

Characteristics of Multiple Viewpoints in Abstract Argumentation*

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Abstract

The study of extension-based semantics within the seminal abstract argumentation model of Dung has largely focused on definitional, algorithmic and complexity issues. In contrast, matters relating to comparisons of representational limits, in particular, the extent to which given collections of extensions are expressible within the formalism, have been under-developed. As such, little is known concerning conditions under which a candidate set of subsets of arguments are “realistic” in the sense that they correspond to the extensions of some argumentation framework AF for a semantics of interest. In this paper we present a formal basis for examining extension-based semantics in terms of the sets of extensions that these may express within a single AF. We provide a number of characterization theorems which guarantee the existence of AFs whose set of extensions satisfy specific conditions and derive preliminary complexity results for decision problems that require such characterizations.

1 Introduction

The last 15 years have seen an enormous effort to design, compare, and implement different semantics for Dung’s abstract argumentation frameworks (Dung 1995), AFs for short. As a result of this extensive study argumentation is now a significant topic within AI research (Bench-Capon and Dunne 2007; Rahwan and Simari 2009). Surprisingly, systematic comparison of AF capabilities respecting *multiple* extensions, and thus their power in modelling multiple viewpoints within a single framework has, so far, been neglected. Nevertheless, an understanding of which extensions can, in principle, coexist when a framework is evaluated with respect to a semantics of interest not only clarifies the “strength” of that semantics but also is a crucial issue in several applications.

In this work, we address this issue by studying the *signatures* of argumentation semantics. A semantics σ maps each argumentation framework F to a set of extensions (i.e. a set of sets of arguments). The signature of σ is defined by

$$\Sigma_\sigma = \{\sigma(F) \mid F \text{ is an AF}\},$$

and gives the collection of all possible sets of extensions an AF can possess under semantics σ . We shall focus on several important semantics namely naive, preferred, semi-stable, stage, stable, and complete semantics (Dung 1995; Verheij 1996; Caminada, Carnielli, and Dunne 2012) and aim at finding simple criteria to decide whether a set $\mathbb{S}$ is contained in Σ_σ . For instance, we will show that each $\mathbb{S} \in \Sigma_{pref}$ satisfies the condition that for each pair of distinct sets A and B from $\mathbb{S}$ there is at least one $a \in A$ and $b \in B$ such that a, b do not occur together in any set in $\mathbb{S}$. Thus, for instance, $\mathbb{S} = \{\{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$ is part of the signature Σ_{pref} while neither $\mathbb{S} \cup \{\{a, d\}\}$ nor $\mathbb{S} \cup \{\{b, c\}\}$ are. This fact can be exploited in a search procedure for enumerating preferred extensions: assume for a given AF F , the three extensions from $\mathbb{S}$ have already been calculated as preferred extensions of F . The procedure can now restrict the search space to find further extensions of F (if they exist) to sets with at least one argument different from a, b, c , and d .

The problem we study here is also essential in many other aspects.

First, our results are important for constructing AFs. Indeed, knowing whether a set $\mathbb{S}$ is contained in Σ_σ is a necessary condition which should be checked before actually looking for an AF F which *realizes* $\mathbb{S}$ under σ , i.e. $\sigma(F) = \mathbb{S}$. This is of high importance when dynamic aspects of argumentation are considered (Falappa et al. 2011). As an example, suppose a framework F possesses as its σ -extensions a set $\mathbb{S}$ and one asks for an adaptation of the framework F such that its σ -extensions are given by $\mathbb{S} \cup \{E\}$, i.e. one extension is to be added. The addition of E to $\mathbb{S}$ may, for instance, be desired by some agent on the grounds that E contains some subset of arguments which it wishes to be collectively accepted by other agents: no extension in $\mathbb{S}$, however, provides support for the subset of interest to be considered justifiable. Furthermore the agent wishing to add E is reluctant to jeopardize the chance of this happening if the modified AF is such that some existing element of $\mathbb{S}$ ceases to be an extension: such an outcome being likely to prejudice other agents against agreeing to changes which admit E . Before considering the adapted framework’s structure, it is obviously crucial to know whether an appropriate framework exists at all, i.e. whether $\mathbb{S} \cup \{E\} \in \Sigma_\sigma$. In a recent paper on revision of AFs (Coste-Marquis et al. 2013), the authors circumvent this issue by allowing revision to result in a set of AFs such that

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the union of their extensions yields the desired outcome. Our results provide exact conditions under which their approach admits a single AF as an outcome of a given revision.

Second, our work adds to the comparison of semantics (see, e.g., (Baroni and Giacomin 2007)) by means of different properties. So far such properties have largely focused on aspects of *single* extensions $S \in \mathbb{S}$ rather than on *sets* of such. An obvious exception being incomparability (the sets in $\mathbb{S}$ are not proper subsets of each other; this property is also known as I-maximality); as we will see, however, all of the standard semantics impose additional (yet distinct) requirements on $\mathbb{S}$ in order for containment in the signature to hold. Furthermore, our results add to the growing body of work considering *generic* treatments of argumentation semantics, that is with respect to shared properties rather than from the perspective of distinguishing features. Previous such examinations of abstract argumentation semantics include the “principle-based evaluation” methods proposed by Baroni and Giacomin (2007) and the concept of “rationality postulates” from (Caminada and Amgoud 2007). For example, we show that most semantics σ are closed under intersection of extensions (more formally, for all AFs F_1, F_2 , there exists an AF F such that $\sigma(F) = \sigma(F_1) \cap \sigma(F_2)$, whenever $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$).

Related work includes studies on enforcing (Baumann 2012; Baumann and Brewka 2010) certain outcomes, where the task is to modify AFs in such a way that desired arguments become acceptable. However, the issue of multiple extensions is not covered. In fact, the work which is closest to our investigations are studies of intertranslatability issues (Dvořák and Spanring 2012; Dvořák and Woltran 2011), where signatures of semantics are put in relation to each other. More precisely, if there is an exact translation such that θ -extensions of the transformed AF coincide with the σ -extensions of the original AF, then θ is at least as expressive as σ , that is $\Sigma_\sigma \subseteq \Sigma_\theta$ in our terms. These results, however, tell us little about the *contents* of Σ_σ and Σ_θ . To summarize, the main contributions of our work are:

- We first identify necessary conditions any set of extensions under a given semantics σ satisfies, i.e. we identify sets of extensions Σ_σ^+ with $\Sigma_\sigma \subseteq \Sigma_\sigma^+$. This not only informs the exact characteristics for the signature of σ , but also determines those sets of extensions that are impossible to be jointly expressed with one AF.
- Then, we provide sufficient conditions for a set of extensions to be realizable under a given semantics σ . For any such set $\mathbb{S}$, we present constructions of canonical frameworks having $\mathbb{S}$ as their σ -extensions. In other words we identify sets of extensions Σ_σ^- with $\Sigma_\sigma^- \subseteq \Sigma_\sigma$. For all semantics σ under consideration (with the exception of complete extensions) our results hold with $\Sigma_\sigma^+ = \Sigma_\sigma^-$. We, thus, obtain exact characterizations of the signatures Σ_σ .
- We apply our results to study the aforementioned property of closure under intersection (of extension-sets). Moreover, we identify limits of disagreement the different semantics face. While the capabilities of semantics differ in cases involving expressing more than two extensions, it turns out that the maximal number of extensions (over n arguments) that can be captured is equal for many semantics.

- One particular application of our results is the problem of recasting, i.e. to decide if the σ -extensions of a given AF can be expressed via another semantics θ . We give some preliminary complexity results of the recasting problem, which go up to Π_2^P -completeness. In contrast, we show that the problem of checking if $\mathbb{S} \in \Sigma_\sigma$ for a given $\mathbb{S}$ is decidable in polynomial time for all exactly characterized semantics.

2 Preliminaries

We assume a countably infinite domain $\mathcal{A}$ of arguments. An *argumentation framework* (AF) is a pair $F = (A, R)$ where $A \subseteq \mathcal{A}$ is non-empty and finite, and $R \subseteq A \times A$ is the attack relation. The collection of all AFs is given as $AF_{\mathcal{A}}$. We write $a \mapsto_R b$ for $(a, b) \in R$ and $S \mapsto_R a$ (resp. $a \mapsto_R S$) if $\exists s \in S$ such that $s \mapsto_R a$ (resp. $a \mapsto_R s$). We drop subscript R in $\mapsto_R$ if there is no ambiguity. For $S \subseteq A$, the *range* of S (wrt. R), denoted S_R^+ , is the set $S \cup \{b \mid S \mapsto_R b\}$.

Given $F = (A, R)$, an argument $a \in A$ is *defended* (in F) by a set $S \subseteq A$ if for each $b \in A$, such that $b \mapsto_R a$, also $S \mapsto_R b$. A set T of arguments is defended (in F) by S if each $a \in T$ is defended by S (in F). The following result is in the spirit of Dung’s fundamental lemma and is used later.

Lemma 1. *Given an AF $F = (A, R)$ and two sets of arguments $S, T \subseteq A$. If S defends itself in F and T defends itself in F , then $S \cup T$ defends itself in F .*

Given an AF $F = (A, R)$, a set $S \subseteq A$ is *conflict-free* (in F), if there are no arguments $a, b \in S$, such that $(a, b) \in R$. We denote the set of all conflict-free sets in F as $cf(F)$. $S \in cf(F)$ is called *admissible* (in F) if S defends itself. We denote the set of admissible sets in F as $adm(F)$.

The semantics we focus in this work are the naive, stable, preferred, complete, stage, and semi-stable extensions. Given $F = (A, R)$ they are defined as subsets of $cf(F)$:

- $S \in naive(F)$, if there is no $T \in cf(F)$ with $T \supset S$
- $S \in stb(F)$, if $S \mapsto a$ for all $a \in A \setminus S$
- $S \in pref(F)$, if $S \in adm(F)$ and $\nexists T \in adm(F)$ s.t. $T \supset S$
- $S \in com(F)$, if $S \in adm(F)$ and $a \in S$ for all $a \in A$ defended by S
- $S \in stage(F)$, if $\nexists T \in cf(F)$ with $T_R^+ \supset S_R^+$
- $S \in sem(F)$, if $S \in adm(F)$ and $\nexists T \in adm(F)$ s.t. $T_R^+ \supset S_R^+$

The objects of our interest are the *signatures* of a semantics.

