

Canonical Logic Programs Are Succinctly Incomparable with Propositional Formulas

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Abstract

Canonical (logic) programs (CP) refer to the class of *normal* programs (LP) augmented with connective *not*, and are equally expressive as *propositional formulas* (PF). In this paper we address the question of whether CP and PF are *succinctly incomparable*. Our main result shows that the PARITY problem *only* has exponential CP representations, while it can be polynomially represented in PF. In other words, PARITY *separates* PF from CP. Simply speaking, this means that exponential size blowup is generally inevitable when translating a set of PF formulas into a (logically) equivalent CP program (without introducing new variables). Furthermore, since it has been shown by Lifschitz and Razborov that there is also a problem which separates CP from PF (assuming $P \not\subseteq NC^1/poly$), it follows that the two formalisms are indeed succinctly incomparable.

1 Introduction

The relationship between (*logic*) programs under *answer set semantics* (ASP) (Lifschitz 2008; Brewka, Eiter, and Truszczyński 2011) and *propositional satisfiability* (SAT) (Biere et al. 2009) gains a lot of attention in the literature. In 2006, Lifschitz and Razborov proved that an exponential size blowup is generally inevitable when translating a *normal program* (LP) to a (logically) equivalent set of *propositional formulas* (PF) (without introducing additional variables). More precisely, they showed that (a variant of) the P-complete problem PathSystem (PATH) has polynomial size LP representations, however, it *cannot* be polynomially represented in PF (assuming $P \not\subseteq NC^1/poly$) (Lifschitz and Razborov 2006), i.e., PATH *separates* LP from PF.

As noted in (Lifschitz and Razborov 2006), PF can be considered as a special case of the class of (*nondisjunctive*) *nested programs* (NLP, without classical negation $\neg$) (Lifschitz 1999). Therefore, NLP is *stronger* than PF in terms of the *succinctness* criterion (or the “*comparative linguistics*” approach) (Gogic et al. 1995):

That is, we consider formalism A to be stronger than formalism B if and only if any knowledge base (KB) in B has an equivalent KB in A that is only polynomially

longer, while there is a KB in A that can be translated to B only with an exponential blowup.

So the following footnote in (Lifschitz 2008) seems convincing:

...ASP appears to be stronger than SAT in the sense of the “comparative linguistics” approach to knowledge representation...

However, ASP involves many kinds of programs, the above statement probably needs further clarification. Particularly, the class of so-called (*nondisjunctive*) *canonical programs* (CP¹) (Lifschitz 1999; Lee 2005; Lee, Lifschitz, and Yang 2013), is a minimal form of ASP that is equally expressive as PF, but looks more likely *not* succinctly stronger. So a question naturally arises: *Does there exist a problem that separates PF from CP?* If there is such a problem, then CP and PF are *succinctly incomparable* (assuming $P \not\subseteq NC^1/poly$).

In this paper we address the question and give a *positive* answer. Our main result shows that the PARITY problem separates PF from CP. Simply speaking, this means exponential size blowup is generally inevitable when translating a set of PF formulas into an equivalent CP program. The PARITY problem asks whether a binary string contains an odd number of 1’s, and it is well-known that (i) $PARITY \in NC^1/poly^2$, i.e., it has polynomial PF representations; (ii) $PARITY \notin AC^0$, i.e., it cannot be represented by *polynomial* size (boolean) circuits with *constant depth* and *unbounded fan-in* (Arora and Barak 2009; Jukna 2012).

To show PARITY separates PF from CP, we provide a procedure that simplifies every PARITY program Π into a shorter program Π' s.t. Π' is equivalent to its *completion* $Comp(\Pi')$ (Erdem and Lifschitz 2003). Such completions are essentially constant depth, unbounded fan-in circuits. According to $PARITY \notin AC^0$, these circuits must be of exponential size, consequently, there are no polynomial size PARITY programs in CP. In the rest of the paper, we shall introduce the basic concepts, key steps of our proof and discuss the importance of succinctness research in the theory and practice of Knowledge Representation (KR).

