

Lightweight Description Logics and Branching Time: A Troublesome Marriage

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Abstract

We study branching-time temporal description logics (BTDLs) based on the temporal logic CTL in the presence of rigid (*time-invariant*) roles and general TBoxes. There is evidence that, if full CTL is combined with the classical $\mathcal{ALC}$ in this way, reasoning becomes undecidable. In this paper, we begin by substantiating this claim, establishing undecidability for a fragment of this combination. In view of this negative result, we then investigate BTDLs that emerge from combining fragments of CTL with lightweight DLs from the $\mathcal{EL}$ and $DL\text{-}Lite$ families. We show that even rather inexpressive BTDLs based on $\mathcal{EL}$ exhibit very high complexity. Most notably, we identify two *convex* fragments which are undecidable and hard for non-elementary time, respectively. For BTDLs based on $DL\text{-}Lite_{bool}^N$, we obtain tight complexity bounds that range from PSPACE to EXPTIME.

1 Introduction

Classical description logics (DLs) such as those that underlie the W3C recommendation OWL, are decidable fragments of first order logic and aim at the representation of and reasoning about *static* knowledge. With the objective of enhancing DLs with means to capture temporal aspects of knowledge, over the last 20 years much effort has been spent on the study of temporal description logics (TDLs) as discussed in detail in the surveys (Artale and Franconi 2000; Lutz, Wolter, and Zakharyashev 2008) and the references therein. TDLs are useful for applications requiring to describe dynamic aspects of the application domain. For example, in the medical domain the description of a term might refer to temporal patterns; hence a medical ontology like SNOMED CT (Baader, Ghilardi, and Lutz 2008) should model the knowledge that, for example, the disease *diabetes* may potentially lead to several ocular disorders in the future.

A prominent approach to TDLs, following Schild's original proposal (Schild 1993), is to combine classical DLs with the standard temporal logics LTL, CTL and CTL* based on a two-dimensional product-like semantics in the style of many-dimensional DLs (Gabbay et al. 2003). In the construction of temporal DLs, besides the logics used in the temporal and DL component, there are a number of further design choices. For example, we can choose whether

we apply temporal operators to concepts, roles or TBoxes. We can moreover define some DL roles or concepts as rigid, meaning that they will not change their interpretation over time: for example, since every human has the same genetic disorders during their life, the relation “has genetic disorder” between humans and diseases should be modelled by a rigid role *hasGeneticDisease*. Consequently, it has been argued that applications of TDLs require rigid roles and general TBoxes; e.g., in temporal data modeling (Artale et al. 2012) or in medical ontologies such as SNOMED CT.

Remarkably, the landscape of the expressivity and computational complexity of linear-time TDLs based on LTL (LTDLs) is well-understood today (Artale et al. 2007; Baader, Ghilardi, and Lutz 2008; Franconi and Toman 2011; Artale et al. 2012). In particular, various design choices have been explored, and both lightweight and expressive DLs have been considered. Notably, it has been shown that LTDLs based on the basic DL $\mathcal{ALC}$ become undecidable in the presence of a general TBox as soon as temporal operators are applied not only to concepts but also to roles, or rigid roles are allowed.

It is interesting to note that, more generally, combinations of DLs and modal logics allowing for rigid roles and general TBoxes tend to have very high computational complexity. Indeed, the standard reasoning problem for DLs is concept satisfiability with respect to a TBox. Via the well-known correspondence of DL combinations with product modal logics (PMLs) (Gabbay et al. 2003, Chapters 3, 14), this reasoning problem relates to the global consequence problem in a product modal logic. For many PMLs, global consequence is known to be undecidable (Gabbay et al. 2003).

In the context of LTDLs, with the aim to attain decidability in the presence of rigid roles and general TBoxes, recent work investigates lightweight DLs from the $\mathcal{EL}$ and $DL\text{-}Lite$ families as the DL component (Artale et al. 2007; 2012). This choice is supported by the fact that lightweight DLs typically allow for tractable reasoning, and motivated by their importance in applications such as ontology-based data access or the representation of huge bio- and medical ontologies (Baader, Brandt, and Lutz 2005; Artale et al. 2009). The results of this work on LTDLs are two-fold: the studied LTDLs based on $DL\text{-}Lite$ are decidable while those based on $\mathcal{EL}$ are undecidable. Undecidability in the $\mathcal{EL}$ case is explained by an interaction of two phenomena: the first is

non-convexity – roughly speaking, the ability to simulate disjunctions, here using properties of linear-time, as $\Diamond X \sqcap \Diamond Y$ implies that one of $\Diamond(X \sqcap \Diamond Y)$ and $\Diamond(Y \sqcap \Diamond X)$ is true, where $\Diamond X$ reads as *eventually* in the future X holds. The second is the known fact that $\mathcal{EL}$ with disjunctions can encode $\mathcal{ALC}$; hence TDLs based on $\mathcal{EL}$ are as hard as the corresponding $\mathcal{ALC}$ variant, which is undecidable in the case of LTDLs.

From the viewpoint of some DL applications, it has been argued that representing the existence of different *possible futures*, inherent to a branching-time structure, is necessary for a more appropriate modeling (Gutiérrez-Basulto, Jung, and Lutz 2012); for example, in medical ontologies where a symptom or disease might evolve in different ways in the future. For instance consider the statement ‘*each patient having the autoimmune disorder diabetes will possibly develop glaucoma in the future*’. In LTDLs this can be modeled as

Patient $\sqcap \exists \text{hasAutoDis.Diabetes} \sqsubseteq \Diamond \exists \text{develops.Glaucoma}$,

meaning that each diabetes patient will *eventually* develop glaucoma, thus excluding the possibility that a glaucoma will never be developed if diabetes is controlled. In branching-time TDLs we can use the existential path quantifier E together with $\Diamond$ to more carefully state that there exists a *possible future* where the patient develops glaucoma, leaving open the existence of other possible futures.

Interestingly, opposite to LTDLs, the study of TDLs based on branching-time temporal logics (BTDLs) has been rather limited; mainly focusing on decidability boundaries obtained in the context of first-order branching temporal logic (Hodkinson, Wolter, and Zakharyashev 2002; Bauer et al. 2004). Only recently tight elementary bounds were presented for combinations with the DLs $\mathcal{ALC}$ and $\mathcal{EL}$ in the case where only *local* roles are allowed (Gutiérrez-Basulto, Jung, and Lutz 2012). Hence, the aim of this paper is to deepen the study of BTDLs by considering rigid roles, that is, we study the case where temporal operators are applied to concepts, and rigid roles and general TBoxes are allowed. In particular, we will analyze the decidability and complexity of the standard reasoning problems. In the light of the known results from the literature, we will focus on BTDLs based on lightweight DLs from the $\mathcal{EL}$ and $DL\text{-}Lite$ families, but we will lay the groundwork by reconsidering $\mathcal{ALC}$.

Our study starts with the combination of CTL and $\mathcal{ALC}$. We show that its fragment with only the $E\Diamond$, $A\Box$ operators is undecidable, combining results for products of ‘transitive’ modal logics with a well-known DL-technique for reasoning about transitive roles (Tobies 2001). Notably, we obtain a more general result, covering a whole family of PMLs.

We continue by investigating several BTDLs based on $\mathcal{EL}$. We identify non-convex BTDLs based on $\mathcal{EL}$ and thus show undecidability of their subsumption problem similarly to the results described above. As a next step, we focus on identifying convex BTDLs based on $\mathcal{EL}$. Studying them is motivated by the fact that convex logics have a good chance to be computationally well-behaved: examples of convex combinations of DLs with modal logics are the tractable combination of $\mathcal{EL}$ with the CTL operator $E\Diamond$ in the case where only *local* roles are allowed (Gutiérrez-Basulto, Jung, and Lutz 2012), and various combinations

of the modal logic S5 with $\mathcal{EL}$, which have been shown to be easier than the $\mathcal{ALC}$ variant (Lutz and Schröder 2010; Gutiérrez-Basulto et al. 2011). For BTDLs, we indeed identify three convex fragments, based on the temporal operator sets $\{E\Box\}$, $\{E\Diamond\}$, and $\{E\Diamond, A\Box\}$. Surprisingly, we then show that these fragments exhibit very high complexity: undecidable and hard for non-elementary for the $E\Box$ and $E\Diamond$ cases, respectively. The proofs of these results are challenging and technically involved because the absence of disjunction makes it difficult to use standard techniques for establishing lower complexity bounds, such as tilings.