Definition 1. The signature Σ_σ of a semantics σ is defined as

$$\Sigma_\sigma = \{\sigma(F) \mid F \in AF_{\mathcal{A}}\}.$$

For characterizing the signatures of the semantics of our interest we will make frequent use of the following concepts.

Definition 2. Given $\mathbb{S} \subseteq 2^{\mathcal{A}}$, $Arg_{\mathbb{S}}$ denotes $\bigcup_{S \in \mathbb{S}} S$ and $Pairs_{\mathbb{S}}$ denotes $\{(a, b) \mid \exists S \in \mathbb{S} : \{a, b\} \subseteq S\}$. $\mathbb{S}$ is called an *extension-set* (over $\mathcal{A}$) if $Arg_{\mathbb{S}}$ is finite. While $|\mathbb{S}|$ denotes the number of extensions in $\mathbb{S}$, $\|\mathbb{S}\|$ stands for $|Arg_{\mathbb{S}}|$.

Observe that for any $a \in Arg_{\mathbb{S}}$, $(a, a) \in Pairs_{\mathbb{S}}$ holds for all extension-sets $\mathbb{S}$; also note that for all considered semantics σ each element $\mathbb{S} \in \Sigma_\sigma$ is an extension-set.

3 Properties of Argumentation Semantics

In this section, we provide necessary conditions for an extension-set $\mathbb{S}$ to be in the signature Σ_σ . To this end, we have to find common properties for $\sigma(F)$ which hold for any AF F . We do so by abstracting away from the syntactical structure of a given framework.

Some properties are rather easy to see: $cf(F)$ satisfies the property that for any $S \in cf(F)$, also $S' \in cf(F)$ for any $S' \subseteq S$. Likewise, for all $\sigma \in \{naive, stb, stage, pref, sem\}$ it obviously holds that $\sigma(F)$ is incomparable for any AF. We define these two properties formally.

Definition 3. Let $\mathbb{S} \subseteq 2^{\mathcal{A}}$. The *downward-closure*, $dcl(\mathbb{S})$, of $\mathbb{S}$ is given by $\{S' \subseteq S \mid S \in \mathbb{S}\}$. We call $\mathbb{S}$ *downward-closed* if $\mathbb{S} = dcl(\mathbb{S})$ and *incomparable* if for each $S, S' \in \mathbb{S}$, $S \subseteq S'$ implies $S = S'$.

However, all considered semantics σ enjoy additional properties. Just consider the incomparable extension-set $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Now suppose there exists an AF F with $\sigma(F) = \mathbb{S}$. Then F must not contain attacks between a and b , a and c , and respectively b and c . But then $\sigma(F)$ typically contains $\{a, b, c\}$.

Stable, Stage, Naive and Conflict-Free Sets

There are several ways to define the required property which excludes sets like $\mathbb{S}$ from above. To characterize stable, stage and naive extensions, we start with a rather strong restriction making use of the concept of $Pairs_{\mathbb{S}}$ (in Section 6, we will discuss an alternative yet equivalent definition in terms of a ternary operator on $\mathbb{S}$). For the other semantics, we will use a strictly weaker criterion.

Definition 4. An extension-set $\mathbb{S} \subseteq 2^{\mathcal{A}}$ is *tight* if for all $S \in \mathbb{S}$ and $a \in Arg_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}$ then there exists an $s \in S$ such that $(a, s) \notin Pairs_{\mathbb{S}}$.

For incomparable $\mathbb{S}$, the definition of $\mathbb{S}$ being tight reduces to check whether for all $S \in \mathbb{S}$ and $a \in Arg_{\mathbb{S}} \setminus S$ there is an $s \in S$ such that $(a, s) \notin Pairs_{\mathbb{S}}$. The idea behind the notion of being tight is that if an argument a does not occur in some extension S there must be a reason for that. The most simple reason one can think of is that there is a conflict between a and some $s \in S$, i.e. a and s do not occur jointly in any extension-set of $\mathbb{S}$ or, in other words, $(a, s) \notin Pairs_{\mathbb{S}}$. Indeed, the set $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ from above is not tight, since there is no reason to, for instance, exclude c from extension $\{a, b\}$ ((a, c) and (b, c) are both contained in $Pairs_{\mathbb{S}}$). On the other hand, the set $\{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}\}$ is easily checked to be tight.

Before stating our first results, we examine certain properties of tight sets which we exploit later.

Lemma 2. For a tight extension-set $\mathbb{S} \subseteq 2^{\mathcal{A}}$ it holds that

1. the $\subseteq$ -maximal elements in $\mathbb{S}$ form a tight set, and
2. if $\mathbb{S}$ is incomparable then each $\mathbb{S}' \subseteq \mathbb{S}$ is tight.

Proof. Consider some tight extension-set $\mathbb{S}$. (1) For each $S \in \mathbb{S}$ and $a \in Arg_{\mathbb{S}}$ it holds that $S \cup \{a\} \notin \mathbb{S}$ if $S \cup \{a\} \notin \mathbb{S}'$, where $\mathbb{S}'$ is the set of $\subseteq$ -maximal elements in $\mathbb{S}$. Hence, as $Pairs_{\mathbb{S}} = Pairs_{\mathbb{S}'}$ and $\mathbb{S}' \subseteq \mathbb{S}$, $\mathbb{S}'$ is tight too. (2) Let $\mathbb{S}$ be incomparable, and consider some $\mathbb{S}' \subseteq \mathbb{S}$. Then

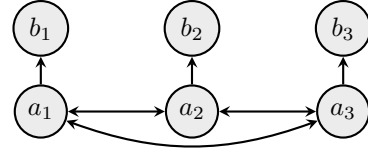


Figure 1: Argumentation framework F used in Example 1.

$Pairs_{\mathbb{S}'} \subseteq Pairs_{\mathbb{S}}$ and, as $\mathbb{S}$ is incomparable, $S \cup \{a\} \notin \mathbb{S}$ iff $S \cup \{a\} \notin \mathbb{S}'$ for all $S \in \mathbb{S}$, $a \in Arg_{\mathbb{S}}$. Thus $\mathbb{S}'$ is tight. $\square$

Proposition 1. For each AF $F = (A, R)$,

1. $cf(F)$ is non-empty, downward-closed and tight;
2. $naive(F)$ is non-empty, incomparable and its downward-closure is tight;
3. $stage(F)$ is non-empty, incomparable and tight;
4. $stb(F)$ is incomparable and tight.

Proof. The properties of being non-empty and incomparable are clear. Likewise, it is easy to see that $cf(F) = dcl(cf(F))$, i.e. $cf(F)$ is downward-closed. To show that $cf(F)$ is tight, let $S \in cf(F)$ and $a \in Arg_{cf(F)}$, such that $S \cup \{a\} \notin cf(F)$. It follows that $S \neq \emptyset$ (otherwise, $\{a\} \in cf(F)$ and since we know $cf(F) = dcl(cf(F))$, this would yield $a \in Arg_{cf(F)}$). Moreover there exists an argument $s \in S$ such that $s \mapsto a$ or $a \mapsto s$. Then $\{a, s\} \notin cf(F)$ and since $cf(F) = dcl(cf(F))$, $\{a, s\} \not\subseteq T$ for any $T \in cf(F)$. It follows that $(a, s) \notin Pairs_{cf(F)}$. Since this applies to all S, a with $S \in cf(F)$ and $a \notin S$, tightness of $cf(F)$ follows. Next, observe that $dcl(naive(F)) = cf(F)$, by definition of the semantics. Thus $dcl(naive(F))$ is tight, and by Lemma 2, $naive(F)$ is tight. With the same lemma, we get that every $\mathbb{S} \subseteq naive(F)$ is tight. By the well-known fact that $stb(F) \subseteq stage(F) \subseteq naive(F)$, the result follows. $\square$

Example 1. For the AF F in Figure 1, we have $\mathbb{S} = stb(F) = stage(F) = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$ and $naive(F) = \mathbb{S} \cup \{\{b_1, b_2, b_3\}\}$. $\mathbb{S}$ is indeed tight. Take, for instance, $E = \{a_1, b_2, b_3\}$; then for each argument t not in E (i.e. b_1, a_2, a_3) there is an argument $s \in E$ such $(s, t) \notin Pairs_{\mathbb{S}}$. In this particular case a_1 plays this role for each t . The other two extensions behave in a symmetric way. However, $dcl(\mathbb{S})$ is not tight. In fact, $\{b_2, b_3\} \in dcl(\mathbb{S})$ and now for b_1 , $\{b_1, b_2, b_3\} \notin dcl(\mathbb{S})$ but (b_1, b_2) and (b_1, b_3) are contained in $Pairs_{dcl(\mathbb{S})} = Pairs_{\mathbb{S}}$. Proposition 1 now gives evidence that an AF G with $naive(G) = \mathbb{S}$ cannot exist.

Preferred, Semi-Stable and Admissible Sets

The following property checks whether each pair of extensions in an extension-set $\mathbb{S}$ has at least one mutual conflict in case its union is not contained in $\mathbb{S}$.

Definition 5. A set $\mathbb{S} \subseteq 2^{\mathcal{A}}$ is called *adm-closed* if for each $A, B \in \mathbb{S}$ the following holds: if $(a, b) \in Pairs_{\mathbb{S}}$ for each $a, b \in A \cup B$, then also $A \cup B \in \mathbb{S}$.

Note that for $a, b \in A$ (likewise $a, b \in B$), $(a, b) \in Pairs_{\mathbb{S}}$ holds by definition. Thus the property of adm-closed is determined by arguments $a \in A \setminus B, b \in B \setminus A$, for $A, B \in \mathbb{S}$.

