

Socratic Proofs for Propositional Linear-Time Logic

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Abstract:

This paper presents a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) and discusses potential automation of its proof search.

1 Introduction

Propositional Linear-Time Logic (PLTL) [6] gained various deductive constructions: axiomatic [5], tableau [11], resolution [4], and natural deduction [1]. In this paper we present a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) [6], [3], [7] abbreviated as PLTL_{SP}. The calculus is based upon the hypersequent calculus [2] and it fits into the framework of Socratic proofs by Wisniewski (cf. [8], [10] and [9]).

2 Logic PLTL_T

We utilise the language of PLTL which extends the language of Classical Propositional Calculus (CPC) by temporal operators: $\mathcal{U}$ (*until*) $\bigcirc$ (*at the next moment in time*), $\Box$ (*always in the future*), and $\Diamond$ (*at sometime in the future or eventually*). The semantics for the temporal part of the logic PLTL_T is defined in the standard way over linear sequence of states, finite in the past, infinite in the future.

In order to formulate PLTL_T we need to extend the language of PLTL with the following signs: $\vdash$, $?$, 1 and 2. Intuitively, $\vdash$ stands for derivability relation and $?$ is a question-forming operator. The numerals 1 and 2 will be used to encode tree-structure of a Socratic transformation.

There are two disjoint categories of wffs: *declarative* wffs (d-wffs) and *erotetic* wffs (e-wffs), or questions. There are also two types of d-wffs: *atomic* d-wffs and *indexed* d-wffs. Atomic d-wffs are expressions of the form $S \vdash A$, where S is a finite sequence (possibly with repetitions) of PLTL-wffs, and A is a PLTL-wff, and if A is an empty formula, then S is a non-empty sequence. Indexed d-wffs are expressions of the form $S \vdash^n A$ or of the form $T \vdash^n$, where $S \vdash A$ and $T \vdash$ are atomic d-wffs of and n is a sequence of 1's or 2's, starting with 1. E-wffs, or questions are expressions of the form $?(\Phi)$, where Φ is a non-empty finite sequence of indexed atomic d-wffs (*constituents* of Φ).

In the formulation of rules we shall use the following classification of PLTL formulae to α and β types:

α	α_1	α_2
$A \wedge B$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \rightarrow B)$	A	$\neg B$
$\Box A$	A	$\bigcirc \Box A$
$\neg \Diamond A$	$\neg A$	$\bigcirc \Box \neg A$
$\neg(A \cup B)$	$\neg B$	$\neg(A \wedge \bigcirc(A \cup B))$

β	β_1	β_2	β_1^*
$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$A \vee B$	A	B	$\neg A$
$A \rightarrow B$	$\neg A$	B	A
$\neg \Box A$	$\neg A$	$\bigcirc \Diamond \neg A$	A
$\Diamond A$	A	$\bigcirc \Diamond A$	$\neg A$
$A \cup B$	B	$A \wedge \bigcirc(A \cup B)$	$\neg B$

Rules for PLTL_T:

$$\begin{aligned}
 L_\alpha : & \frac{?(\Phi; S' \alpha' T \vdash^n C; \Psi)}{?(\Phi; S' \alpha_1' \alpha_2' T \vdash^n C; \Psi)} \\
 L_\beta : & \frac{?(\Phi; S' \beta_1' T \vdash^{n1} C; S' \beta_2' T \vdash^{n2} C; \Psi)}{?(\Phi; S' \neg A' T \vdash^n C; \Psi)} \\
 L_{\neg\neg} : & \frac{?(\Phi; S' A' T \vdash^n C; \Psi)}{?(\Phi; S' \neg \bigcirc A' T \vdash^n C; \Psi)} \\
 L_{\neg\bigcirc} : & \frac{?(\Phi; S' \bigcirc \neg A' T \vdash^n C; \Psi)}{?(\Phi; S' \bigcirc \neg A' T \vdash^n C; \Psi)} \\
 R_\alpha : & \frac{?(\Phi; S \vdash^n \alpha; \Psi)}{?(\Phi; S \vdash^{n1} \alpha_1; S \vdash^{n2} \alpha_2; \Psi)} \\
 R_\beta : & \frac{?(\Phi; S' \beta_1^* \vdash^n \beta_2; \Psi)}{?(\Phi; S \vdash^n \neg \neg A; \Psi)} \\
 R_{\neg\neg} : & \frac{?(\Phi; S \vdash^n A; \Psi)}{?(\Phi; S \vdash^n \neg \bigcirc A; \Psi)} \\
 R_{\neg\bigcirc} : & \frac{?(\Phi; S \vdash^n \bigcirc \neg A; \Psi)}{?(\Phi; S \vdash^n \bigcirc \neg A; \Psi)}
 \end{aligned}$$

If none of the above rules is applicable to a PLTL formula B , such a formula is called *marked*. If all PLTL-formulas within an indexed formula $S \vdash^n A$ are marked, such a formula is called a *state*.