Finally, we investigate BTDLs based on $DL\text{-}Lite_{bool}^N$, showing that the technique developed by Artale et al. (2012) to prove decidability of LTDLs based on $DL\text{-}Lite$ is robust in the sense that it can be carried over to *most* BTDLs based on CTL. We obtain tight complexity bounds, namely PSPACE- and EXPTIME-completeness.

Throughout the paper, we will omit most technical proof details which, together with additional technical notation, can be found in the long version available online: <http://tinyurl.com/kr14tdl>

2 Preliminaries

We introduce $CTL_{\mathcal{ALC}}$, a temporal DL based on the classical DL $\mathcal{ALC}$. The other temporal DLs studied in this paper are fragments or variants of $CTL_{\mathcal{ALC}}$, introduced later as needed. Let N_C and N_R be countably infinite sets of *concept names* and *role names*, respectively. We assume that N_R is partitioned into two countably infinite sets N_{rig} and N_{loc} of *rigid role names* and *local role names*, respectively. $CTL_{\mathcal{ALC}}\text{-concepts}$ C are defined by the following grammar

$$C := \top \mid A \mid \neg C \mid C \sqcap D \mid \exists r.C \mid E\Box C \mid E\Diamond C \mid E(CUD)$$

where A ranges over N_C , r over N_R . As usual, we use $\perp$ to abbreviate the concept $\neg\top$, $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$, and $\forall r.C$ for $\neg\exists r.\neg C$. We additionally have the following temporal abbreviations (Clarke, Grumberg, and Peled 1999): $E\Diamond C = E(\top UC)$, $A\Box C = \neg E\Diamond \neg C$, $A\Diamond C = \neg E\Box \neg C$, $A(CUD) = \neg E(\neg DU(\neg C \wedge \neg D)) \wedge \neg E\Box \neg D$. We moreover consider the *strict* ($\cdot^<$) versions of the temporal operators: $E\Box^< C = E\Box E\Box C$, $E(CU^< D) = E\Box E(CUD)$.

The semantics of classical, non-temporal DLs, such as $\mathcal{ALC}$ is given in terms of *interpretations* of the form $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, where Δ is a non-empty set called the *domain* and $\cdot^{\mathcal{I}}$ is an *interpretation function* that maps each $A \in N_C$ to a subset $A^{\mathcal{I}} \subseteq \Delta$ and each $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta \times \Delta$. The semantics of $CTL_{\mathcal{ALC}}$ is given in terms of temporal interpretations, which are infinite trees in which every node is associated with a classical interpretation. For the purposes of this paper, a *tree* is a directed graph $T = (W, E)$ where $W \subseteq (\mathbb{N} \setminus \{0\})^*$ is a prefix-closed non-empty set of *nodes* and $E = \{(w, wc) \mid wc \in W, w \in W^*, c \in \mathbb{N}\}$ a set of *edges*; we generally assume that $wc \in W$ and $c' < c$ implies $wc' \in W$ and that E is a total relation. The node $\varepsilon \in W$ is the *root* of T . For brevity and since E can be reconstructed from W , we will usually identify T with W . In the context of temporal DLs we refer to nodes of T as *time points or worlds*.

A *temporal interpretation* is a structure $\mathcal{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ where $T = (W, E)$ is a tree, and for each $w \in W$, $\mathcal{I}_w$ is an interpretation with domain Δ , such that $r^{\mathcal{I}_w} = r^{\mathcal{I}_{w'}}$ for all $r \in \mathbb{N}_{\text{rig}}$ and $w, w' \in W$. We usually write $A^{\mathcal{I}, w}$ instead of $A^{\mathcal{I}_w}$, and intuitively $d \in A^{\mathcal{I}, w}$ means that in the interpretation $\mathcal{I}$, the object d is an instance of the concept name A at time point w . Moreover, note that we make the *constant domain assumption*, that is, all time points share the same domain Δ . Intuitively, this means that objects are not created or destroyed over time.

We now define the semantics of $\text{CTL}_{\mathcal{ALC}}$ -concepts. A *path* in a tree $T = (W, E)$ starting at a node w is a minimal set $\pi \subseteq W$ such that $w \in \pi$ and for each $w' \in \pi$, there is a $c \in \mathbb{N}$ with $w'c \in \pi$. We use $\text{Paths}(w)$ to denote the set of all paths starting at the node w . For a path $\pi = w_0w_1w_2 \dots$ and $i \geq 0$, we use $\pi[i]$ to denote w_i . The mapping $\cdot^{\mathcal{I}, w}$ is extended from concept names to $\text{CTL}_{\mathcal{ALC}}$ -concepts as follows:

$$\begin{aligned} \top^{\mathcal{I}, w} &= \Delta; & (C \sqcap D)^{\mathcal{I}, w} &= C^{\mathcal{I}, w} \cap D^{\mathcal{I}, w}; \\ (\exists r.C)^{\mathcal{I}, w} &= \{d \in \Delta \mid \exists e.(d, e) \in r^{\mathcal{I}, w} \wedge e \in C^{\mathcal{I}, w}\}; \\ (\text{EO}C)^{\mathcal{I}, w} &= \{d \in \Delta \mid d \in C^{\mathcal{I}, \pi[1]} \text{ for some } \pi \in \text{Paths}(w)\}; \\ (\text{E} \square C)^{\mathcal{I}, w} &= \{d \in \Delta \mid \forall j \geq 0. d \in C^{\mathcal{I}, \pi[j]} \text{ for some } \pi \in \text{Paths}(w)\}; \\ (\text{E}(CUD))^{\mathcal{I}, w} &= \{d \in \Delta \mid \exists j \geq 0. (d \in D^{\mathcal{I}, \pi[j]} \wedge \\ &\quad (\forall 0 \leq k < j. d \in C^{\mathcal{I}, \pi[k]})) \\ &\quad \text{for some } \pi \in \text{Paths}(w)\}. \end{aligned}$$

A general $\text{CTL}_{\mathcal{ALC}}$ -TBox $\mathcal{T}$ is a finite set of *concept inclusions* (CIs) $C \sqsubseteq D$ with $C, D \text{ CTL}_{\mathcal{ALC}}$ concepts.

A temporal interpretation $\mathcal{I}$ is a *model* of a concept C if $C^{\mathcal{I}, \varepsilon} \neq \emptyset$; it is a *model* of $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}, w} \subseteq D^{\mathcal{I}, w}$ for all $w \in W$; it is a model of a TBox $\mathcal{T}$ if it satisfies all CIs in $\mathcal{T}$. Thus, a TBox $\mathcal{T}$ is interpreted globally in the sense that it has to be satisfied at *every* time point.

We say that a concept C is *satisfiable with respect to a* $\text{CTL}_{\mathcal{ALC}}$ -TBox $\mathcal{T}$ if there is a common model of C and $\mathcal{T}$. We then consider the following *concept satisfiability problem*: given a concept C and a TBox $\mathcal{T}$, decide whether there is a common model of C and $\mathcal{T}$.

We will denote fragments of $\text{CTL}_{\mathcal{ALC}}$ by putting the available temporal operators as a superscript; for example, $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ denotes the fragment having as the only operators available $\text{E}\diamond$ and $\text{A}\square$.

3 Undecidability of CTL combined with $\mathcal{ALC}$

We begin our study by looking at $\text{CTL}_{\mathcal{ALC}}$. To our knowledge, there are no decidability or complexity results specifically for this combination in the literature. However, there are two results about logics that are close to $\text{CTL}_{\mathcal{ALC}}$: On the one hand, it follows from results by (Hodkinson, Wolter, and Zakharyashev 2002) that a combination of the two-variable guarded fragment of first-order logic (which subsumes $\mathcal{ALC}$) with CTL is undecidable, but that logic is more expressive than our $\text{CTL}_{\mathcal{ALC}}$: for example, it allows for CIs in the scope of temporal operators. On the other hand, there is a close correspondence between $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ and the

PML $\text{S4} \times \text{K}$, whose classical satisfiability problem is decidable (Gabbay et al. 2003), though hard for non-elementary time (Göller, Jung, and Lohrey 2012). This correspondence is based on the fact that $\text{S4} \times \text{K}$ has the tree model property and thus $\text{S4} \times \text{K}$ is the logic of all bi-modal Kripke frames (relational structures) that are the product of the reflexive and transitive closure of a tree (S4) and a tree (K). Therefore, in $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ with one rigid role and no further roles, the CTL branching time structures correspond to the S4 -dimension and $\mathcal{ALC}$ corresponds to K .