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¹Extends LP with connective *not*.

² $NC^1/poly$ (or *non-uniform* NC^1) *exactly* contains languages computable (representable) by polynomial size PF formulas.

2 Background

Canonical Programs

The following notations are adopted from (Lifschitz 1999; Lee 2005). A *rule element* e is defined as

$$e := \top \mid \perp \mid x \mid \text{not } x \mid \text{not not } x$$

in which $\top, \perp$ are 0-ary connectives, x is a (*boolean*) *variable* (or an *atom*) and *not* is a unary connective³. A (*nondisjunctive canonical*) *rule* is an expression of the form

$$H \leftarrow B \quad (1)$$

where the *head* H is either a variable or the connective $\perp$, and the *body* B is a finite set of rule elements. A *canonical program* Π is a finite set of rules. E.g., the following is a canonical program:

$$\begin{aligned} x_1 &\leftarrow \text{not not } x_1, & x_3 &\leftarrow \text{not } x_1, \text{not } x_2, \\ x_2 &\leftarrow \text{not not } x_2, & x_3 &\leftarrow x_1, x_2. \end{aligned} \quad (2)$$

A canonical program Π is *normal* if it contains no connectives *not not*. A normal program Π is *basic* if it contains no occurrences of connective *not*.

The *satisfaction relation* $\models$ between a set of variables I and a rule element is defined as follows:

- $I \models \top$ and $I \not\models \perp$,
- $I \models x$ iff $I \models \text{not not } x$ iff $x \in I$,
- $I \models \text{not } x$ iff $x \notin I$.

Say I satisfies a set of rule elements B if I satisfies each rule element in B . We say I is *closed* under a program Π , if I is closed under every rule in Π , i.e., for each rule $H \leftarrow B \in \Pi$, $I \models H$ whenever $I \models B$. Let Π be a basic program and let $Cn(\Pi)$ denotes the *minimal* set (in terms of set inclusion) closed under Π , we say I is an *answer set* of Π if $I = Cn(\Pi)$. Note that a basic program has exactly one answer set.

The *reduct* Π^I of a program Π w.r.t. I is a set of rules obtained from Π via: (i) Replacing each *not not* x with $\top$ if $I \models x$, and with $\perp$ otherwise; (ii) Replacing each *not* x with $\top$ if $I \not\models x$, and with $\perp$ otherwise. Observe that Π^I must be a basic program. We say I is an answer set of Π if $I = Cn(\Pi^I)$, i.e., I is an answer set of Π^I . E.g., the following single rule program:

$$x \leftarrow \text{not not } x \quad (3)$$

has two answer sets $\emptyset$ and $\{x\}$.

For a set of rule elements B , define $\text{var}(B) = \{e \in B : e \text{ is a variable}\}$. E.g., $\text{var}(\{x_1, \text{not } x_2, \text{not not } x_3\}) = \{x_1\}$. The *signature* $\text{sig}(\Pi)$ of a program Π is the set of all involved variables in Π . By $\text{Ans}(\Pi)$ we denote the set of all answer sets of Π . E.g., if Π is (2), then $\text{sig}(\Pi) = \{x_1, x_2, x_3\}$ and $\text{Ans}(\Pi) = \{\{x_1, x_2, x_3\}, \{x_1\}, \{x_2\}, \{x_3\}\}$. The *size* $|\Pi|$ of Π is the number of its rules. As a convention, Π_n refers to a program with signature $\{x_1, \dots, x_n\}$, i.e., $\text{sig}(\Pi_n) = \{x_1, \dots, x_n\}$.

It is easy to see that by using rules of the form (3) and appropriate *constraints* of the form $\perp \leftarrow B$, it is easy to give an arbitrary set of answer sets over $\text{sig}(\Pi_n)$, in other words, CP has exactly the same expressive power as PF.