Lemma 3. *Every tight extension-set is also adm-closed.*

Proof. Consider some tight extension-set $\mathbb{S}$ and let $A, B \in \mathbb{S}$. We need to show that in case $(a, b) \in \text{Pairs}_{\mathbb{S}}$ for each $a, b \in A \cup B$, then also $A \cup B \in \mathbb{S}$. Let $B = \{b_1, \dots, b_n\}$. As $\mathbb{S}$ is tight and $(a, b_1) \in \text{Pairs}_{\mathbb{S}}$ for all $a \in A$, $A \cup \{b_1\} = A_1 \in \mathbb{S}$ must hold. We can do this for each $b_i \in B$ and get $A_{n-1} \cup \{b_n\} = (A \cup B) \in \mathbb{S}$. $\square$

Observe that for incomparable $\mathbb{S}$ the property of being adm-closed reduces to check for each $A, B \in \mathbb{S}$ ($A \neq B$), whether there exist $a, b \in A \cup B$ such that $(a, b) \notin \text{Pairs}_{\mathbb{S}}$.

We can give results analogous to those of Lemma 2.

Lemma 4. *For an adm-closed extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$,*

1. *the $\subseteq$ -maximal elements in $\mathbb{S}$ form an adm-closed set,*
2. *if $\mathbb{S}$ is incomparable then each $\mathbb{S}' \subseteq \mathbb{S}$ is adm-closed, and*
3. *$\mathbb{S} \cup \{\emptyset\}$ is adm-closed.*

Proof. (1) basically follows from the fact that $\text{Pairs}_{\mathbb{S}} = \text{Pairs}_{\mathbb{S}'}$ where $\mathbb{S}'$ is the set of $\subseteq$ -maximal elements in $\mathbb{S}$. For (2) recall that for incomparable $\mathbb{S}$, adm-closed reduces to check for each $A, B \in \mathbb{S}$ with $A \neq B$, whether there exist $a, b \in A \cup B$ such that $(a, b) \notin \text{Pairs}_{\mathbb{S}}$. It is easy to see that this property still holds for $\mathbb{S}' \subseteq \mathbb{S}$ (since then, $\text{Pairs}'_{\mathbb{S}} \subseteq \text{Pairs}_{\mathbb{S}}$). (3) holds, since $S \cup \emptyset \in \mathbb{S}$ for each $S \in \mathbb{S}$ ensures that $\mathbb{S} \cup \{\emptyset\}$ is adm-closed by definition. $\square$

The following proposition finally shows the role the property of being adm-closed plays in terms of admissible, preferred and semi-stable semantics.

Proposition 2. *For each AF $F = (A, R)$,*

1. *$\text{adm}(F)$ is adm-closed and contains $\emptyset$;*
2. *$\text{pref}(F)$ is non-empty, incomparable and adm-closed;*
3. *$\text{sem}(F)$ is non-empty, incomparable and adm-closed.*

Proof. ad 1) By definition, $\emptyset$ is always admissible. We show that $\text{adm}(F)$ is adm-closed. Towards a contradiction, assume $B, C \in \text{adm}(F)$ such that for all $b, c \in B \cup C$, $(b, c) \in \text{Pairs}_{\text{adm}(F)}$, but $B \cup C \notin \text{adm}(F)$. From Lemma 1 we know that $B \cup C$ defends itself in F . So for $B \cup C \notin \text{adm}(F)$ there must be a conflict in $B \cup C$, i.e. there is an attack $(b, c) \in R$ such that $\{b, c\} \subseteq B \cup C$. But then, for all $D \in \text{adm}(F)$, $\{b, c\} \not\subseteq D$. Hence, $(b, c) \notin \text{Pairs}_{\text{adm}(F)}$, a contradiction.

ad 2+3) By definition, both semantics $\sigma \in \{\text{pref}, \text{sem}\}$ yield at least one extension and for any $S, S' \in \sigma(F)$, $S \subseteq S'$ implies $S = S'$. By Lemma 4 and definition of preferred extensions, the fact that $\text{pref}(F)$ is adm-closed follows from the already shown property of $\text{adm}(F)$ being adm-closed. Since $\text{sem}(F) \subseteq \text{pref}(F)$ holds for all AFs F , Lemma 4 immediately shows that $\text{sem}(F)$ is adm-closed, as well. $\square$

Example 2. Consider the framework F in Figure 2 and let $A = \{a, b\}$, $B = \{a, c, e\}$, $C = \{b, d, e\}$, and $\mathbb{S} = \{A, B, C\}$. We have $\text{pref}(F) = \text{sem}(F) = \mathbb{S}$. $\mathbb{S}$ is adm-closed, since for each pair of extensions, there exists a pair of arguments not contained in $\text{Pairs}_{\mathbb{S}}$: $b, c \in A \cup B$ and $(b, c) \notin \text{Pairs}_{\mathbb{S}}$; $a, d \in A \cup C$ and $(a, d) \notin \text{Pairs}_{\mathbb{S}}$; $c, d \in B \cup C$ and $(c, d) \notin \text{Pairs}_{\mathbb{S}}$. However, we also observe

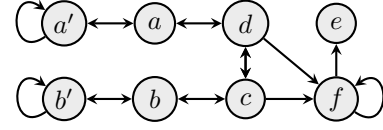


Figure 2: Argumentation framework F used in Example 2.

that $\mathbb{S}$ is not tight, since $A \cup \{e\} \notin \mathbb{S}$ but both (a, e) and (b, e) are contained in $\text{Pairs}_{\mathbb{S}}$. This shows that the reverse of Lemma 3 does not hold.

Complete Sets

Finally, we turn to the complete semantics. Compared to being adm-closed, we need a further relaxation.

Definition 6. Given an extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ and $E \subseteq \mathfrak{A}$, we define the completion-sets $\mathbb{C}_{\mathbb{S}}(E)$ of E in $\mathbb{S}$ as the set of $\subseteq$ -minimal sets $S \in \mathbb{S}$ with $E \subseteq S$.

In words, completion sets just give the “next” (in terms of supersets of A) elements contained in $\mathbb{S}$.

Definition 7. A set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ is called *com-closed* if for each $\mathbb{T} \subseteq \mathbb{S}$ the following holds: if $(a, b) \in \text{Pairs}_{\mathbb{S}}$ for each $a, b \in \text{Arg}_{\mathbb{T}} = \bigcup_{T \in \mathbb{T}} T$, then there exists a unique completion-set $S \in \mathbb{C}_{\mathbb{S}}(\text{Arg}_{\mathbb{T}})$. For a com-closed extension-set $\mathbb{S} \subseteq 2^{\mathfrak{A}}$ and $E \subseteq \mathfrak{A}$, we denote the unique element of $\mathbb{C}_{\mathbb{S}}(E)$ by $C_{\mathbb{S}}(E)$.

Lemma 5. *Each adm-closed extension-set is com-closed.*

Proof. Consider an adm-closed extension-set $\mathbb{S}$ and $\mathbb{T} \subseteq \mathbb{S}$. Then $(a, b) \in \text{Pairs}_{\mathbb{S}}$ for each $a, b \in \text{Arg}_{\mathbb{T}}$ implies $\text{Arg}_{\mathbb{T}} \in \mathbb{S}$, i.e. $\mathbb{C}_{\mathbb{S}}(\text{Arg}_{\mathbb{T}}) = \{\text{Arg}_{\mathbb{T}}\}$. $\square$

In case of incomparable sets, the notions of adm-closed and com-closed coincide. In anticipation of the following result, this coincidence reflects the fact that all preferred extensions are complete.

Proposition 3. *For each AF F , $\text{com}(F)$ is a non-empty, com-closed extension-set with $(\bigcap_{S \in \text{com}(F)} S) \in \text{com}(F)$.*

Proof. First note that there is always at least one complete extension, namely the grounded extension. Moreover the grounded extension is the unique $\subseteq$ -minimal complete extension and hence $(\bigcap_{S \in \text{com}(F)} S) \in \text{com}(F)$. Finally consider a set of complete extensions $\mathbb{E} \subseteq \text{com}(F)$ such that $(a, b) \in \text{Pairs}_{\text{com}(F)}$ for each $a, b \in \text{Arg}_{\mathbb{E}}$. By Lemma 1, $\text{Arg}_{\mathbb{E}}$ is an admissible set and thus can be extended to a unique complete extension $E' \supseteq \text{Arg}_{\mathbb{E}}$ by iteratively adding all defended arguments. Therefore $\text{com}(F)$ is com-closed. $\square$

Example 3. Consider the AF $F = (\{a, a', b, b', c\}, \{(x, x'), (x', x), (x', x'), (x', c) \mid x \in \{a, b\}\})$. We have $\text{com}(F) = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$, which is com-closed, in particular, as $\mathbb{C}_{\text{com}(F)}(\{a\} \cup \{b\}) = \{\{a, b, c\}\}$. Observe that since $(a, b) \in \text{Pairs}_{\text{com}(F)}$, but $\{a, b\} \notin \text{com}(F)$, $\text{com}(F)$ is not adm-closed. This also shows that the reverse direction of Lemma 5 does not hold.

4 Realizability

In the previous section we have given necessary characteristics for the extension-sets $\mathbb{S} \in \Sigma_\sigma$, where $\sigma \in \{cf, adm, naive, stb, stage, pref, sem, com\}$. Except for the complete semantics, we will now show that these characteristics are also sufficient.

To this end, we need the concept of *realizability*: an extension-set $\mathbb{S} \subseteq 2^{\mathcal{A}}$ is σ -realizable if there is an AF $F \in AF_{\mathcal{A}}$, such that $\sigma(F) = \mathbb{S}$. We start with the following concept of a canonical argumentation framework, which will underlie all subsequent results on realizability.

Definition 8. Given an extension-set $\mathbb{S}$, we define the canonical argumentation framework for $\mathbb{S}$ as

$$F_{\mathbb{S}}^{cf} = (Arg_{\mathbb{S}}, (Arg_{\mathbb{S}} \times Arg_{\mathbb{S}}) \setminus Pairs_{\mathbb{S}}).$$

The idea behind the framework is simple: we draw a relation between two arguments iff they do *not* occur jointly in any set $S \in \mathbb{S}$. Thus, for any $\mathbb{S}$, $F_{\mathbb{S}}^{cf}$ is symmetric and has no self-attacking arguments (recall that for each $a \in Arg_{\mathbb{S}}$, $(a, a) \in Pairs_{\mathbb{S}}$). As an example, consider $\mathbb{T} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}, \{b_1, b_2, b_3\}\}$. $F_{\mathbb{T}}^{cf}$ has the same structure as the AF from Figure 1, but with all attacks being symmetric. We obtain $stb(F_{\mathbb{T}}^{cf}) = naive(F_{\mathbb{T}}^{cf}) = \mathbb{T}$. When we consider $\mathbb{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, i.e. $\mathbb{S} = \mathbb{T} \setminus \{\{b_1, b_2, b_3\}\}$, we obtain the same framework $F_{\mathbb{S}}^{cf} = F_{\mathbb{T}}^{cf}$, since $Pairs_{\mathbb{S}} = Pairs_{\mathbb{T}}$. In terms of naive semantics, this is not problematic, since $\mathbb{S}$ (as discussed in Example 1) cannot be realized via naive semantics at all. However, this observation readily suggests that realizing $\mathbb{S}$ with, say, stable semantics, requires additional concepts.

