from H_F . The result of the applying of this partition operation is denoted by $\pi(H_F, \sigma)$. If M is the result of removing all the quantifiers from F then $[M \cdot \sigma]/\pi(H_F, \sigma)$ is defined as the result of the replacement of every term that occupies an argument place in $M \cdot \sigma$ by the class containing this term and being considered as a constant in the formula $[M \cdot \sigma]/\pi(H_F, \sigma)$.

The specificity of any logic with equality under consideration is that the deducibility of an initial sequent $\rightarrow F$ in a usual sequent logic with the equality (for example, in Gentzen's logic $\mathbf{LK}$ or $\mathbf{LJ}$ with equality) is equivalent to the deducibility of the sequent $\rightarrow \forall x(x = x) \supset F$ in an appropriate calculus $\mathbf{SC}^=$ under consideration containing only Kanger's rules for equality handling.

Theorem 2. Let $\mathbf{SC}^=$ be a first-order intuitionistic modal sequent logic with equality and maybe the cut. For a closed formula F , the sequent $\rightarrow \forall x(x = x) \supset F$ is deducible in $\mathbf{SC}^=$ if and only if there exist a sequence of tautological and Herbrand expansions of $\rightarrow \forall x(x = x) \supset F$ leading to the construction of a formula F' and a substitution σ of terms from a Herbrand universe $H_{F'}$ for all the dummies of M being the result of removing all the quantifiers from F' such that

- (i) there exists a proof tree Tr for $[M \cdot \sigma]/\pi(H_F, \sigma)$ in the propositional fragment of $\mathbf{SC}^=$,
- (ii) σ is an admissible substitution for F' , and
- (iii) tree Tr is compatible with the substitution σ .

For other classes of logics with equality, we have:

- (1) In the case of intuitionistic (modal) logics containing the eliminable cut, reminder about tautological expansion can be removed from Theorem 2.
- (2) In the case of classical modal logics containing the ineliminable cut, the item (iii) can be removed Theorem 2.
- (3) In the case of classical modal logics (or simply classical logic) containing the eliminable cut, both the reminder about tautological expansion and the item (iii) can be removed from Theorem 2.
- (4) In the case of logics containing the ineliminable cut and admitting skolemization, the items (ii) and (iii) can be removed from Theorem 2, if skolemization is applied.
- (5) In the case of logics containing the eliminable cut and admitting skolemization, both the reminder about tautological expansion and the items (ii) and (iii) can be removed from Theorem 2, if skolemization is applied.

Applying Theorem 2 to Gentzen's calculi $\mathbf{LK}$ and $\mathbf{LJ}$ with equality [1], we obtain the Herbrand theorems stating that the deducibility of sequents in $\mathbf{LK}$ and $\mathbf{LJ}$ with equality is equivalent to the deducibility of certain quantifier-free sequents in their propositional parts not requiring equality rule applications.

When proving the results, all the reasoning is made on deducibility. If for a certain sequent calculus under consideration, there is a theorem on its soundness and completeness, we automatically obtain a corresponding Herbrand theorem on validity. The calculi $\mathbf{LK}$ and $\mathbf{LJ}$ (with and without equality) can serve as such examples.

References

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