³By (Lifschitz 1999), *not not not* x can be replaced by *not* x .

Problem Representation and Succinctness

A (*binary*) *string* is a finite sequence of *bits* from $\{0, 1\}$. A string w of length n (i.e., $w \in \{0, 1\}^n$) defines a subset of variables $\{x_1, \dots, x_n\}$. E.g., 1010 stands for $\{x_1, x_3\}$. Therefore, a set of variables I and a string w can be regarded as the same. A *problem* (or *language*) L is a set of strings.

Definition 2.1 (Problem Representation). A *problem* L can be represented in a class of programs (or formulas, etc) $\mathcal{C}$ (i.e., $L \in \mathcal{C}$), if there exists a sequence of programs $\{\Pi_n\}$ ($n = 1, 2, \dots$) in $\mathcal{C}$ that computes L , i.e., for every string $w \in \{0, 1\}^n$,

$$w \in L \Leftrightarrow w \in \text{Ans}(\Pi_n).$$

Say L has polynomial representations in $\mathcal{C}$ (i.e., $L \in \text{Poly-}\mathcal{C}$), if $L \in \mathcal{C}$ and $|\Pi_n|$ is bounded by a polynomial $p(n)$.

Recall that the PARITY problem is the set of binary strings with an odd number of 1's. By PARITY_n we denote PARITY strings of length n . Clearly, (2) represents PARITY_3 since its answer sets are 111, 100, 010 and 001.

Definition 2.2 (Succinctness (Gogic et al. 1995; French et al. 2013)). Let $\mathcal{C}, \mathcal{C}'$ be two classes of programs such that for every problem L , $L \in \mathcal{C} \Leftrightarrow L \in \mathcal{C}'$. Say $\mathcal{C}$ is at least as succinct as $\mathcal{C}'$ (i.e., $\mathcal{C}' \preceq \mathcal{C}$), if for every problem L ,

$$L \in \text{Poly-}\mathcal{C}' \Rightarrow L \in \text{Poly-}\mathcal{C}.$$

If $L \in \text{Poly-}\mathcal{C}$ but $L \notin \text{Poly-}\mathcal{C}'$ (i.e., $\mathcal{C} \not\preceq \mathcal{C}'$), then L separates $\mathcal{C}$ from $\mathcal{C}'$. If $\mathcal{C}' \preceq \mathcal{C}$ and $\mathcal{C} \not\preceq \mathcal{C}'$, then $\mathcal{C}$ is strictly more succinct than $\mathcal{C}'$ (i.e., $\mathcal{C}' \prec \mathcal{C}$). Moreover, $\mathcal{C}, \mathcal{C}'$ are succinctly incomparable if there is a problem L that separates $\mathcal{C}$ from $\mathcal{C}'$, and vice versa (i.e., $\mathcal{C} \not\preceq \mathcal{C}'$ and $\mathcal{C}' \not\preceq \mathcal{C}$).

PARITY Representations and Completion

Note that (2) suggests a “pattern” for PARITY_n : The first part of the program (e.g., the “*not not*” rules in (2)) generates all possible strings of length $n - 1$, the second part identifies the last bit to form an odd string of length n . Following the pattern, we can give a sequence of *exponential* size CP programs for PARITY, i.e., $\text{PARITY} \in \text{CP}$.

On the other hand, it is a textbook result that PARITY has polynomial size PF representations with classical connectives $\{\wedge, \vee, \neg\}$, i.e., $\text{PARITY} \in \text{NC}^1/\text{poly}$ (or Poly-PF). Remind that the size of a formula is the number of its connectives. E.g., $(x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2)$ represents PARITY_2 , and the formula can be recursively applied for arbitrary PARITY_n . It is also well-known that $\text{PARITY} \notin \text{AC}^0$, i.e., PARITY cannot be represented by a sequence of polynomial size circuits $\{C_n\}$ in which these circuits C_n have unbounded fan-in and constant depth.