The correspondence with PMLs, however, is only of restricted use for our study of the concept satisfiability problem *relative to global TBoxes*: this problem corresponds to the global consequence problem in modal logic which, to the best of our knowledge, has not been studied for $\text{S4} \times \text{K}$. Therefore, the above result for classical satisfiability cannot easily be transferred to $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$. We will in fact show that concept satisfiability relative to global TBoxes in $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ is undecidable, exploiting undecidability of satisfiability for a family of transitive PMLs which includes $\text{S4} \times \text{K4}$ (Gabelaia et al. 2005). We prove that the global consequence problem of $\text{S4} \times \text{K}$ can encode the satisfiability problem of $\text{S4} \times \text{K4}$, using a technique by Tobies (2001).

We first rephrase a consequence of Gabelaia et al.'s result in BTDL notation and will then sketch how to employ Tobies's technique to establish undecidability of $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$.

Theorem 1 (Gabelaia et al. 2005) *The following problem is undecidable for $\mathcal{L} \in \{\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}, \text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square, \text{A}\square^<}\}$. Given an $\mathcal{L}$ -concept C that uses at most one rigid role r and no other roles, is C satisfiable in a temporal interpretation $\mathcal{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ where $r^{\mathcal{I}, w}$ is transitive for all w ?*

We sketch how to employ Tobies's technique to transfer this result to our standard concept satisfiability problem for $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ and $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square, \text{A}\square^<}$.

Theorem 2 *Concept satisfiability w.r.t. general TBoxes for $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ and $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square, \text{A}\square^<}$ with rigid roles is undecidable.*

Proof. We only sketch the proof for $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$; the strict variant can be treated using the same arguments. Consider a $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ -concept C that uses at most one rigid role r and no other roles, and assume that C is in negation normal form (NNF), i.e., negation occurs only in front of atomic concepts in C . We use $\text{cl}(C)$ to denote the set of NNFs of all subconcepts of C and of their negations, and $\sim D$ to denote the NNF of $\neg D$ for any $D \in \text{cl}(C)$. For every $\forall r.D \in \text{cl}(C)$, reserve a fresh concept name X_D . We define a translation $\cdot^{\text{tr}}$ from $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ -concepts to $\text{CTL}_{\mathcal{ALC}}^{\text{E}\diamond, \text{A}\square}$ -concepts.

$$\begin{aligned} A^{\text{tr}} &= A^{\text{tr}} & (\forall r.D)^{\text{tr}} &= X_D \\ (\neg A)^{\text{tr}} &= \neg A^{\text{tr}} & (\exists r.D)^{\text{tr}} &= \neg X_{\sim D} \\ (D_1 \sqcap D_2)^{\text{tr}} &= D_1^{\text{tr}} \sqcap D_2^{\text{tr}} & (\text{E}\diamond D)^{\text{tr}} &= \text{E}\diamond D \\ (D_1 \sqcup D_2)^{\text{tr}} &= D_1^{\text{tr}} \sqcup D_2^{\text{tr}} & (\text{A}\square D)^{\text{tr}} &= \text{A}\square D \end{aligned}$$

Let $\mathcal{T}_C$ be the following TBox.

$$\mathcal{T}_C = \{X_D \equiv \forall r.D^{\text{tr}}, X_D \sqsubseteq \forall r.X_D \mid \forall r.D \in \text{cl}(C)\}$$

It suffices to show that C is satisfiable in an interpretation that makes r transitive everywhere iff C^{tr} is satisfiable w.r.t. $\mathcal{T}_C$ in $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond, \mathbf{A}\square}$. For the “ $\Rightarrow$ ” direction, take a model $\mathfrak{J}$ of C with r being transitive, and additionally interpret X_D as $\forall r.D$. Then a standard induction shows that the modified $\mathfrak{J}$ satisfies C^{tr} , and transitivity of r ensures that it is a model of $\mathcal{T}_C$. For the “ $\Leftarrow$ ” direction, take a model $\mathfrak{J}$ of C^{tr} and $\mathcal{T}_C$ and extend the interpretation of r in every world to its transitive closure. Then $\mathfrak{J}$ being a model of $\mathcal{T}_C$ can be used to show that the modified $\mathfrak{J}$ still satisfies C . $\square$

In fact, we are able to prove a more general result of independent interest in the field of PMLs. All notation used in the following theorem is standard for PMLs (Gabbay et al. 2003), except for $\mathfrak{C}^+$, which refers to the class of all transitive frames in the frame class $\mathfrak{C}$.

Theorem 3 *For any classes $\mathfrak{C}_1, \mathfrak{C}_2$ of frames, where $\mathfrak{C}_2$ contains only transitive frames and both $\mathfrak{C}_1^+, \mathfrak{C}_2$ contain frames of arbitrarily large finite or infinite depth, the global satisfiability problem for $\text{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is undecidable.*

4 CTL combined with $\mathcal{EL}$

In this section, we consider fragments of $\text{CTL}_{\mathcal{EL}}$, the fragment of $\text{CTL}_{\mathcal{ALC}}$ that disallows the constructor $\neg$ (and thus also the abbreviations $C \sqcup D, \forall r.C, \mathbf{A}\square, \dots$). As an example, consider the following $\text{CTL}_{\mathcal{EL}}$ -TBox:

$$\exists \text{hasAutoDis.Diabetes} \sqsubseteq \mathbf{E}\diamond(\exists \text{develops.Glaucoma} \sqcap \mathbf{E}\diamond \exists \text{hasTreat.EyeDrops})$$

$$\mathbf{E}\diamond \exists \text{hasTreat.EyeDrops} \sqsubseteq \mathbf{E}\diamond \exists \text{hasEffect.ChangeEyeColor}$$

It says that everyone with diabetes may eventually develop a glaucoma which may eventually be treated using eye drops, and that such a treatment may lead to a change in eye color.

Because of the absence of negation, satisfiability in $\text{CTL}_{\mathcal{EL}}$ is trivial; as in non-temporal $\mathcal{EL}$, we therefore consider subsumption as the central reasoning problem. Formally, a concept D *subsumes* a concept C w.r.t. a $\text{CTL}_{\mathcal{EL}}$ TBox $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $C^{\mathfrak{J}} \subseteq D^{\mathfrak{J}}$ for all temporal interpretations $\mathfrak{J}$ that are a model of $\mathcal{T}$. For example, the TBox above implies that $\exists \text{hasAutoDis.Diabetes} \sqsubseteq \mathbf{E}\diamond \exists \text{hasEffect.ChangeEyeColor}$. We then consider the following *concept subsumption problem*: given concepts C, D and a $\text{CTL}_{\mathcal{EL}}$ TBox $\mathcal{T}$, decide whether $\mathcal{T} \models C \sqsubseteq D$.

In the rest of the section we focus on the study of several fragments of $\text{CTL}_{\mathcal{EL}}$. In this context, we view each of the operators $\mathbf{E}\diamond, \mathbf{E}\square, \mathbf{A}\diamond, \mathbf{A}\square, \mathbf{E}\mathcal{U}$ and their strict versions as primitive instead of as an abbreviation.

4.1 Non-Convex Fragments

We next show undecidability of certain fragments of $\text{CTL}_{\mathcal{EL}}$ using techniques developed in the context of *linear-time* temporal extensions of $\mathcal{EL}$ (Artale et al. 2007). In particular, the non-convexity of these extensions, and therefore their capability to ‘express’ disjunction, is exploited to show that they are as complex as the corresponding $\mathcal{ALC}$ variant.

Before we present the undecidability results we recall the standard notion of convexity.

