

# Tableau Development for a Bi-Intuitionistic Tense Logic\*

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**Abstract:** Motivated by the theory of relations on graphs and applications to spatial reasoning, we present a bi-intuitionistic logic BISK<sub>T</sub> with tense operators. The logic is shown to be decidable and have the effective finite model property. We present a sound, complete and terminating tableau calculus for the logic and use the MetTeL system to obtain an implementation.

## 1 Introduction

In image processing the operations of mathematical morphology are used to derive new images from given ones. These new images approximate the input images in useful ways. The operations transform sets of pixels to sets of pixels in a way which depends on a parameter which is effectively a relation on the set of all pixels. Motivated by the aim of developing similar approximation operations for subgraphs of a graph, we have developed a theory of relations on graphs [5]. These are ordinary relations on the set of all edges and nodes of the graph which satisfy a stability condition. A generalisation of these relations is used here to provide accessibility relations for a modal logic. Semantically, formulae can represent subgraphs of a graph, or more generally, downward closed sets in a pre-order. A particular novel feature of the semantics is the use of a weaker form of the converse operation on relations. This has resulted in a bi-intuitionistic tense logic, called BISK<sub>T</sub>, in which the four modalities are not mutually independent. In exploring the properties of this logic we have made essential use of tableau systems generated using the MetTeL software [1, 8].

## 2 Bi-intuitionistic stable tense logic

BISK<sub>T</sub> is a modal bi-intuitionistic logic with four modalities  $\Box$ ,  $\blacklozenge$ ,  $\lozenge$  and  $\blacksquare$ . The remaining connectives are  $\perp$ ,  $\neg$  (intuitionistic negation),  $\neg$  (dual negation),  $\wedge$ ,  $\vee$ ,  $\rightarrow$  (intuitionistic implication) and  $\succ$  (dual implication).

Kripke frames for BISK<sub>T</sub> consist of a pre-order  $H$  interacting with an accessibility relation  $R$  via a stability condition. A binary relation  $R$  on a set  $U$  is *stable*, if  $H ; R ; H \subseteq R$ , where  $;$  denotes relational composition. The semantics interprets formulae as  $H$ -sets, i.e.,

the downwardly closed sets of the pre-order, and the bi-intuitionistic connectives are interpreted as usual. This means, for example,  $\neg$  is interpreted as pseudo-complement with respect to  $H$ ,  $\rightarrow$  as relative pseudo-complement (i.e., intuitionistic implication) with respect to  $H$ , etc.  $\Box$  and  $\blacklozenge$  are interpreted respectively as the standard box modality and the standard backward looking diamond over  $R$ . The pair  $\blacksquare$  and  $\lozenge$  are interpreted similarly as the standard box modality and the standard backward looking diamond, but this time over the left converse of  $R$ . The *left converse* of a stable relation  $R$  is  $\smile R = H ; \check{R} ; H$ , where  $\check{R}$  is the (ordinary) converse of  $R$ .

Unlike in (bi-)intuitionistic logics and other (bi-)intuitionistic modal logics, where all connectives are independent from each other, in BISK<sub>T</sub> the white  $\Box$  and white  $\lozenge$  are related as follows:

$$\lozenge\varphi \equiv \neg\Box\neg\varphi.$$

We have shown:

**Theorem 1** *BISK<sub>T</sub> is decidable, it has the effective finite model property and the computational complexity is PSPACE-complete.*

The proof is by showing that BISK<sub>T</sub> can be embedded into a modal logic, called  $Kt(H, R)$ , which itself can be embedded into the guarded fragment, which is known to be decidable and has the effective finite model property [2].  $Kt(H, R)$  is a traditional modal logic with forward and backward looking modal operators defined by  $H$  and  $R$  as accessibility relations. The frame conditions are reflexivity and transitivity of  $H$ , and stability of  $R$  with respect to  $H$ . As both embeddings have linear complexity and define in fact an effective translation into a subfragment of the guarded fragment, which is PSPACE-complete, the result follows.

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### 3 Tableau development

Since the accessibility relations in the Kripke models of BISKI involve converse relations it is natural to use a semantic tableau method. In particular, we use a labelled signed tableau approach based on an explicit tableau system because this ensures proof confluence. This means there is no need for backtracking over the proof search, and there is more flexibility in defining search heuristics in an implementation. These are aspects that make it harder to develop implicit tableau systems where the rules operate on formulae of the logic and do not include any meta-logical entities referring to semantics. An additional advantage of semantic tableau calculi is that they return concrete models for satisfiable formulae.

Semantic tableau deduction calculi are easy to develop. We follow the methodology of tableau synthesis and refinement as introduced in [3, 7] to develop a tableau calculus for BISKI. Tableau synthesis amounts to a normalisation process of the definition of the semantics of the logic. In the case of BISKI atomic rule refinement was sufficient to obtain a set of tableau rules with reduced branching. Soundness and completeness of the obtained calculus follows from the results of the general framework in [3, 7]. To guarantee termination we use the unrestricted blocking technique [3, 4]. Compared to other blocking techniques it imposes no restrictions and is generic, so independent of any logic and can even be used for undecidable logics. Unrestricted blocking has the property that adding it to a sound and complete semantic tableau calculus guarantees termination, for any logic that has the finite model property. Since we proved that BISKI has the finite model property the tableau calculus is thus terminating.

Implementing a prover requires lots of specialist knowledge and there are various non-trivial obstacles to overcome, but using the MetTeL tableau prover generator requires just feeding in the rules of the calculus into the tool which then fully automatically generates an implementation in Java [1, 8]. Our tableau calculus is in exactly the form as accepted by MetTeL and unrestricted blocking is supported by MetTeL. We have therefore implemented the calculus using MetTeL. MetTeL turned out to be useful to experiment with several initial versions of the calculus. In combination with the tableau synthesis method it was easy to run tests on a growing collection of problems with different provers for several preliminary versions of formalisations of bi-intuitionistic tense logics before

settling on a definition. MetTeL has also allowed us to experiment with different refinements of the rules and different variations of blocking.

### 4 Concluding remarks

Further details and a detailed description of the tableau calculus are presented in the long version [6] of this abstract. The MetTeL specification of the tableau calculus and the generated prover for BISKI can be downloaded from <http://staff.cs.manchester.ac.uk/~schmidt/publications/biskit13/>, where also performance graphs and the problems used can be found.

### References

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