Simply speaking, a sequence of polynomial size PF formulas $\{\phi_n\}$ represents an AC^0 language, if the number of variables occur in a *conjunction* or *disjunction* of ϕ_n is *unrestricted*, and the distance of the longest path from the root to a leaf in the tree structure of each ϕ_n is *fixed*. E.g., PARITY cannot be represented by a sequence of polynomial size CNFs $\{\psi_n\}$ since $\{\psi_n\}$ represents an AC^0 language. For more details about circuits, please see (Arora and Barak 2009; Jukna 2012).

The *completion* $Comp(\Pi)$ (Erdem and Lifschitz 2003) of a CP program Π consists of a set (or a conjunction) of PF formulas (we slightly abuse the connective $\equiv$): (i) $x \equiv \tilde{B}_1 \vee \tilde{B}_2 \vee \dots \vee \tilde{B}_m$, where $x \leftarrow B_1, \dots, x \leftarrow B_m$ are all rules in Π with head x , and each $\tilde{B}_i$ is the conjunction of rule elements in B_i with connective *not* replaced by $\neg$; (ii) $x \equiv \perp$, if x is not a head of any rule in Π ; (iii) $\neg \tilde{B}$, if a rule $\perp \leftarrow B$ is in Π .

Proposition 2.1. *The completion $Comp(\Pi)$ of an arbitrary canonical program Π is a constant depth, unbounded fan-in circuit whose size is polynomially bounded by $|\Pi|$.*

It is well-known that an answer set of Π is also a model of its completion, but the inverse generally does not hold. E.g., the completion $\{x_1 \equiv \neg x_2 \vee (x_2 \wedge x_1), x_2 \equiv \neg x_1 \vee (x_1 \wedge x_2)\}$ of the $PARITY_2$ program:

$$\begin{aligned} x_1 &\leftarrow \text{not } x_2, & x_2 &\leftarrow \text{not } x_1, \\ x_1 &\leftarrow x_2, \text{not not } x_1, & x_2 &\leftarrow x_1, \text{not not } x_2, \end{aligned} \quad (4)$$

has an even string model 11, i.e., (4) is not *completion-equivalent*. In the next section, we will show how to simplify a $PARITY$ program to be completion-equivalent.

3 Simplifying $PARITY$ Programs

Let B be a finite set of rule elements built on signature $V = \{x_1, \dots, x_n\}$. By $S(B)$ we denote the set $\{I \subseteq V : I \models B\}$. E.g., let $V = \{x_1, x_2, x_3, x_4\}$ and $B = \{x_2, \text{not } x_3, \text{not not } x_4\}$, then $S(B) = \{\{x_1, x_2, x_4\}, \{x_2, x_4\}\} = \{1101, 0101\}$. We say B is *consistent* if there is a set of variables I s.t. $I \models B$. We say B *covers* a variable $x \in V$ iff $x \in B$ or $\text{not } x \in B$ or $\text{not not } x \in B$. If B covers every variable in V then B *fully covers* V . E.g., $B = \{x_1, \text{not } x_2, \text{not not } x_3\}$ fully covers $V = \{x_1, x_2, x_3\}$. Clearly, B is consistent and fully covers V iff $S(B)$ contains a unique string.

In the following we assume that all rules $x \leftarrow B$ have consistent body B and $x \notin \text{var}(B)$. It is easy to see that the assumption does not affect the generality.