Definition 1 *A logic $\mathcal{L}$ is called convex if, for all $\mathcal{L}$ -TBoxes $\mathcal{T}$ and all $\mathcal{L}$ -concepts $C, D_1, \dots, D_n$, $n \geq 2$, whenever $\mathcal{T} \models C \sqsubseteq D_1 \sqcup \dots \sqcup D_n$, then $\mathcal{T} \models C \sqsubseteq D_i$ for some $i \in \{1, \dots, n\}$.*

Theorem 4 *Concept subsumption w.r.t. general TBoxes is undecidable for*

- | | | |
|--|---|---|
| (a) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{A}\diamond}$ | (b) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{E}\square}$ | (c) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\mathcal{U}}$ |
| (d) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\mathcal{U}^<}$ | (e) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond^<, \mathbf{E}\square^<}$ | (f) $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{E}\square}$ |

Proof. The proof is in two steps. First, we show the non-convexity (in opposition to convexity) of these fragments of $\text{CTL}_{\mathcal{EL}}$. Second, we reduce from satisfiability w.r.t. TBoxes in the corresponding $\text{CTL}_{\mathcal{ALC}}$ fragment.

Lemma 5 *The fragments (a)–(f) of $\text{CTL}_{\mathcal{EL}}$ are non-convex.*

Proof. The non-convexity of (a)–(d) was previously established in (Gutiérrez-Basulto, Jung, and Lutz 2012), for the witnesses see Figure 1. In the same place, the fragments (e) and (f) were stated to be convex, too; however, we show here that this is not the case. Interestingly, this shows that the convexity of $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond}$, which we show later (Theorem 6), is rather fragile since $\mathbf{E}\diamond$ and $\mathbf{E}\square$ allow only for existential quantification and there is no possibility of talking about ‘next’ time points, which in principle might make us think of $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{E}\square}$ as a temporal analog of $\mathcal{EL}$. Alas, the presence of both operators makes it non-convex. We begin by presenting the witnesses used to prove non-convexity and their intuition.

To show non-convexity of $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{E}\square}$ we use the following witness.

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\diamond(D \sqcap A), D \sqsubseteq \mathbf{E}\diamond(C \sqcap A)\} \text{ and } D_1 = \mathbf{E}\diamond(C \sqcap D), D_2 = \mathbf{E}\square \mathbf{E}\diamond A.$$

Intuitively, $\mathcal{T}$ stipulates that every instance d of C has the possibility of eventually being an instance of D and A , and thereafter has the possibility of eventually being an instance of C and A , but there is a choice on whether d becomes an instance of C and D at the same time point, or whether d infinitely alternates between C and D .

Although the disjunction expressed in $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond, \mathbf{E}\square}$ relies on the non-strict interpretation of $\mathbf{E}\diamond$ and $\mathbf{E}\square$, the following shows that the strict version is also non-convex.

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\diamond^< B, B \sqsubseteq D \sqcap \mathbf{E}\square^< D\} \text{ and } D_1 = \mathbf{E}\diamond^< \mathbf{E}\diamond^< B, D_2 = \mathbf{E}\square^< D$$

Intuitively, every instance d of C has two choices on becoming an instance of B for the first time, namely, either it does it at the immediate successor, or at some point after it.

Formally, in both cases we show non-convexity by proving that $\mathcal{T} \models C \sqsubseteq \sqcup D_i$, but $\mathcal{T} \not\models C \sqsubseteq D_i$. $\square$

Having established the non-convexity of the above logics we can devise a reduction from the satisfiability problem w.r.t. TBoxes in the corresponding $\mathcal{ALC}$ variant. First, recall that $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond, \mathbf{A}\square}$ and $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond^<, \mathbf{A}\square^<}$ are undecidable (Theorem 2). Note moreover that all fragments above contain at least the $\mathbf{E}\diamond$ or $\mathbf{E}\diamond^<$ operator. Hence all the $\mathcal{ALC}$ variants of these fragments are undecidable.

$E \Diamond A \Diamond$	$\mathcal{T} = \emptyset, C = A \Diamond A \sqcap A \Diamond B$ $D_1 = E \Diamond (A \sqcap E \Diamond B), D_2 = E \Diamond (B \sqcap E \Diamond A)$
$E \Diamond E \Diamond$	$\mathcal{T} = \emptyset, C = E \Diamond A, D_1 = A, D_2 = E \Diamond E \Diamond A$
EU	$\mathcal{T} = \emptyset, C = E(AUB), D_1 = B, D_2 = A$
$EU^<$	$\mathcal{T} = \emptyset, C = E(AU^<B), D_1 = E(AU^<A),$ $D_2 = E(BU^<B)$

Figure 1: Non-convexity witnesses

The crucial point of the reduction is the elimination of disjunction, that is, encoding CIs $C \sqsubseteq A \sqcup B$ present in the $\mathcal{ALC}$ variants. To this aim, we use the non-convexity witnesses (cf. witnesses presented above and Fig. 1) which, as discussed, allow to describe ‘choices’.

4.2 Convex Fragments

In this section, we will show that the logics $CTL_{\mathcal{EL}}^{E \Diamond}, CTL_{\mathcal{EL}}^{E \Diamond \Diamond},$ and $CTL_{\mathcal{EL}}^{E \Diamond A \sqcap}$ are convex, which suggests that they have limited expressive power. Indeed, convex logics are typically computationally well-behaved (Oppen 1980; Baader, Brandt, and Lutz 2005; Lutz and Wolter 2012). Our convexity result is contrasted by two observations: first, the linear-time TDL based on $\Diamond$ and $\mathcal{EL}$ with rigid roles is non-convex (Artale et al. 2007) which, intuitively, is due to the linear structure of time. Second, we will show later that $CTL_{\mathcal{EL}}^{E \Diamond}$ is undecidable because its expressive power is sufficient to encode the halting problem for two-counter machines, which is equivalent to that of Turing machines.

Theorem 6 $CTL_{\mathcal{EL}}^{E \Diamond}, CTL_{\mathcal{EL}}^{E \Diamond \Diamond},$ and $CTL_{\mathcal{EL}}^{E \Diamond A \sqcap}$ are convex.

Proof. (sketch) We start with $CTL_{\mathcal{EL}}^{E \Diamond}$ and first observe that $CTL_{\mathcal{EL}}^{E \Diamond}$, if restricted to a single role that is rigid, is a notational variant of a fragment of the product modal logic $KD \times K$, where K is the basic (uni-)modal logic and KD is the modal logic of the class of frames where every state has a successor. We can therefore use standard results and machinery from product modal logics – namely unravelling tolerance, the standard translation to first-order logic, and first-order axiomatizations of product frames – to achieve two things: we will translate $CTL_{\mathcal{EL}}^{E \Diamond}$ -TBoxes and -concepts into a fragment of first-order logic that generalizes the Horn fragment, and we will axiomatize the rigidity of roles and the properties of the temporal successor relation in the same fragment. This will allow us to encode logical consequence in $CTL_{\mathcal{EL}}^{E \Diamond}$, even with multiple non-rigid roles, as first-order formulas in this fragment. Using the fact that such formulas are preserved under a standard operation on structures called *direct product*, it will then be easy to establish convexity.

In the following, we sketch some details of this argument. Consider a $CTL_{\mathcal{EL}}^{E \Diamond}$ -TBox $\mathcal{T}$ and $CTL_{\mathcal{EL}}^{E \Diamond}$ -concepts $C, D_1, \dots, D_n$, $n \geq 2$. Let $r_1, \dots, r_k$ be the rigid roles

occurring in $\mathcal{T}, C, D_1, \dots, D_n$, and let $r_{k+1}, \dots, r_\ell$ be the non-rigid roles. We denote by $ST(\mathcal{T})$ the standard translation of $\mathcal{T}$ into first-order logic. Assuming $\mathcal{T}$ to be in a normal form that extends the usual one for $\mathcal{EL}$ (cf. Baader, Brandt, and Lutz 2005), it is easy to see that $ST(\mathcal{T})$ is equivalent to a conjunction of embedded implicational dependencies (EIDs) (Fagin 1982), i.e., to a conjunction of FO-sentences

$$\forall x_1 \dots x_m. ((E_1 \wedge \dots \wedge E_n) \rightarrow \exists x_{m+1} \dots x_k. (F_1 \wedge \dots \wedge F_\ell))$$

where each E_i is a relational formula $Px_{j_1} \dots x_{j_d}$ and each F_i is either a relational formula or an equality $x_j = x_{j'}$.