The following is a state-prestate rule:

$$S - P : \frac{?(\Phi; S \vdash^n A; \Psi)}{?(\Phi; S^\circ \vdash^n A^\circ; \Psi)}$$

where $S \vdash^n A$ is a state and S° (resp. A°) results from S (resp. A) by replacing all the formulas of the form $\bigcirc B$ with B and deleting all the remaining formulas. Every formula

of the form $S^* \vdash^m A^*$, where n is an initial subsequence of m or m is an initial subsequence of n , is called a *pre-state* (cf. [11]).

Definition 1. Let $\mathbf{q} = \langle Q_1, \dots, Q_r \rangle$ be a finite sequence of questions of $\mathbf{P}^*$. Let Q_g, Q_{h-1}, Q_h ($1 \leq g < h-1 \leq r$) be elements of the sequence $\mathbf{q}$. Let $S_j \vdash^n A_j$ be a constituent of Q_g and let $S_k \vdash^m A_k$ be a constituent of Q_h such that $S_j = S_k$, $A_j = A_k$ and the sequence n is an initial subsequence of the sequence m . Let $S_l \vdash^i A_l$ be a constituent of Q_{h-1} such that $S_k \vdash^m A_k$ is obtained from $S_l \vdash^i A_l$ by application of a PT^* -rule. Then $S_j \vdash^n A_j, \dots, S_l \vdash^i A_l$ form a loop (a sequence of atomic d-wffs of $\mathbf{P}^*$... etc.), and $S_k \vdash^m A_k$ is called a loop-generating formula.

Socratic transformations are sequences of questions that aim at deciding derivability of formulae from sets of formulae.

Definition 2. A finite sequence $\langle Q_1, \dots, Q_r \rangle$ of questions of $\mathbf{P}^*$ is a Socratic transformation of $S \vdash A$ iff the following conditions hold: (i) $Q_1 = ?(S \vdash^1 A)$; (ii) Q_i (where $i = 2, \dots, r$) results from Q_{i-1} by applying a PT^* -rule.

Definition 3. A constituent ϕ of a question Q_i is called successful iff one of the following holds: (a) ϕ is of the form $T'B'U \vdash^n B$, or (b) ϕ is of the form $T'B'U' \vdash^n B'W \vdash^n C$, or (c) ϕ is of the form $T' \vdash^n B'U'B'W \vdash^n C$.

Definition 4. A Socratic transformation $\langle Q_1, \dots, Q_r \rangle$ of $S \vdash A$ is completed iff for each constituent ϕ of Q_r at least one of the following conditions hold: (a) no rule is applicable to PLTL-formulas in ϕ , or (b) ϕ is successful, or (c) ϕ is a loop-generating formula.

Definition 5. A formula B is called an eventuality in $S \vdash^n A$ iff one of the following holds: (i) B is a term of S and there exists a PLTL-formula C such that $B = \Diamond C$, or (ii) there exists a PLTL-formula C such that $B = A = \Box C$.

Definition 6. A completed Socratic transformation $\mathbf{q} = \langle Q_1, \dots, Q_r \rangle$ is a Socratic proof of $S \vdash A$ iff: (a) all the constituents of Q_n are successful, or (b) for each non-successful constituent ϕ of Q_n , ϕ is a loop-generating formula and the loop generated by ϕ contains a pre-state with an unfulfilled eventuality.

The presented system is sound and complete. Proofs of these theorems involve construction of a canonical model with maximal consistent sets of formulae as its states.

3 Examples

In the examples below by highlighting we indicate a formula which is analyzed at the current step. Double underlining of a formula reflects that it is a state. The question following the one containing a state is obtained by state-prestate rule.

Example 1

1. $?(\vdash^1 \underline{\Box p \rightarrow p})$
2. $?(\underline{\Box p} \vdash^1 p)$
3. $?(p, \Box \Box p \vdash^1 p)$

Example 2

1. $?(\vdash^1 \underline{\Box p \rightarrow \Box p})$
2. $?(\underline{\Box p} \vdash^1 \Box p)$
3. $?(p, \Box \Box p \vdash^1 \Box p)$
4. $?(\underline{\Box p} \vdash^1 p)$
5. $?(p, \Box \Box p \vdash^1 p)$

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