Stable, Stage, Naive and Conflict-Free Sets

Proposition 4. (1) For each downward-closed and tight extension-set $\mathbb{S} \neq \emptyset$, $cf(F_{\mathbb{S}}^{cf}) = \mathbb{S}$. (2) For each incomparable extension-set $\mathbb{S} \neq \emptyset$, where $dcl(\mathbb{S})$ is tight, $naive(F_{\mathbb{S}}^{cf}) = \mathbb{S}$.

Proof. (1) Let $F_{\mathbb{S}}^{cf} = (Arg_{\mathbb{S}}, R_{\mathbb{S}})$. $cf(F_{\mathbb{S}}^{cf}) \subseteq \mathbb{S}$: Observe that for each $E \in cf(F_{\mathbb{S}}^{cf})$, $(a, b) \in Pairs_{\mathbb{S}}$ for all $a, b \in E$ by construction of $R_{\mathbb{S}}$. Now suppose there exists $E' \in cf(F_{\mathbb{S}}^{cf})$ such that $E' \not\subseteq \mathbb{S}$. Wlog let E' be $\subseteq$ -minimal with this property. Then $E' = S \cup \{c\}$ for some $S \in \mathbb{S}$. As $\mathbb{S}$ is tight there is an $s \in S$ such that $(s, c) \notin Pairs_{\mathbb{S}}$, a contradiction to the above observation. $cf(F_{\mathbb{S}}^{cf}) \supseteq \mathbb{S}$: Follows immediately by the fact that $\forall (a, b) \in Pairs_{\mathbb{S}} : (a, b) \notin R_{\mathbb{S}}$.

(2) First, note that $F_{dcl(\mathbb{S})}^{cf} = F_{\mathbb{S}}^{cf}$, since $Arg_{dcl(\mathbb{S})} = Arg_{\mathbb{S}}$ and $Pairs_{dcl(\mathbb{S})} = Pairs_{\mathbb{S}}$. $cf(F_{\mathbb{S}}^{cf}) = dcl(\mathbb{S})$ follows from (1). By construction of $dcl(\mathbb{S})$, the $\subseteq$ -maximal sets in $dcl(\mathbb{S})$ are given by $\mathbb{S}$ and as naive sets are just $\subseteq$ -maximal conflict-free, $naive(F_{\mathbb{S}}^{cf}) = \mathbb{S}$ follows. $\square$

We proceed with stable and stage semantics. Stable semantics are the only semantics that can realize $\mathbb{S} = \emptyset$. Note that $\mathbb{S} = \emptyset$ is easily *stb*-realizable, for instance with the framework $(\{a\}, \{(a, a)\})$. In Proposition 1 the only difference between stable and stage semantics was the case $\mathbb{S} = \emptyset$.

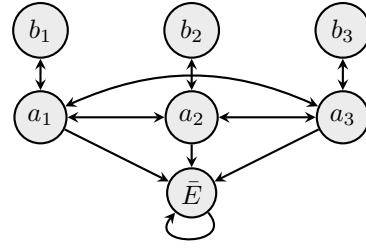


Figure 3: Excluding $\{b_1, b_2, b_3\}$ from $stb(F_{\mathbb{S}}^{cf})$.

The next result shows that this indeed is the only difference between the signatures for stable and stage semantics.

The idea of the construction below is to suitably extend the canonical framework from Definition 8 such that undesired sets are excluded. Coming back to our example with $\mathbb{S} = \{\{a_1, b_2, b_3\}, \{a_2, b_1, b_3\}, \{a_3, b_1, b_2\}\}$, recall that $F_{\mathbb{S}}^{cf}$ had one such undesired naive (and stable) extension, $E = \{b_1, b_2, b_3\}$. To get rid of it we add a new argument which is attacked by all other sets from $\mathbb{S}$ but not by E , see Figure 3 for illustration. The next definition generalizes this idea which is due to (Dvořák and Spanring 2012).

Definition 9. Given an extension-set $\mathbb{S}$ and its canonical framework $F_{\mathbb{S}}^{cf} = (A_{\mathbb{S}}^{cf}, R_{\mathbb{S}}^{cf})$, let $\mathbb{X} = stb(F_{\mathbb{S}}^{cf}) \setminus \mathbb{S}$. We define $F_{\mathbb{S}}^{st} = (A_{\mathbb{S}}^{cf} \cup \{\bar{E} \mid E \in \mathbb{X}\}, R_{\mathbb{S}}^{cf} \cup \{(\bar{E}, \bar{E}), (a, \bar{E}) \mid E \in \mathbb{X}, a \in Arg_{\mathbb{S}} \setminus E\})$.

Proposition 5. For each non-empty, incomparable and tight extension-set $\mathbb{S}$, $stb(F_{\mathbb{S}}^{st}) = stage(F_{\mathbb{S}}^{st}) = \mathbb{S}$.

Proof. Since $\mathbb{S}$ is non-empty, showing existence of an AF F with $stb(F) = \mathbb{S}$ is sufficient (for each F with $stb(F) \neq \emptyset$, $stb(F) = stage(F)$ holds).

Moreover, for each incomparable and tight extension-set $\mathbb{S}$, it holds that $\mathbb{S} \subseteq stb(F_{\mathbb{S}}^{cf})$: Towards a contradiction, let $S \in \mathbb{S}$ such that $S \notin stb(F_{\mathbb{S}}^{cf})$. By construction of $F_{\mathbb{S}}^{cf}$, $S \in cf(F_{\mathbb{S}}^{cf})$. Thus $\exists a \in Arg_{\mathbb{S}} : S \not\vdash a$. But then by construction, $(a, s) \in Pairs_{\mathbb{S}}$, for all $s \in S$; this contradicts $\mathbb{S}$ being tight.

Now let $\mathbb{X} = stb(F_{\mathbb{S}}^{cf}) \setminus \mathbb{S}$ and consider $F_{\mathbb{S}}^{st}$. We show that $stb(F_{\mathbb{S}}^{st}) = \mathbb{S}$. $stb(F_{\mathbb{S}}^{st}) \subseteq \mathbb{S}$: Let $E \in stb(F_{\mathbb{S}}^{st})$. Then also $E \in stb(F_{\mathbb{S}}^{cf}) = \mathbb{X} \cup \mathbb{S}$. But if $E \in \mathbb{X}$, by Definition 9, $E \not\vdash \bar{E}$ and also $\bar{E} \notin E$, thus $E \notin stb(F_{\mathbb{S}}^{st})$. Hence, $E \in \mathbb{S}$. $stb(F_{\mathbb{S}}^{st}) \supseteq \mathbb{S}$: Let $E \in \mathbb{S}$. We already know that $E \in stb(F_{\mathbb{S}}^{cf})$. Now consider $F_{\mathbb{S}}^{st}$. E still attacks all arguments in $Arg_{\mathbb{S}} \setminus E$. Now consider an arbitrary argument $\bar{E}'$ for $E' \in \mathbb{X}$. $\bar{E}'$ is attacked by all arguments $a \in Arg_{\mathbb{S}} \setminus E'$ and as E and E' are both stable in $F_{\mathbb{S}}^{cf}$ (and thus incomparable) at least one of these arguments must be contained in E . Hence $E \in stb(F_{\mathbb{S}}^{st})$. $\square$

Preferred, Semi-Stable and Admissible Sets

Towards a suitable canonical AF for admissibility-based semantics we introduce the following technical concept.

Definition 10. Given an extension-set $\mathbb{S}$, the *defense-formula* $\mathcal{D}_a^{\mathbb{S}}$ of an argument $a \in Arg_{\mathbb{S}}$ in $\mathbb{S}$ is defined as $\bigvee_{S \in \mathbb{S} \text{ s.t. } a \in S} \bigwedge_{s \in S \setminus \{a\}} s$. $\mathcal{D}_a^{\mathbb{S}}$ given as (a logically equivalent) CNF is called *CNF-defense-formula* $CD_a^{\mathbb{S}}$ of a in $\mathbb{S}$.

Intuitively, $\mathcal{D}_a^{\mathbb{S}}$ describes the conditions for the argument a being in an extension. Each disjunct represents a set of arguments which jointly allows a to “join” an extension, i.e. represents a collection of arguments defending a . Further note that $\mathcal{D}_a^{\mathbb{S}} = \top$ if and only if $\{a\} \in \mathbb{S}$.

Example 4. Consider $\mathbb{T} = \{\emptyset, \{a\}, \{b, c\}, \{a, c, d\}\}$. Then $\mathcal{D}_a^{\mathbb{T}} = \top$, $\mathcal{D}_b^{\mathbb{T}} = c$, $\mathcal{D}_c^{\mathbb{T}} = b \vee (a \wedge d)$ and $\mathcal{D}_d^{\mathbb{T}} = a \wedge c$. The corresponding CNF-defense-formulas¹ are $\mathcal{CD}_a^{\mathbb{T}} = \{\}$, $\mathcal{CD}_b^{\mathbb{T}} = \{\{c\}\}$, $\mathcal{CD}_c^{\mathbb{T}} = \{\{a, b\}, \{b, d\}\}$, and $\mathcal{CD}_d^{\mathbb{T}} = \{\{a\}, \{c\}\}$.

Lemma 6. Given an extension-set $\mathbb{S}$ and an argument $a \in \text{Arg}_{\mathbb{S}}$, then for each $S \subseteq \text{Arg}_{\mathbb{S}}$ with $a \in S$: $(S \setminus \{a\})$ is a model of $\mathcal{D}_a^{\mathbb{S}}$ (resp. $\mathcal{CD}_a^{\mathbb{S}}$) iff there exists an $S' \subseteq S$ with $a \in S'$ such that $S' \in \mathbb{S}$.