Furthermore, we need to express rigidity of the roles $r_1, \dots, r_k$ as well as the property that the temporal “direct successor” relation is a total tree. For rigidity, we borrow from the theory of product modal logics (Gabbay et al. 2003): we have to say that each pair (R_i, S) of binary predicates representing the rigid role r_i and the temporal successor relation is left-commutative, right-commutative and satisfies the Church-Rosser property. For example, left-commutativity is given by the following EID.

$$LC = \bigwedge_{i=1}^k \forall xyz (R_i xy \wedge Syz \rightarrow \exists u (Sxu \wedge R_i uz))$$

The other two properties can be expressed via analogous EIDs RC and CRP . For enforcing the total tree, we only need to say that S is total because standard translations of $CTL_{\mathcal{EL}}^{E \Diamond}$ -TBoxes are preserved under unravelling and under taking point-generated substructures. Totality of S can be expressed by the EID $D = \forall x \exists y (Sxy)$.

We now consider the conjunction

$$\varphi(\mathcal{T}) = LC \wedge RC \wedge CRP \wedge D \wedge ST(\mathcal{T})$$

which, by classical results (Chang and Keisler 1990; Fagin 1982) is preserved under *direct products*: If n structures $\mathcal{M}_1, \dots, \mathcal{M}_n$ satisfy an EID, then so does their direct product, which has as its domain the cross-product of the domains of $\mathcal{M}_1, \dots, \mathcal{M}_n$ and interprets every m -ary predicate P as the set of m -tuples whose projections to the domain of each $\mathcal{M}_i$ are in the interpretation of P .

In order to show convexity of $CTL_{\mathcal{EL}}^{E \Diamond}$ via the contrapositive, we observe that the statement

$$\mathcal{T} \not\models C \sqsubseteq D_i \text{ for all } i \text{ implies } \mathcal{T} \not\models C \sqsubseteq D_1 \sqcup \dots \sqcup D_n$$

is equivalent to

$$\begin{aligned} \varphi(\mathcal{T}) \not\models \forall x. (Cx \rightarrow D_i x) \text{ for all } i & \text{ implies} \\ \varphi(\mathcal{T}) \not\models \forall x. (Cx \rightarrow D_1 x \vee \dots \vee D_n x). \end{aligned}$$

The claim of this property can be established easily, consulting the direct product of the n structures witnessing the non-entailments in the hypotheses.

To carry this reasoning over to $CTL_{\mathcal{EL}}^{E \Diamond \Diamond}$ and $CTL_{\mathcal{EL}}^{E \Diamond A \sqcap}$, we replace totality in the above argument with reflexivity and transitivity. Consequently, we have to replace the conjunct D in $\varphi(\mathcal{T})$ with the conjunction of the EIDs $T = \forall x (xR_h x)$ and $4 = \forall xyz (xR_h y \wedge yR_h z \rightarrow xR_h z)$. Unfortunately, $ST(\mathcal{T})$ now contains translations of axioms containing $A \sqcap$, which are not generally equivalent to EIDs. Using additional arguments, however, we can show that $ST(\mathcal{T})$ for these two

logics is then preserved under direct products of structures where the relation S representing the temporal successor relation is interpreted as a total relation. Therefore, the above $\varphi(\mathcal{T})$ remains preserved under direct products of such structures. Since $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ and $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{A}\square}$ are restricted to such structures, the above argument goes through.

It is worth recalling that convexity of both $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ and $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ does not imply that $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{EO}}$ is convex: in order to translate a TBox using both the EO and the EO operator into FO, we would have to use two binary relations S_{EO} and S_{EO} and state that S_{EO} is the reflexive transitive closure of S_{EO} , which is not expressible in FO. $\square$

We continue our study by showing that $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ lacks the finite model property (FMP): consider the TBox $\mathcal{T}$ consisting of the following concept inclusions

$$\begin{aligned} A \sqsubseteq \exists r.A, \quad A \sqsubseteq \text{EO}B, \quad \exists r.B \sqsubseteq B', \\ \exists r.B' \sqsubseteq B', \quad \text{EO}(B \sqcap B') \sqsubseteq C, \quad \exists r.C \sqsubseteq C, \end{aligned}$$

with r a rigid role. It is not hard to see that $\mathcal{T} \not\models A \sqsubseteq C$ but for every finite model $\mathfrak{J}$ of $\mathcal{T}$, $\mathfrak{J} \models A \sqsubseteq C$. In fact, we show that this also holds if the two occurrences of EO in $\mathcal{T}$ are replaced by EO . Therefore, the following holds.

Theorem 7 $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ and $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{EO}}$ lack the FMP.

Interestingly, in the same way the TBox presented above enforces a temporal tree with infinite branching degree, relying on rigid roles.

We next study the complexity of subsumption relative to general TBoxes in $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ and $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{EO}}$. In Theorem 2, we already showed that the $\mathcal{ALC}$ variant of the latter is undecidable. For the former, observe that the corresponding $\mathcal{ALC}$ -version is a notational variant of the product logic $\mathbf{K} \times \mathbf{K}$, for which the global consequence problem (and thus reasoning relative to global TBoxes) is known to be undecidable.

We will prove undecidability for $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ and a nonelementary lower bound for $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{EO}}$, thus showing that even the convex fragments have a high complexity. These high complexity results are surprising because convex logics often exhibit better computational properties. Specifically, the lack of expressiveness prohibits reductions that were previously used to show undecidability particularly for product modal logics, for instance reductions from tiling problems: without disjunction we cannot express that in some point of the grid one of the tile types has to be true. However, we show how to exploit rigidity of the roles for proving these results. Let us remark that the proven lower bounds remain valid when we adopt the *expanding domain assumption* (Gabbay et al. 2003); in particular, the same proofs work.

4.3 Undecidability with the Next Operator

We proceed to show undecidability of the combination of $\mathcal{EL}$ with EO .

Theorem 8 Concept subsumption w.r.t. general $\text{CTL}_{\mathcal{EL}}^{\text{EO}, \text{EO}}$ -TBoxes is undecidable.

We present a reduction from the reachability problem for two-counter machines (TCM). A TCM is a pair (Q, P) with states $Q = \{0, \dots, n\}$, where 0 is the initial state and n is the halting state, and corresponding instructions $P = \{I_0, \dots, I_{n-1}\}$. Additionally, there are two registers 1 and 2. For all $i < n$, we have that

- either $I_i = \text{inc}(c, j)$ is an *incrementation instruction* with $c \in \{1, 2\}$ a register and j the subsequent state;
- or $I_i = \text{dec}(c, j, k)$ is a *decrementation instruction* with $c \in \{1, 2\}$ a register and j the subsequent state if register c contains value 0 and k the subsequent state otherwise.

A configuration is a triple (i, n, m) where i is the current state and n and m are the contents of registers 1 and 2, respectively. We write $(i, n_1, n_2) \Rightarrow_M (j, m_1, m_2)$ if one of the following holds:

- $I_i = \text{inc}(c, j)$, $m_c = n_c + 1$ and $m_{\bar{c}} = n_{\bar{c}}$ (where $\bar{c} = 3 - c$);
- $I_i = \text{dec}(c, j, k)$, $m_c = n_c = 0$ and $m_{\bar{c}} = n_{\bar{c}}$;
- $I_i = \text{dec}(c, k, j)$, $n_c > 0$, $m_c = n_c - 1$, and $m_{\bar{c}} = n_{\bar{c}}$.

The computation of M is the unique longest configuration sequence $(p_0, n_0, m_0) \Rightarrow_M (p_1, n_1, m_1) \Rightarrow_M \dots$ such that $p_0 = n_0 = m_0 = 0$. The halting problem for TCM, i.e., to decide whether a given TCM M reaches state n , is undecidable (Minsky 1967).

For our purposes, it is convenient to define a variant of the halting problem. Fix some TCM $M = (Q, I)$ with $Q = \{0, \dots, n\}$. An extended state (*e-state*) x for M is either $x = q$ for some $q \in Q$ with I_q an increment instruction or a pair $x = (q, f)$ with $f \in \{0, +\}$ and I_q a decrement instruction. Denote with $\bar{x}$ the state q if $x = q \in Q$ or $x = (q, f)$. A finite sequence of e-states $x_0, \dots, x_k$ is called *halting* if $x_k = n$ and there are numbers $n_0, m_0, \dots, n_k, m_k$ such that $x_0 = n_0 = m_0 = 0$ and $(\bar{x}_i, n_i, m_i) \Rightarrow_M (\bar{x}_{i+1}, n_{i+1}, m_{i+1})$ for all i , and additionally, if $x_i = (q, 0)$ and $I_q = \text{dec}(c, \cdot, \cdot)$ then either $c = 1$ and $n_i = 0$ or $c = 2$ and $m_i = 0$, and if $x_i = (q, +)$ and $I_q = \text{dec}(c, \cdot, \cdot)$, then either $c = 1$ and $n_i > 0$ or $c = 2$ and $m_i > 0$. It is easy to verify that M halts iff there is a halting sequence of e-states.

For the reduction let M be as above and assume without loss of generality that I_0 is an inc-instruction. Moreover, define a set of rigid roles, consisting of one role for each e-state:

$$\text{ROL} = \{r_i \mid I_i = \text{inc}(\cdot, \cdot)\} \cup \{r_{i0}, r_{i+} \mid I_i = \text{dec}(\cdot, \cdot, \cdot)\}.$$

Finally, introduce concept names $Q_0, \dots, Q_n$ representing the states of M , for each $c \in \{1, 2\}$ concepts C_c marking elements that contribute to the value in register c and concepts Tail_c and Head_c that mark the bounds of register c .

We construct a $\text{CTL}_{\mathcal{EL}}^{\text{EO}}$ -TBox $\mathcal{T}_M$ and identify two concept names A, B such that:

Lemma 9 $\mathcal{T}_M \models A \sqsubseteq B$ iff there is a halting sequence of e-states.

The idea is (i) to generate all sequences of e-states for M using the roles in ROL and (ii) to check if one of them is halting. Point (i) is done using the concept inclusions

$$A \sqsubseteq \exists r_0.(S \sqcap A), \quad A \sqsubseteq \exists r.A \text{ for all } r \in \text{ROL} \setminus \{r_0\}$$

where the first inclusion additionally ensures that after creating an r_0 successor, some special concept name S is satisfied. Intuitively, domain elements satisfying A in some world stipulate an infinite tree where the path to each node corresponds to *the reverse of a sequence of e-states*. For example, the branch $r_{3+}r_{20}r_{2+}r_1r_0$ of such a tree corresponds to the sequence $0, 1, (2, +), (2, 0), (3, +)$ of e-states for M .

For achieving Point (ii), we simulate the behavior of M *backwards* over the tree mentioned above. A configuration (i, n_1, n_2) of M will be represented in the temporal dimension by a time point that has a (time) successor for each register: one successor labeled with Head_1 and another with Head_2 . Both successors have some (not necessarily direct) successor that is labeled with Tail_c and all worlds between Tail_c and Head_c are labeled with C_c . Further, the distance between Tail_c and Head_c is precisely the value n_c of the register c . Moreover, all these worlds are labeled with the current state Q_i .

Note that a valid computation of M always starts in state 0 and that by the above CI all corresponding nodes are labeled with S . Thus, we can use

$$S \sqsubseteq \mathbf{EO} \prod_{c \in \{1,2\}} \mathbf{EO}(\text{Head}_c \sqcap \text{Tail}_c \sqcap Q_0)$$

in order to enforce (the encoding of) an initial configuration of M starting in every node labeled with S . The begin of the counter never changes along a computation:

$$\exists r. \text{Head}_c \sqsubseteq \text{Head}_c \text{ for all } r \in \text{ROL}, c \in \{1, 2\}.$$

It remains to describe how the ends Tail_c , the states Q_i , and the intermediate labels C_c are updated according to the instructions I . For instructions $I_i = \text{inc}(c, j)$, we add the inclusions

$$\begin{aligned} \exists r_i. (Q_i \sqcap \text{Tail}_c) &\sqsubseteq Q_j \sqcap C_c \sqcap \mathbf{EO}(Q_j \sqcap \text{Tail}_c) \\ \exists r_i. (Q_i \sqcap C_c) &\sqsubseteq Q_j \sqcap C_c \\ \exists r_i. (Q_i \sqcap \text{Tail}_{\bar{c}}) &\sqsubseteq Q_j \sqcap \text{Tail}_{\bar{c}} \\ \exists r_i. (Q_i \sqcap C_{\bar{c}}) &\sqsubseteq Q_j \sqcap C_{\bar{c}} \end{aligned}$$

For instructions $I_i = \text{dec}(c, j, k)$, add the inclusions

$$\begin{aligned} \exists r_{i0}. (Q_i \sqcap \text{Head}_c \sqcap \text{Tail}_c) &\sqsubseteq Q_j \sqcap \text{Tail}_c \\ \exists r_{i0}. (Q_i \sqcap C_{\bar{c}}) &\sqsubseteq Q_j \sqcap C_{\bar{c}} \\ \exists r_{i0}. (Q_i \sqcap \text{Tail}_{\bar{c}}) &\sqsubseteq Q_j \sqcap \text{Tail}_{\bar{c}} \\ \exists r_{i+}. (Q_i \sqcap C_c \sqcap \mathbf{EO}\text{Tail}_c) &\sqsubseteq Q_k \sqcap \text{Tail}_c \\ \exists r_{i+}. (Q_i \sqcap C_c \sqcap \mathbf{EO}C_c) &\sqsubseteq Q_k \sqcap C_c \\ \exists r_{i+}. (Q_i \sqcap C_{\bar{c}}) &\sqsubseteq Q_k \sqcap C_{\bar{c}} \\ \exists r_{i+}. (Q_i \sqcap \text{Tail}_{\bar{c}}) &\sqsubseteq Q_k \sqcap \text{Tail}_{\bar{c}} \end{aligned}$$

Note that in order to correctly update a configuration along some role r_i or r_{if} , $f \in \{0, +\}$, it is required that (1) the state of the configuration is in fact i , i.e., labeled with Q_i , and (2) for decrementation instructions $\text{dec}(c, \cdot, \cdot)$ the mentioned conditions are satisfied. For example, if $f = 0$, then the value currently encoded in register c has to be zero. If this is not the case, none of the above CIs for register c can

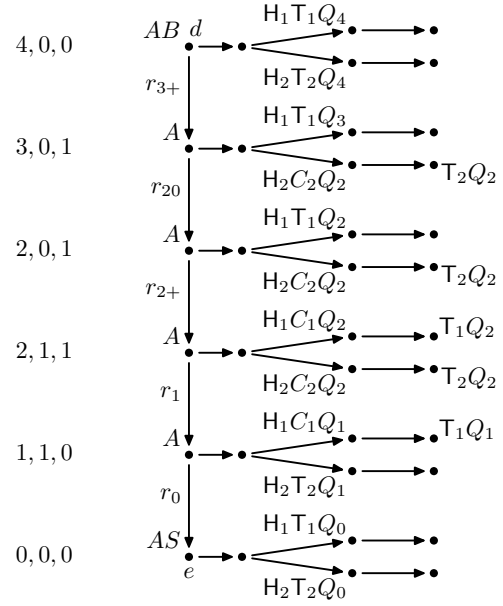


Figure 2: Example computation for $\mathcal{T}_M$.

be applied; thus, no information is propagated for register c and the sequence under consideration does not correspond to a valid computation.

Finally, a halting configuration can be detected by checking whether the encoding of some configuration has label Q_n in both registers. This is expressed by the CI

$$\mathbf{EO}(\mathbf{EO}(\text{Head}_1 \sqcap Q_n) \sqcap \mathbf{EO}(\text{Head}_2 \sqcap Q_n)) \sqsubseteq B$$

in order to label accepting configurations with B .

Example 1 Let $M = (Q, I)$ be a TCM with $Q = \{0, \dots, 4\}$ and $I_0 = \text{inc}(1, 1)$, $I_1 = \text{inc}(2, 2)$, and $I_2 = \text{dec}(1, 3, 2)$, $I_3 = \text{dec}(2, 2, 4)$. It is easy to check that M halts and that the witnessing sequence of e-states is $0, 1, (2, +), (2, 0), (3, +), 4$. The construction of $\mathcal{T}_M$ implies that for any instance d of A , there is another domain element that is reachable via the sequence of roles $r_{3+}, r_{20}, r_{2+}, r_1, r_0$. This sequence is depicted in Figure 2. Note that by $\mathcal{T}_M$, e is labeled with S and starting from e we find the encoding of the initial configuration of M (the temporal relation is drawn horizontally and H, T abbreviate Head, Tail). $\mathcal{T}_M$ induces the configurations from bottom to top. The encoding of the configurations is found in the right side of the picture; the corresponding configurations are depicted on the left side. It can be seen that the top-most configuration is actually halting and enforces finally label B in point d . Note that only the consequences of one sequence of e-states are depicted. In particular, in a full model, all instances of A are also instances of B .