Proof. The if-direction follows straight forwardly by definition of $\mathcal{D}_a^{\mathbb{S}}$, since the conjunction of the elements of $S' \setminus \{a\}$ forms a term of $\mathcal{D}_a^{\mathbb{S}}$ for each $S' \in \mathbb{S}$ with $a \in S'$.

To show the only-if-direction consider some $S \subseteq \text{Arg}_{\mathbb{S}}$ with $a \in S$ where $S \setminus \{a\}$ is a model of $\mathcal{D}_a^{\mathbb{S}}$. If $\mathcal{D}_a^{\mathbb{S}} = \top$ then it holds that $\{a\} \in \mathbb{S}$. For $S \setminus \{a\}$ to be a model of $\mathcal{D}_a^{\mathbb{S}} \neq \top$, there must be some term $\tau \in \mathcal{D}_a^{\mathbb{S}}$, whose elements form a subset of $S \setminus \{a\}$. Consider such a term $\tau \in \mathcal{D}_a^{\mathbb{S}}$. Then by construction of $\mathcal{D}_a^{\mathbb{S}}$ there is some $S' \in \mathbb{S}$ with $a \in S'$, where $S' \setminus \{a\}$ coincides with the elements of τ . So $S' \subseteq S$. $\square$

We make use of the defense formulas in the following construction.

Definition 11. Given an extension-set $\mathbb{S}$, the *canonical defense-argumentation-framework* $F_{\mathbb{S}}^{\text{def}} = (A_{\mathbb{S}}^{\text{def}}, R_{\mathbb{S}}^{\text{def}})$ extends the canonical AF $F_{\mathbb{S}}^{\text{cf}} = (\text{Arg}_{\mathbb{S}}, R_{\mathbb{S}}^{\text{cf}})$ as follows:

$$A_{\mathbb{S}}^{\text{def}} = \text{Arg}_{\mathbb{S}} \cup \bigcup_{a \in \text{Arg}_{\mathbb{S}}} \{\alpha_{a\gamma} \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}\}, \text{ and}$$

$$R_{\mathbb{S}}^{\text{def}} = R_{\mathbb{S}}^{\text{cf}} \cup \bigcup_{a \in \text{Arg}_{\mathbb{S}}} \{(b, \alpha_{a\gamma}), (\alpha_{a\gamma}, \alpha_{a\gamma}), (\alpha_{a\gamma}, a) \mid \gamma \in \mathcal{CD}_a^{\mathbb{S}}, b \in \gamma\}.$$

$F_{\mathbb{S}}^{\text{def}}$ consists of all arguments given in the extension-set plus a certain amount of additional arguments. Each $\alpha_{a\gamma}$ attacks argument a and is attacked by all arguments occurring as literals in clause γ of the CNF-defense-formula of a . So in $F_{\mathbb{S}}^{\text{def}}$ for a to be defended from $\alpha_{a\gamma}$ it takes at least one argument of these occurring as atoms in clause γ of $\mathcal{CD}_a^{\mathbb{S}}$.

Example 5. Again consider extension-set $\mathbb{T} = \{\emptyset, \{a\}, \{b, c\}, \{a, c, d\}\}$. We have given the CNF-defense-formulas in Example 4. $F_{\mathbb{T}}^{\text{def}}$ is given by the AF in Figure 4. Considering, for instance, argument c , where $\mathcal{CD}_c^{\mathbb{T}} = \{\{a, b\}, \{b, d\}\}$, one can see that in $F_{\mathbb{T}}^{\text{def}}$ it takes a or b in order to defend c from $\alpha_{c\{a,b\}}$, and b or d in order to defend c from $\alpha_{c\{b,d\}}$.

Proposition 6. For each adm-closed extension-set $\mathbb{S}$ where $\emptyset \in \mathbb{S}$, $\text{adm}(F_{\mathbb{S}}^{\text{def}}) = \mathbb{S}$.

¹In what follows, we write CNF formulas in clause form.

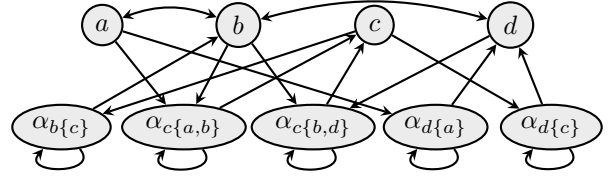


Figure 4: AF $F_{\mathbb{T}}^{\text{def}}$ used in Example 5.

Proof. $\mathbb{S} \subseteq \text{adm}(F_{\mathbb{S}}^{\text{def}})$: Let $S \in \mathbb{S}$. If $S = \emptyset$, the assertion trivially holds. If $S = \{a\}$, then $\mathcal{CD}_a^{\mathbb{S}}$ is the empty set of clauses. By definition of $R_{\mathbb{S}}^{\text{def}}$, a is then defended in $F_{\mathbb{S}}^{\text{def}}$ and thus $S \in \text{adm}(F_{\mathbb{S}}^{\text{def}})$. Thus let $S \in \mathbb{S}$ contain at least two arguments. By construction, S is conflict-free in $F_{\mathbb{S}}^{\text{def}}$. It remains to show that each $s \in S$ is defended by S in $F_{\mathbb{S}}^{\text{def}}$. Let $s \in S$. We know from Lemma 6 that $S \setminus \{s\}$ is a model of $\mathcal{CD}_s^{\mathbb{S}}$. Hence, each clause $\gamma \in \mathcal{CD}_s^{\mathbb{S}}$ contains at least one variable $t_{\gamma} \in S \setminus \{s\}$. Thus, by construction of $R_{\mathbb{S}}^{\text{def}}$, $(S \setminus \{s\}) \mapsto \alpha_{s\gamma}$ for each $\gamma \in \mathcal{CD}_s^{\mathbb{S}}$, i.e. S defends s in $F_{\mathbb{S}}^{\text{def}}$.

$\text{adm}(F_{\mathbb{S}}^{\text{def}}) \subseteq \mathbb{S}$: Let $S \in \text{adm}(F_{\mathbb{S}}^{\text{def}})$. For $S = \emptyset$, $S \in \mathbb{S}$ by definition. If $S = \{a\}$, then $a \mapsto b$ whenever some argument b attacks a . Hence, $\mathcal{CD}_a^{\mathbb{S}} = \{\}$, i.e. $\mathcal{D}_a^{\mathbb{S}} = \top$ must hold, therefore $S \in \mathbb{S}$. Now assume S contains at least two arguments. As S is conflict-free in $F_{\mathbb{S}}^{\text{def}}$, the construction of $R_{\mathbb{S}}^{\text{def}}$ guarantees that for all $a, b \in S$, $(a, b) \in \text{Pairs}_{\mathbb{S}}$. Let $s \in S$ with $\{s\} \notin \text{adm}(F_{\mathbb{S}}^{\text{def}})$. Then we have $\alpha_{s\gamma} \mapsto s$ for each $\gamma \in \mathcal{CD}_s^{\mathbb{S}}$. Since s is defended by S , for each $\gamma \in \mathcal{CD}_s^{\mathbb{S}}$, $\exists t_{\gamma} \in (S \setminus \{s\}) : t_{\gamma} \mapsto \alpha_{s\gamma}$. By definition of $F_{\mathbb{S}}^{\text{def}}$, thus t_{γ} occurs in the clause γ . It follows that $T = \{t_{\gamma} \mid \gamma \in \mathcal{CD}_s^{\mathbb{S}}\}$ is a model of $\mathcal{CD}_s^{\mathbb{S}}$ and $\mathcal{D}_s^{\mathbb{S}}$. Then by Lemma 6 there is some $S'_s \subseteq T \cup \{s\}$ (note that also $S'_s \subseteq S$ as $s \in S$ and for each $t_{\gamma} \in T$ also $t_{\gamma} \in S$) with $s \in S'_s$ such that $S'_s \in \mathbb{S}$. Recall also that in case $\{s\} \in \text{adm}(F_{\mathbb{S}}^{\text{def}})$, we know that $\{s\} \in \mathbb{S}$ (say $S'_s = \{s\}$). Now since $(a, b) \in \text{Pairs}_{\mathbb{S}}$ for all $a, b \in S$ and since $\mathbb{S}$ is adm-closed, it must hold that $S'_{s_1} \cup S'_{s_2} \in \mathbb{S}$ for any $s_1, s_2 \in S$. Hence $S = (\bigcup_{s_i \in S} S'_{s_i}) \in \mathbb{S}$. $\square$

The construction also works for further semantics.

Proposition 7. For each non-empty, incomparable and adm-closed extension-set $\mathbb{S}$ there exist F, F' with $\text{pref}(F) = \mathbb{S}$ and $\text{sem}(F') = \mathbb{S}$.

Proof. By Proposition 6 and Lemma 4.3, $\text{adm}(F_{\mathbb{S}}^{\text{def}}) = \mathbb{S} \cup \{\emptyset\}$. Since $\mathbb{S}$ is incomparable, $\text{pref}(F_{\mathbb{S}}^{\text{def}}) = \mathbb{S}$. Moreover, as shown by Dvořák and Woltran (2011), for each AF F there is an AF F' such that $\text{pref}(F) = \text{sem}(F')$. $\square$

Each extension-set $\mathbb{S}$ realizable under the admissible semantics, can also be realized under the complete semantics: extend $F_{\mathbb{S}}^{\text{def}}$ for each $a \in \text{Arg}_{\mathbb{S}}$ by a self-attacking argument with an attack from and to a . As Example 3 already suggests, the complete semantics is however capable of realizing more diverse extension-sets. However, the property of being com-closed (cf. Definition 7) so far only gives an approximation of the sets which can realized.

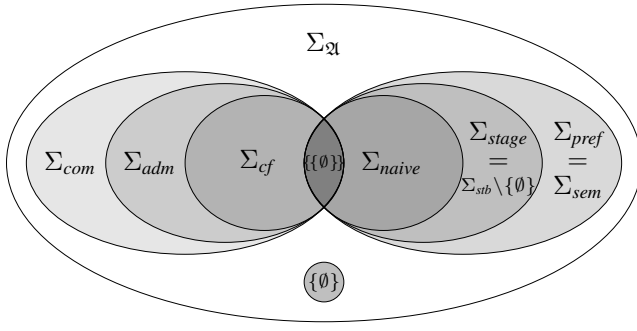


Figure 5: Relations between signatures.