4.4 Hardness for Non-Elementary Time with the Transitive Future Operator

We begin by noting that the reduction presented in the previous section does not easily transfer to the case when we consider the transitive future operator $\mathbf{E}\Diamond$ instead of the next

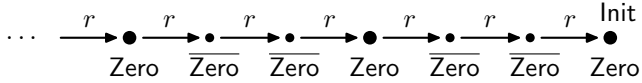
operator $\mathbf{EO}$. The main problem is that we cannot update the concept Tail_c for a decrement instruction because the fact ‘there is a future with Tail_c but no C_c until then’ is not expressible. This does not come as a surprise since it is known that the impossibility of talking about next time points brings considerable technical difficulties. For example, for proving lower bounds for products of transitive logics quite intricate techniques are required (Gabelaia et al. 2005).

In this section, we show that also the combination of $\mathbf{EO}$ with $\mathcal{EL}$ exhibits high computational complexity.

Theorem 10 *Concept subsumption w.r.t. general $\text{CTL}_{\mathcal{EL}}^{\mathbf{EO}}$ -TBoxes is inherently nonelementary.*

In order to prove the non-elementary lower bound, we use a technique in the style of (Stockmeyer 1974), which has already been used in the context of many-dimensional modal logics; for instance, for concept satisfiability for $\text{LTL}_{\mathcal{ALC}}$ without TBoxes (Gabbay et al. 2003). In a nutshell, this technique consists of two steps: the first step is to find some encoding for numbers and show that arbitrarily big elementary numbers can be ‘enforced’. In the second step, a standard reduction is used to show, e.g., k -EXPTIME-hardness of the problem for every $k \geq 1$.

We will proceed along these lines. For step 1, the encoding of a number n , we will provide a TBox $\mathcal{T}$ such that in any model of $\mathcal{T}$ containing a sequence of domain elements connected via a rigid role r the following holds. If some element of the sequence, say d , is an instance of the concept name Init , then every element that can reach d via $k \cdot n$ steps along r is an instance of a concept name Zero , and all other elements are instances of a concept name $\overline{\text{Zero}}$. For $n = 3$, this sequence can be depicted as follows.



Formally, we say that a TBox $\mathcal{T}$ counts modulo n if there exist concept names Init , Zero , $\overline{\text{Zero}}$, Fail , and a rigid role r such that for all $i \geq 0$ there are models $\mathcal{J}$ of $\mathcal{T}$ and $\exists r^i.\text{Init}^1$ with $\text{Fail}^{\mathcal{J},w} = \emptyset$ for all worlds w , and in all such models $\mathcal{J}$ it holds for all $i \geq 0$:

- (i) $\mathcal{J} \models \exists r^i.\text{Init} \sqsubseteq \text{Zero}$ iff $i \equiv 0 \pmod n$;
- (ii) $\mathcal{J} \models \exists r^i.\text{Init} \sqsubseteq \overline{\text{Zero}}$ iff $i \not\equiv 0 \pmod n$.

Intuitively, the concept Fail is necessary as negation is not available; it is used to ensure that complementary concept names behave in a complementary way. The use of Fail is straightforward, so in the proof sketches below we will concentrate on points (i) and (ii).

We next show that there are ‘small TBoxes that count modulo large numbers’. For making this precise, let us introduce the function $\exp : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$:

$$\exp(1, n) := 2^n; \quad \exp(\ell + 1, n) := \exp(\ell, n) \cdot 2^{\exp(\ell, n)}.$$

Lemma 11 *For each $k, n \geq 1$, there is a polynomially sized (in k, n) $\text{CTL}_{\mathcal{EL}}^{\mathbf{EO}}$ -TBox $\mathcal{T}_{k,n}$ that counts modulo $\exp(k, n)$.*

Proof. (sketch) The proof is by induction on k . For $k = 1$, we describe an $\mathcal{EL}$ -TBox that counts modulo 2^n . To

¹ $\exists r^i$ abbreviates sequences of $\exists r$ with length i .

this aim, we introduce concept names $X_1, \overline{X_1}, \dots, X_n, \overline{X_n}$, $C_1, \overline{C_1}, \dots, C_n, \overline{C_n}$ modeling n bits and carry bits (and their negations), respectively, of a counter and take the CIs

$$\begin{aligned} \exists r.X_1 &\sqsubseteq \overline{X_1} \sqcap C_1 & \exists r.\overline{X_1} &\sqsubseteq X_1 \sqcap \overline{C_1} \\ C_i \sqcap \exists r.X_{i+1} &\sqsubseteq \overline{X_{i+1}} \sqcap C_{i+1} \\ C_i \sqcap \exists r.\overline{X_{i+1}} &\sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}} \\ \overline{C_i} \sqcap \exists r.X_{i+1} &\sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}} \\ \overline{C_i} \sqcap \exists r.\overline{X_{i+1}} &\sqsubseteq \overline{X_{i+1}} \sqcap \overline{C_{i+1}} \end{aligned}$$

For initializing the counter and for propagating Fail we include the following CIs for $1 \leq i \leq n$:

$$\begin{aligned} \text{Init}_1 &\sqsubseteq \text{Zero}_1, & \text{Zero}_1 &\equiv \overline{X_1} \sqcap \dots \sqcap \overline{X_n}, \\ X_i &\sqsubseteq \overline{\text{Zero}_1}, & X_i \sqcap \overline{X_i} &\sqsubseteq \text{Fail}, & C_i \sqcap \overline{C_i} &\sqsubseteq \text{Fail} \end{aligned}$$

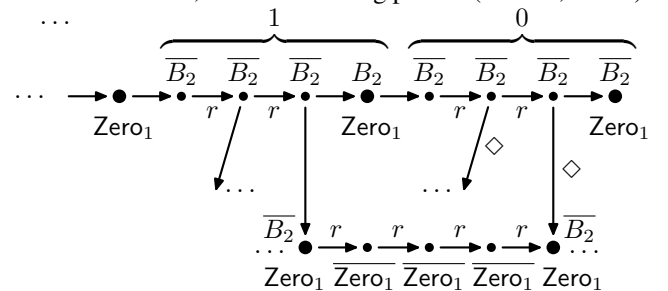
It should be clear that the TBox $\mathcal{T}_{1,n}$ consisting of all these CIs counts modulo 2^n .

For the induction step $k > 1$, a number is now represented in binary in $\exp(k-1, n)$ consecutive domain elements along the r -chain; we use additional concepts B_k ($\overline{B_k}$) expressing that a bit is set (not set). We proceed ‘inductively’:

- (1) If Init_k is satisfied in some domain element (in some world), then enforce that the next $\exp(k-1, n)$ domain elements satisfy $\overline{B_k}$; this encodes the number 0.
- (2) If a sequence of $\exp(k-1, n)$ domain elements encodes some number, say M , then the preceding sequence of $\exp(k-1, n)$ domain elements is enforced to encode the number $M+1$.

Clearly, for both steps (1) and (2) it is required to determine $\exp(k-1, n)$ consecutive elements, which is possible due to the induction hypothesis. In particular, for step (2) we mark the begin of the encoding of some number using $\mathcal{T}_{k-1,n}$: a new number begins whenever the concept Zero_{k-1} is satisfied. Also by induction hypothesis, we can mark all domain elements that are *not* the begin of some number.

For the actual incrementation necessary in Point (2), it suffices to communicate between domain elements that have distance (again: along r) precisely $\exp(k-1, n)$ as these domain elements represent the same bit positions in two successive numbers. This is realized by enforcing in each element of the chain a possible future where the bit of this element is stored. In the domain element having precisely distance $\exp(k-1, n)$, this bit is accessed and the correct value is enforced, see the following picture ($M = 0$, $k = 2$).



It remains to mark the elements of the sequence with Zero_k and $\overline{\text{Zero}_k}$, respectively. First, note that Zero_k can hold only

at the begin of the encoding of some number. So in all ‘non-begin’ elements we imply $\overline{\text{Zero}}_k$. For distinguishing the begin elements it suffices to detect the encoding of a number where all $\text{exp}(k-1, n)$ bits are set to 1. Then, starting from the next domain element we will find the encoding of a 0, and we mark its begin with Zero_k . $\square$

In the second step we employ the TBoxes $\mathcal{T}_{k,n}$ to reduce the word-problem for deterministic Turing machines with k -fold exponential space restriction to subsumption in $\text{CTL}_{\mathcal{EL}}^{\text{E}\Diamond}$. Theorem 10 is then a direct corollary.