Example 6. Let $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}, \{a, d, e\}, \{b, d, f\}, \{x, c\}, \{x, d\}\}$. $\mathbb{S}$ is com-closed and satisfies $(\bigcap_{S \in \mathbb{S}} S) = \emptyset \in \mathbb{S}$. We argue that $\mathbb{S}$ is not com-realizable. Towards a contradiction consider an AF F such that $\text{com}(F) = \mathbb{S}$. First as $\{x, c\}, \{x, d\} \in \mathbb{S}$ and no superset of $\{x, c, d\}$ is in $\mathbb{S}$, there must be a conflict between c and d . In case $(c, d) \in R$ and $(d, c) \notin R$, (x, d) is not admissible (as $(x, c) \in \text{Pairs}_{\mathbb{S}}$). Similar for $(d, c) \in R$ and $(c, d) \notin R$. Thus $(c, d), (d, c) \in R$. Now consider $\{a, b\}$ which must be admissible as $\{a\}, \{b\}$ are admissible and $(a, b) \in \text{Pairs}_{\mathbb{S}}$. But as $\{a, b\} \notin \mathbb{S}$ it has to defend argument c ($C_{\mathbb{S}}(\{a, b\}) = \{a, b, c\}$). But we have $(d, c) \in R$ and $(a, d), (b, d) \in \text{Pairs}_{\mathbb{S}}$. Hence $\{a, b\}$ cannot defend c .

5 Signatures & Limits of Disagreement

We now state our main result, i.e. exact characterizations of the signatures for the semantics dealt with in this paper.

Theorem 1. *The signatures for the considered semantics are given by the following collections of extension-sets.*

$$\begin{aligned} \Sigma_{cf} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is downward-closed and tight}\} \\ \Sigma_{naive} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable and } \text{dcl}(\mathbb{S}) \text{ is tight}\} \\ \Sigma_{stb} &= \{\mathbb{S} \mid \mathbb{S} \text{ is incomparable and tight}\} \\ \Sigma_{stage} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable and tight}\} \\ \Sigma_{adm} &= \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is adm-closed and contains } \emptyset\} \\ \Sigma_{pref} &= \Sigma_{sem} = \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is incomparable and adm-closed}\} \end{aligned}$$

All relations of the theorem are immediate from results in the previous section and the corresponding characteristics from Propositions 1 and 2. Moreover, $\Sigma_{com} \subset \{\mathbb{S} \neq \emptyset \mid \mathbb{S} \text{ is com-closed and } (\bigcap_{S \in \mathbb{S}} S) \in \mathbb{S}\}$, cf. Proposition 3.

By inspecting the respective properties, we can now immediately put the signatures of different semantics in relation to each other. For illustration see Figure 5, where $\Sigma_{\mathcal{A}}$ just denotes the set of all extensions-sets over $\mathcal{A}$, and the singleton $\{\emptyset\}$, i.e. the extension-set containing no extension, belongs to Σ_{stb} .

Theorem 2. *The following relations hold*

$$\begin{aligned} \Sigma_{naive} &\subset \Sigma_{stage} \subset \Sigma_{sem} = \Sigma_{pref}, & \Sigma_{stb} &= \Sigma_{stage} \cup \{\emptyset\} \\ \Sigma_{cf} &\subset \Sigma_{adm} \subset \Sigma_{com} \\ \{\text{dcl}(\mathbb{S}) \mid \mathbb{S} \in \Sigma_{naive}\} &= \Sigma_{cf}, & \Sigma_{adm} &\supset \{\mathbb{S} \cup \{\emptyset\} \mid \mathbb{S} \in \Sigma_{pref}\} \end{aligned}$$

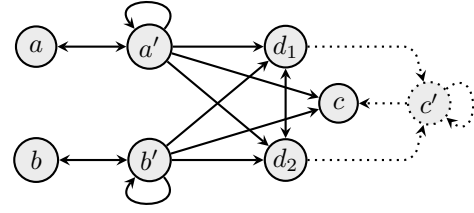


Figure 6: AFS F_1, F_2 with $(\text{com}(F_1) \cap \text{com}(F_2)) \notin \Sigma_{com}$.

Proof. In what follows, we make implicit use of the results from Theorem 1. First, by Lemma 2.1, $\Sigma_{naive} \subseteq \Sigma_{stage}$; $\Sigma_{naive} \neq \Sigma_{stage}$ is witnessed by Example 1. Relations $\Sigma_{stage} \subseteq \Sigma_{sem}$ and $\Sigma_{cf} \subseteq \Sigma_{adm}$ follow from Lemma 3. $\Sigma_{stage} \neq \Sigma_{sem}$ is by Example 2. $\Sigma_{cf} \neq \Sigma_{adm}$ is seen by extension-set $\{\emptyset, \{a, b\}\}$ which is adm-closed but not downward-closed. The fact that adm-realizability implies com-realizability and Example 3 show $\Sigma_{adm} \subset \Sigma_{com}$. The relations in the last line follow from the definition of $\text{dcl}(\cdot)$ and Lemma 4.3 respectively. $\square$

With our results on signatures at hand, we now provide some interesting implications. The first result examines the question whether the intersection of two extension-sets under a given semantics σ , can always be realized.

Theorem 3. *For each $\sigma \in \{cf, naive, stb, stage, pref, sem, adm\}$ it holds that for any AFS F_1, F_2 there exists an AF F such that $\sigma(F) = \sigma(F_1) \cap \sigma(F_2)$ if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$.²*

Proof. (Sketch.) We have to show that $\sigma(F_1) \cap \sigma(F_2)$ satisfies the properties according to the signature of the particular semantics. Existence of F is then a consequence of Theorem 1. The result for *stb*, *stage*, *pref*, *sem* follows from Lemma 2.2 and Lemma 4.2. The argument for *adm* is as follows (*cf* and *naive* are handled in a similar way). Assume $\mathbb{S} = \text{adm}(F_1) \cap \text{adm}(F_2)$ is not adm-closed, i.e. there are $A, B \in \mathbb{S}$ such that for all $a, b \in (A \cup B)$, $(a, b) \in \text{Pairs}_{\mathbb{S}}$, but $(A \cup B) \notin \mathbb{S}$. Then there is $i \in \{1, 2\}$, such that $A, B \in \text{adm}(F_i)$ but $(A \cup B) \notin \text{adm}(F_i)$. But then, $\text{adm}(F_i)$ is not adm-closed, a contradiction to Proposition 2. $\square$

The complete semantics turns out to be the only semantics not closed under this form of intersection. Consider $\mathbb{S} = \{\emptyset, \{a\}, \{b\}, \{a, b, c, d_1\}, \{a, b, c, d_2\}\}$, $\mathbb{S}_1 = \mathbb{S} \cup \{\{a, b\}\}$, and $\mathbb{S}_2 = \mathbb{S} \cup \{\{a, b, c\}\}$. $\mathbb{S}_1$ and $\mathbb{S}_2$ are realizable under the complete semantics. Corresponding AFS are depicted in Figure 6: $\mathbb{S}_1$ are the complete extensions of the entire AF, and $\mathbb{S}_2$ the ones of the AF without the dotted part. However, $\mathbb{S} = \mathbb{S}_1 \cap \mathbb{S}_2$ is not com-closed and therefore, by Proposition 3, no AF F exists such that $\text{com}(F) = \mathbb{S}$.

Our next results concern limits in expressing multiple extensions. Our first result is positive in a sense that as long as two (maximal) sets of arguments are involved, all semantics are capable to deliver such extensions. Recall that three sets (e.g. $\mathbb{S} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$) are already problematic.

Proposition 8. *For any incomparable extension-set $\mathbb{S} \neq \emptyset$ with $|\mathbb{S}| \leq 2$, $\mathbb{S} \in \Sigma_{\sigma}$ for $\sigma \in \{naive, stb, stage, pref, sem\}$.*

² $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$ is not a necessary condition for $\sigma = stb$.

Proof. By Theorem 2, it suffices to show the claim for naive semantics. By Theorem 1, we need to show that $dcl(\mathbb{S})$ is tight, which trivially holds for $|\mathbb{S}| = 1$. For $\mathbb{S} = \{S_1, S_2\}$, let $S \in dcl(\mathbb{S})$ and $a \in Arg_{\mathbb{S}}$ such that $S \cup \{a\} \notin dcl(\mathbb{S})$. Wlog. assume $S \subseteq S_1$. Then, $a \in S_2 \setminus S_1$ and $S \not\subseteq S_2$, i.e. there is some $s \in S \setminus S_2$. Since $|\mathbb{S}| = 2$, $(a, s) \notin Pairs_{\mathbb{S}}$. $\square$

How many extensions can maximally be achieved by an AF under a semantics σ ? Such results ease checking $\mathbb{S} \in \Sigma_{\sigma}$ whenever the cardinality of $\mathbb{S}$ exceeds a certain number.

Research in this direction has recently been initiated by Baumann and Strass (2014). They proposed a function giving the maximal number of stable extensions an AF with n arguments can have. In accordance to realizability, we are interested in the number of extensions as a function of a fixed amount of arguments occurring in any extension.

While our results on signatures give insights about the extent of structural diversity a semantics can express, the maximal number of extensions gives, to some degree, an answer to how much quantitative disagreement a semantics can express.

Definition 12. Given a semantics σ , we define the diversity-function $\Delta_{\sigma}(n) = \max_{F \in AF_{\mathbb{A}}, \|\sigma(F)\| = n} |\sigma(F)|$.

It is easy to see that $\Delta_{\sigma}(n) = 2^n$ for $\sigma \in \{cf, adm, com\}$. For the other semantics, we first give a technical lemma.

Lemma 7. For any AF F , there is a symmetric AF F^{sym} with $\|pref(F)\| = \|pref(F^{sym})\|$ and $|pref(F)| \leq |pref(F^{sym})|$.

Proof (Sketch). Transform $F = (A, R)$ by (1) removing all arguments $a \notin Arg_{pref(F)}$ together with adjacent attacks, and (2) adding (b, a) to R if $(a, b) \in R$. Obviously conflict-freeness and defense is preserved, i.e. any set admissible in F is admissible in F^{sym} . Moreover, as only attacks to and from arguments not occurring in any preferred extension of F are removed, any conflict between two $E_1, E_2 \in pref(F)$ survives the translation, therefore there must be two different $E'_1, E'_2 \in pref(F^{sym})$ with $E_1 \subseteq E'_1$ and $E_2 \subseteq E'_2$. Hence $|pref(F)| \leq |pref(F^{sym})|$. As $Arg_{pref(F^{sym})}$ coincides with the arguments of F^{sym} (by symmetry of F^{sym}), it follows that $\|pref(F)\| = \|pref(F^{sym})\|$. $\square$

Using the recent results by Baumann and Strass (2014), Lemma 7 and Theorem 1 allow to specify the function Δ_{σ} for all incomparable semantics σ .

Theorem 4. For $\sigma \in \{naive, stb, stage, pref, sem\}$, we have

$$\Delta_{\sigma}(n) = \begin{cases} 1 & n = 1 \\ 3^s & n \geq 2 \wedge n = 3s \\ 4 \cdot 3^{s-1} & n \geq 2 \wedge n = 3s + 1 \\ 2 \cdot 3^s & n \geq 2 \wedge n = 3s + 2. \end{cases}$$

Proof. Consider a semantics $\sigma \in \{naive, stb, stage, pref, sem\}$ and an AF F having maximal diversity, i.e. $|\sigma(F)| = \Delta_{\sigma}(\|\sigma(F)\|)$. Since $\Sigma_{\sigma} \subseteq \Sigma_{pref}$, we can find an AF F' with $pref(F') = \sigma(F)$. By Lemma 7 we can find a symmetric $F^{sym} = (A^{sym}, R^{sym})$ such that $\|pref(F^{sym})\| = \|\sigma(F)\|$ and $|pref(F^{sym})| = |\sigma(F)|$ (as we assume F maximally diverse). In this symmetric AF it holds that $pref(F^{sym}) = \sigma(F^{sym})$. As $\|\sigma(F^{sym})\| = |A^{sym}|$ we can generalize the result of (Baumann and Strass 2014) for $\Delta_{stb}(n)$ to all semantics σ . $\square$

6 Complexity

In this section, we first consider the problem of checking membership in a signature for semantics σ , i.e. given an extension-set $\mathbb{S}$, does $\mathbb{S} \in \Sigma_{\sigma}$ hold? For most of the semantics, it is not hard to see that this can be done in polynomial time in the size of $\mathbb{S}$. The only exception is the naive semantics, since the characterization in Theorem 1 makes use of $dcl(\mathbb{S})$ which is not polynomially bounded in the size of $\mathbb{S}$.

We provide an alternative characterization based on the ternary majority operator maj_3 : given $S_1, S_2, S_3 \subseteq \mathbb{A}$, $maj_3(S_1, S_2, S_3) = (S_1 \cap S_2) \cup (S_2 \cap S_3) \cup (S_1 \cap S_3)$.

Proposition 9. For an incomparable extension-set $\mathbb{S}$ it holds that $dcl(\mathbb{S})$ is tight iff for all $S_1, S_2, S_3 \in \mathbb{S}$ there is an $S \in \mathbb{S}$, such that $maj_3(S_1, S_2, S_3) \subseteq S$.

Proof. Suppose for all $S_1, S_2, S_3 \in \mathbb{S}$ there is some $S \in \mathbb{S}$ such that $maj_3(S_1, S_2, S_3) \subseteq S$. Towards a contradiction assume that the downward-closure of $\mathbb{S}$ is not tight, i.e. there exist $S' \in dcl(\mathbb{S})$ and $a \in Arg_{\mathbb{S}} = Arg_{dcl(\mathbb{S})}$, such that $(S' \cup \{a\}) \notin dcl(\mathbb{S})$ and for all $s \in S'$, $(a, s) \in Pairs_{\mathbb{S}} = Pairs_{dcl(\mathbb{S})}$. Assume $|S'| = 1$, i.e. $S' = \{s\}$. As $(a, s) \in Pairs_{\mathbb{S}}$ by assumption, $\{a, s\} \in dcl(\mathbb{S})$, a contradiction to $S' \cup \{a\} \notin dcl(\mathbb{S})$. Hence $|S'| > 1$, i.e. $S' = \{s_1, \dots, s_n\}$ with $n > 1$. By assumption, $\{s_1, \dots, s_n, a\} \notin \mathbb{S}$, but $(a, s_i) \in Pairs_{\mathbb{S}}$ for each $s_i \in S'$. Hence, for each $s_i \in S'$ there is some $S_i \in \mathbb{S}$ with $\{a, s_i\} \subseteq S_i$ for $i = 1 \dots n$. Moreover $maj_3(S_i, S_j, S') \supseteq \{s_i, s_j, a\}$ for each $i, j \in \{1, \dots, n\}$ since $s_i \in S' \cap S_i$, $s_j \in S' \cap S_j$, and $a \in S_i \cap S_j$. Therefore there is some $S_{ij} \in \mathbb{S}$ with $S_{ij} \supseteq \{s_i, s_j, a\}$. Now for some $k \in \{1, \dots, n\}$, we get $S_{ijk} = maj_3(S_{ij}, S_k, S') \supseteq \{s_i, s_j, s_k, a\}$ and $S_{ijk} \in \mathbb{S}$. Following this procedure for all $1 \dots n$ yields a $T \in \mathbb{S}$ with $T \supseteq \{s_1, \dots, s_n, a\}$, a contradiction to $S' \cup \{a\} \notin dcl(\mathbb{S})$.

On the other hand consider some extension-set $\mathbb{S}$ where $dcl(\mathbb{S})$ is tight and assume, towards a contradiction, sets $S_1, S_2, S_3 \in \mathbb{S}$ such that $maj_3(S_1, S_2, S_3) \not\subseteq S$, for all $S \in \mathbb{S}$. Now, consider the maximal $S' \subset maj_3(S_1, S_2, S_3)$ with $S' \in dcl(\mathbb{S})$, i.e. $\exists a \in maj_3(S_1, S_2, S_3) \setminus S'$ such that $S' \cup \{a\} \notin dcl(\mathbb{S})$. As $S' \cup \{a\} \subseteq maj_3(S_1, S_2, S_3)$, all $s \in (S' \cup \{a\})$ are also contained in $((S_1 \cap S_2) \cup (S_2 \cap S_3) \cup (S_1 \cap S_3))$. Thus for each $s \in S'$ there is some $S_i \in \{S_1, S_2, S_3\}$ with $\{s, a\} \subseteq S_i$. Hence, $(s, a) \in Pairs_{\mathbb{S}}$, so that $dcl(\mathbb{S})$ is not tight, a contradiction. $\square$

Testing the majority criterion is done in polynomial time by looping over all triples (S_1, S_2, S_3) stemming from $\mathbb{S}$. We obtain the following theorem.

Theorem 5. For semantics $\sigma \in \{cf, naive, stb, stage, adm, pref, sem\}$, given $\mathbb{S}$, testing $\mathbb{S} \in \Sigma_{\sigma}$ is in polynomial time.

Next we exploit our results to give a preliminary complexity analysis in terms of the problem of *recasting*: given an AF $F_1 \in AF_{\mathbb{A}}$ and two semantics σ_1, σ_2 , decide whether there exists an $F_2 \in AF_{\mathbb{A}}$, such that $\sigma_1(F_1) = \sigma_2(F_2)$. By the very nature of signatures, this is equivalent to test $\sigma_1(F_1) \in \Sigma_{\sigma_2}$. Table 1 summarizes our results. An entry for row σ_1 and column σ_2 gives the complexity of deciding whether $\sigma_1(F) \in \Sigma_{\sigma_2}$. C-h abbreviates hardness for class C; C-c abbreviates completeness for class C; "trivial" means

Table 1: Complexity of the recasting problem.

	<i>cf</i>	<i>adm</i>	<i>com</i>		
<i>cf</i>	-	trivial	trivial		
<i>adm</i>	coNP-c	-	trivial		
<i>com</i>	coNP-c	coNP-c	-		
	<i>naive</i>	<i>stb</i>	<i>stage</i>	<i>pref</i>	<i>sem</i>
<i>naive</i>	-	trivial	trivial	trivial	trivial
<i>stb</i>	coNP-h	-	NP-c	NP-c	NP-c
<i>stage</i>	coNP-h	trivial	-	trivial	trivial
<i>pref</i>	coNP-h	Π_2^P -c	Π_2^P -c	-	trivial
<i>sem</i>	coNP-h	Π_2^P -c	Π_2^P -c	trivial	-

that each instance is a “Yes”-instance. For both tables, these “trivial” entries are immediate from Theorem 2.

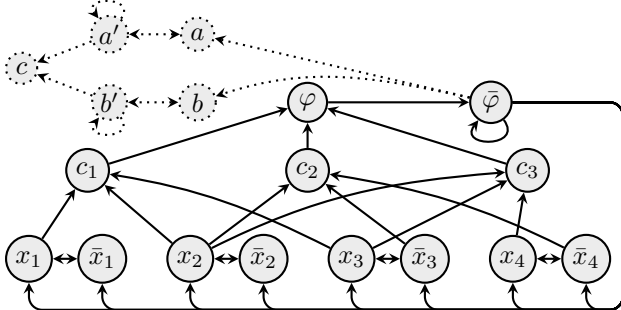
Theorem 6. *The complexity results depicted in Table 1 hold.*

Proof. (Sketch.) Due to the space constraints, we exemplify only one of the coNP-membership arguments for the upper table and provide a non-deterministic algorithm disproving, given an AF F , $com(F) \in \Sigma_{adm}$. Here, we check (i) $\emptyset \notin com(F)$ or (ii) $com(F)$ is not *adm*-closed. Clearly (i) can be done in polynomial time. For (ii) notice that if $com(F)$ is not *adm*-closed then there exist sets $A, B \in com(F)$ such that $A \cup B \in cf(F)$ but $A \cup B \notin com(F)$. But then there exists a set C with $A \cup B \subset C \in com(F)$. So to disprove $com(F)$ being *adm*-closed we guess sets A, B, C and test whether $A, B, C \in com(F)$, $A \cup B \subset C$ but $A \cup B \notin com(F)$.