Theorem 12 *Concept subsumption w.r.t. general $\text{CTL}_{\mathcal{EL}}^{\text{E}\Diamond}$ -TBoxes is k -EXPSPACE-hard for every $k \geq 1$.*

Proof. (sketch) Fix any $k \geq 1$ and let M be a deterministic Turing machine which runs within $\text{exp}(k, p(n))$ space for inputs of length n for some polynomial $p(n)$. Let w be some input of length n and set $N := p(n)$. By Lemma 11, there is a TBox $\mathcal{T}_{k,N}$ that counts modulo $\text{exp}(k, N)$ and whose size is polynomial in k, N and hence also polynomial in k, n .

For the reduction, we first include the CI $A \sqsubseteq \exists r.A$ that generates an infinite and rigid r -chain. For every domain element satisfying A , we enforce a possible future that simulates M (again backwards) along the r -chain: $A \sqsubseteq \text{E}\Diamond \text{Init}_M$. A configuration of M is represented by a sequence of $\text{exp}(k, N)$ domain elements connected via r using concept names for every symbol from the tape alphabet, for every state, and for the head. The borders between two configurations are marked using $\mathcal{T}_{k,N}$. We proceed inductively along the lines of the proof of Lemma 11:

- (1') enforce the initial configuration of M (including w);
- (2') if a sequence of $\text{exp}(k, N)$ domain elements between two borders encodes a configuration of M , the preceding sequence of $\text{exp}(k, N)$ domain elements is enforced to encode the successor configuration.

Obviously, step (1') can be done using a polynomial TBox starting from the concept Init_M . For enforcing tape symbols in step (2') it suffices to communicate between domain elements that have an r -distance of precisely $\text{exp}(k, N)$. Finally, for updating the head position, we might have to communicate between elements having distance $\text{exp}(k, N) \pm 1$. The previous ideas can be easily adapted to this case.

It remains to add the CI $\text{E}\Diamond Q_{acc} \sqsubseteq B$ in order to detect an accepting configuration. It is routine to show that M accepts w iff the constructed TBox implies $A \sqsubseteq B$. $\square$

5 Decidability of CTL with DL-Lite_{bool}^N

In this section, we show that the technique developed by (Artale et al. 2012) to prove decidability of TDLs based on *linear-time* temporal logic and DL-Lite is robust in the sense that it can be carried over to *some* TDLs based on CTL and DL-Lite_{bool}^N (Artale et al. 2009).

$\text{CTL-DL-Lite}_{bool}^N$ concepts C, D are defined by the following grammar:

$$\perp \mid A \mid \geq q r \mid \neg C \mid C \sqcap D \mid \text{E}\Diamond C \mid \text{E}\Box C \mid \text{E}(CUD),$$

where q is a positive integer given in binary, A ranges over concept names and r ranges over $\{r, r^- \mid r \in \text{N}_R\}$.

The semantics for Booleans and temporal operators is defined as in Section 2. We interpret *inverse roles* and *number restrictions* in a temporal interpretation $\mathcal{I}$ as follows:

$$\begin{aligned} (r^-)^{\mathcal{I}, w} &= \{(d', d) \mid (d, d') \in r^{\mathcal{I}, w}\}; \\ (\geq q r)^{\mathcal{I}, w} &= \{d \mid \#\{d' \mid (d, d') \in r^{\mathcal{I}, w}\} \geq q\}. \end{aligned}$$

Our following result shows that three fragments of $\text{CTL-DL-Lite}_{bool}^N$ behave in a similar way as temporal extensions of DL-Lite_{bool}^N based on linear time: reasoning in these fragments is not harder than in the component logics.

Theorem 13 *Concept satisfiability w.r.t. general TBoxes is EXPTIME-complete*

- 1. for $\text{CTL-DL-Lite}_{bool}^N$ without local roles,
- 2. for $\text{CTL}^{\text{EU}, \text{E}\Box} \text{-DL-Lite}_{bool}^N$,
- and PSPACE-complete
- 3. for $\text{CTL}^{\text{E}\Diamond} \text{-DL-Lite}_{bool}^N$.

The fragments of points 2,3 do not allow the operator $\text{E}\Diamond$.

Proof. (sketch) For the upper bound, the proof follows the two-step technique proposed by (Artale et al. 2012). First, we reduce from $\text{CTL-DL-Lite}_{bool}^N$ to the one-variable fragment QCTL^1 of *first-order branching temporal logic* QCTL (Hodkinson, Wolter, and Zakharyashev 2002). Then, the result is further reduced to a QCTL^1 -formula without occurrences of existential quantifiers, which is essentially a CTL-formula. This way we get the same upper bounds as for the propositional CTL fragments. The lower bounds are inherited from the corresponding fragments (Meier et al. 2009).

The elimination of the quantifiers in the second step does not work for full $\text{CTL-DL-Lite}_{bool}^N$ because the original shifting technique (Artale et al. 2012, Lemma 4.2) relies on the past being unbounded, which is not the case for the standard CTL semantics. However, the problem can be circumvented by (1) disallowing local roles – which makes shifting superfluous – or (2) restricting the temporal operators to EU and $\text{E}\Box$ which are tolerant to a certain variant of unravelling into the temporal direction. Unravelling tolerance is related to stutter invariance (Lamport 1983) and does not apply to $\text{E}\Diamond$ and therefore to the strict variants $\text{E}\Box^<$ and $\text{EU}^<$.

We note that the only temporal operators used to encode the TBox and rigidity of roles in the first step are $\text{E}\Diamond$ and $\text{A}\Box$, which are available in $\text{CTL}^{\text{E}\Diamond} \text{-DL-Lite}_{bool}^N$. Hence our translation works for $\text{CTL}^{\text{E}\Diamond} \text{-DL-Lite}_{bool}^N$, too. $\square$

6 Conclusions and Future Work

In this paper, we have made progress towards understanding the landscape of the computational complexity and expressivity of BTDLs allowing for rigid roles and general TBoxes. In particular, we showed for some BTDLs based on $\mathcal{EL}$ that we cannot easily conclude undecidability via non-convexity, as in the case of similar LTDs, since they can be shown to be convex. However, we have been able to squelch the hope that these convex logics are practical because we established a non-elementary lower bound and undecidability, respectively.

The problems left open include determining the precise complexity of $\text{CTL}_{\mathcal{EL}}^{\text{E}\diamond}$ and of unrestricted $\text{CTL-DL-Lite}_{\text{bool}}^N$. For the latter, we plan to pursue an automata-based approach in the style of (Gutiérrez-Basulto, Jung, and Lutz 2012).

Our investigation provides several research directions that are worth exploring in the near future – for example, one could consider reasoning in the presence of *acyclic TBoxes* instead of general ones, as usually the computational properties of DLs over acyclic TBoxes are better. Furthermore, complexity might also drop when assuming *expanding domains* instead of constant domains.

For BTDLs based on *DL-Lite*, the investigation can be broadened in two ways: by considering restrictions such as *DL-Lite_{core}* or by studying the effect of adding role inclusions. In fact, LTDLs based on *DL-Lite_{core}* are less complex than their *DL-Lite_{bool}* counterpart (Artale et al. 2012). Another possibility is the study of BTDLs with the more expressive *temporal roles*; e.g., where $\text{E}\diamond$ and $\text{A}\square$ are applied to roles. LTDLs with temporal roles are undecidable (Artale et al. 2012).

Finally, we believe that our results, together with the suggested restriction to acyclic and empty TBoxes, can have an impact on the study of PMLs, given the known correspondence between PMLs and combinations of DLs and modal-like logics with rigid roles and empty TBoxes: it seems that PMLs with good computational properties may be found among their *positive fragments*, some of which are notational variants of $\text{CTL}_{\mathcal{EL}}^{\text{E}\diamond}$, $\text{CTL}_{\mathcal{EL}}^{\text{E}\diamond}$ and $\text{CTL}_{\mathcal{EL}}^{\text{E}\diamond, \text{A}\square}$. For instance, the satisfiability problems of $\text{K} \times \text{K}$ and $\text{S4} \times \text{K}$ are non-elementary, and there is hope that the complexity of their positive fragments is only elementary.

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