For coNP-hardness, we first give the result for $\sigma_1 \in \{adm, com\}$ and $\sigma_2 = cf$. We use a standard reduction from CNF formulas $\varphi(X) = \bigwedge_{c \in C} c$ with each clause $c \in C$ a disjunction of literals from X to $F_\varphi = (A_\varphi, R_\varphi)$ given as

$$\begin{aligned}
 A_\varphi &= \{\varphi, \bar{\varphi}\} \cup C \cup X \cup \bar{X} \\
 R_\varphi &= \{(c, \varphi) \mid c \in C\} \cup \{(x, \bar{x}), (\bar{x}, x) \mid x \in X\} \cup \\
 &\quad \{(x, c) \mid x \text{ occurs in } c\} \cup \{(\bar{x}, c) \mid \neg x \text{ occurs in } c\} \cup \\
 &\quad \{(\varphi, \bar{\varphi}), (\bar{\varphi}, \varphi)\} \cup \{(\bar{\varphi}, x), (\varphi, \bar{x}) \mid x \in X\}.
 \end{aligned}$$

We illustrate the framework F_φ (ignore the dashed part for the moment) for the CNF-formula $\varphi = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$.



It holds that φ is satisfiable iff there is an $S \neq \emptyset$ in $\sigma_1(F_\varphi)$ iff there is an $S \in \sigma_1(F_\varphi)$ with $\varphi \in S$ (Dimopoulos and Torres 1996). Now if $adm(F_\varphi) = com(F_\varphi) = \{\emptyset\}$ then $\sigma_1(F_\varphi)$ is clearly *cf*-realizable. On the other hand $\{\varphi\} \notin adm(F)$ (and thus $\{\varphi\} \notin com(F)$). So if there is

an $S \in \sigma_1(F_\varphi)$ with $\varphi \in S$ then $\sigma_1(F_\varphi)$ is not downward closed and thus not *cf*-realizable. Thus we obtain that φ is satisfiable iff $\sigma_1(F_\varphi)$ is not *cf*-realizable.

The result for $\sigma_1 = com$ and $\sigma_2 = adm$ can be shown via an extension of above reduction to $F'_\varphi = (A'_\varphi, R'_\varphi)$ with $A'_\varphi = A_\varphi \cup \{a, a', b, b', c\}$ and $R'_\varphi = R_\varphi \cup \{(a, a'), (a', a), (a', a'), (a', c), (b, b'), (b', b), (b', b'), (b', c), (\bar{\varphi}, a), (\bar{\varphi}, b)\}$. The AF F'_φ for the example CNF is illustrated in the above figure including the dashed part.

We turn to second table and use $\Sigma_{stb} = \Sigma_{stage} \cup \{\emptyset\}$, $\Sigma_{stage} \subset \Sigma_{sem} = \Sigma_{pref}$ from our main theorems. Thus, $stb(F) \in \Sigma_\sigma$ ($\sigma \in \{stage, pref, sem\}$) iff $stb(F) \neq \emptyset$. Deciding whether an AF has a stable extension is known to be NP-complete.

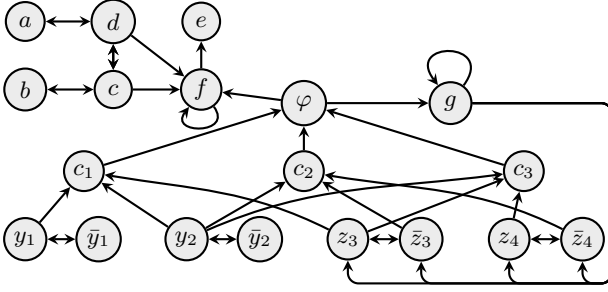
Next, we consider the coNP-hardness of recasting to *naive* semantics. Again, we extend F_φ from above and define $F_\varphi^* = (A_\varphi^*, R_\varphi^*)$ using an additional gadget (see Figure 1) via $A_\varphi^* = A_\varphi \cup \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and $R_\varphi^* = (R_\varphi \setminus \{(\bar{\varphi}, \bar{\varphi})\}) \cup \{(\bar{\varphi}, a_i), (\bar{\varphi}, b_i) \mid 1 \leq i \leq 3\} \cup \{(\bar{\varphi}, \varphi)\} \cup \{(a_i, a_j), (a_i, b_i) \mid 1 \leq i, j \leq 3, i \neq j\}$. It is known that φ is satisfiable iff $\{\bar{\varphi}\} \cup C$ is not the unique preferred extension of F_φ^* (Dunne 2009) (the additional part in the AF does not affect the claim). As $\{\bar{\varphi}\} \cup C$ is also a stable extension this result extends to *stb*, *sem* and *stage* semantics. Let $\sigma \in \{stb, pref, sem, stage\}$. If φ is unsatisfiable then $\{\bar{\varphi}\} \cup C$ is the only $S \in \sigma(F_\varphi^*)$. Otherwise, for each model M of φ , there exist $S_1, S_2, S_3 \in \sigma(F_\varphi^*)$, such that $maj_3(S_1, S_2, S_3) = M \cup \bar{X} \setminus \bar{M} \cup \{\varphi, b_1, b_2, b_3\}$ is a maximal conflict-free set but not admissible and thus not in $\sigma(F_\varphi^*)$. By Proposition 9 and Theorem 1, $\sigma(F_\varphi^*) \notin \Sigma_{naive}$. Hence φ is satisfiable iff $\sigma(F_\varphi^*)$ is not *naive*-realizable.

Finally, we consider the Π_2^P -entries, i.e. $\sigma_1 \in \{pref, sem\}$, $\sigma_2 \in \{stb, stage\}$. Since $\sigma_1(F) \neq \emptyset$ for any AF F , and $\Sigma_{stb} = \Sigma_{stage} \cup \{\emptyset\}$, we can stick to $\sigma_2 = stb$. Membership is by an algorithm that, given an $F = (A, R)$, disproves $\sigma_1(F) \in \Sigma_{stb}$, that is it looks for counter examples (violating tightness) from $\sigma_1(F)$ as follows: guess sets $S \subseteq A$, $\{A_s \subseteq A \mid s \in S\}$ and $a \in A \setminus B$; check $S \in \sigma_1(F)$ and $S \cup \{a\} \notin \sigma_1(F)$, and for all $s \in B$ check $A_s \in \sigma_1(F)$ and $\{a, s\} \subseteq A_s$ (for all checks, an NP-oracle is sufficient (Dunne and Bench-Capon 2002; Dvořák and Woltran 2010)). For sets passing all the checks we have that S and a contradict tightness as $S \cup \{a\} \notin \sigma_1(F)$ but for each $s \in S$, $(a, s) \in Pairs_{\sigma_1(F)}$. By Theorem 1, $\sigma_1(F) \notin \Sigma_{stb}$.

We show Π_2^P -hardness for $\sigma_1 = pref$ (*pref* semantics can be efficiently reduced to *sem* semantics (Dvořák and Woltran 2011), the result for $\sigma_1 = sem$ thus follows): Given QBF $\Phi = \forall Y \exists Z \varphi(Y, Z)$, where φ is a CNF $\bigwedge_{c \in C} c$ with each c a disjunction of literals from $X = Y \cup Z$, let $F_\Phi = (A_\Phi, R_\Phi)$ with $A_\Phi = \{\varphi, g\} \cup C \cup X \cup \bar{X} \cup \{a, b, c, d, e, f\}$ and

$$\begin{aligned}
 R_\Phi &= \{(c, \varphi) \mid c \in C\} \cup \{(x, \bar{x}), (\bar{x}, x) \mid x \in X\} \cup \\
 &\quad \{(x, c) \mid x \text{ occurs in } c\} \cup \{(\bar{x}, c) \mid \neg x \text{ occurs in } c\} \cup \\
 &\quad \{(\varphi, g), (g, g)\} \cup \{(g, z), (g, \bar{z}) \mid z \in Z\} \cup \\
 &\quad \{(a, d), (d, a), (b, c), (c, b), (c, d), (d, c), (c, f), \\
 &\quad (d, f), (f, e), (f, f), (\varphi, f)\}
 \end{aligned}$$

We illustrate F_Φ for the QBF $\Phi = \forall y_1, y_2 \exists z_3, z_4 ((y_1 \vee y_2 \vee z_3) \wedge (y_2 \vee \neg z_3 \vee \neg z_4) \wedge (y_2 \vee z_3 \vee z_4))$.



F_Φ links the reduction from (Dunne and Bench-Capon 2002) with an AF similar to Figure 2. It holds that $\text{pref}(F_\Phi) \in \Sigma_{\text{stb}}$ iff φ is contained in all $E \in \text{pref}(F_\Phi)$. $\square$

7 Discussion

In this work, we initiated a study on the characteristics the sets of extensions w.r.t. a given semantics satisfy. For the semantics naive, stable, stage, preferred, and semi-stable we have an exact picture fully describing their signatures. These results also tell about the limits of global disagreement (a notion introduced in (Booth et al. 2012)) that can be modelled within AFS, e.g. our results show that preferred and semi-stable semantics are able to express more disagreement than stage semantics: $\Sigma_{\text{stage}} \subset \Sigma_{\text{pref}} = \Sigma_{\text{sem}}$.

Besides an exact characterization of the signature of complete semantics, future work includes an investigation of other important semantics, in particular resolution-based grounded (Baroni, Dunne, and Giacomin 2011) and cf2-semantics (Baroni, Giacomin, and Guida 2005), and an according extension of our complexity analysis. On the one hand, exact bounds for the recasting problem involving naive extensions is still missing. On the other hand, the question of how complexity may be affected when $\mathbb{S}$ is specified through some other concise form, e.g. a propositional formula whose models are exactly $S \in \mathbb{S}$ is of some interest and the subject of current investigation. We also want to explore how our results allow for pruning the search space in algorithms for abstract argumentation. Finally, since we have viewed semantics here only in an extension-based way, it would also be of high interest to extend our studies to labelling-based semantics (Caminada and Gabbay 2009).

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