Lemma 3.1. *Let Π_n be a $PARITY_n$ program. If there is a rule $x \leftarrow B$ in Π_n s.t. $\text{not not } x \in B$ and $S(B)$ contains a unique even string, then dropping $x \leftarrow B$ from Π_n results in a $PARITY_n$ program Π'_n .*

E.g., by Lemma 3.1, (4) can be simplified to:

$$x_1 \leftarrow \text{not } x_2, \quad x_2 \leftarrow \text{not } x_1. \quad (5)$$

Lemma 3.2. *Let Π_n be a $PARITY_n$ program. If there is a rule $x \leftarrow B$ in Π_n s.t. $\text{not not } x \in B$ and $S(B)$ contains a unique odd string, then dropping $\text{not not } x$ from B results in a $PARITY_n$ program Π'_n .*

Standard Programs and the Main Theorem

A $PARITY_n$ program Π_n is *standard* if for each rule $x \leftarrow B \in \Pi_n$, $\text{not not } x \notin B$ whenever $S(B \cup \{x\})$ contains a unique string. E.g., (2) and (5) are standard programs. By Lemma 3.1 and 3.2, it is not hard to see:

Proposition 3.1. *Let Π_n be a $PARITY_n$ program. Then there is a standard $PARITY_n$ program Π'_n s.t. $|\Pi'_n| \leq |\Pi_n|$.*

Moreover, note that (2) and (5) are completion-equivalent $PARITY$ programs. This is guaranteed by the following proposition:

Proposition 3.2. *Let Π_n be a standard $PARITY_n$ program. Then Π_n is equivalent to its completion $Comp(\Pi_n)$.*

The proof idea of Proposition 3.2 is that every standard $PARITY_n$ program Π_n can be equivalently rewritten to Π'_n by replacing each $x \in \text{var}(B)$ with $\text{not not } x$ for every rule body B in Π_n . By the Lin-Zhao Theorem (Lin and Zhao 2004) or the (generalized) *Fages Theorem* (Erdem and Lifschitz 2003; You, Yuan, and Zhang 2003), Π'_n is equivalent to its completion $Comp(\Pi'_n)$. And then the Proposition follows from the fact that $Comp(\Pi'_n) = Comp(\Pi_n)$.

Lemma 3.3 (Main Lemma). *Let Π_n be a $PARITY_n$ program. Then there is a $PARITY_n$ program Π'_n s.t. Π'_n is equivalent to $Comp(\Pi_n)$ and $|\Pi'_n| \leq |\Pi_n|$.*

Theorem 3.1 ($PARITY \notin \text{Poly-CP}$). *$PARITY$ has no polynomial size CP representations.*

Proof. Assume the contrary that there is a sequence of $PARITY_n$ programs $\{\Pi_n\}$ s.t. $|\Pi_n|$ is bounded by a polynomial $p(n)$. By Lemma 3.3, there is a sequence of completion-equivalent $PARITY_n$ programs $\{\Pi'_n\}$ in which $|\Pi'_n|$ is also bounded by the polynomial $p(n)$. By Proposition 2.1, $\{\Pi'_n\}$ represents a language in AC^0 . This contradicts $PARITY \notin AC^0$. $\square$

Corollary 3.1. *$PARITY$ separates PF from CP.*

Corollary 3.2. *Suppose $P \not\subseteq NC^1/\text{poly}$. Then CP and PF are succinctly incomparable.*

4 Discussion and Conclusion

In this paper we show that the $PARITY$ problem separates PF from CP, and the two formalisms are succinctly incomparable (assuming $P \not\subseteq NC^1/\text{poly}$). Interestingly, our main result may at first appear counter-intuitive: the P-complete problem PATH has Poly-CP representations, while this does not hold for an “easy” problem $PARITY$. Actually, there is no contradiction. As noted in (Abiteboul, Hull, and Vianu 1995; Dantsin et al. 2001), a *complete* problem in a complexity class can be represented in a formalism $\mathcal{C}$, *does not* imply that *all* problems in that class can be represented in $\mathcal{C}$.

Generally speaking, the research of succinctness (Gogic et al. 1995; Coste-Marquis et al. 2004; Grohe and Schweikardt 2005; French et al. 2013) gives us a deeper understanding about KR formalisms, for it reveals their (in)abilities of concisely representing different problems under the condition that the encoded models are the *same*. This is particularly interesting if the formalisms are *equally expressive* and share the same reasoning complexity.

E.g., in addition to CP and NLP, programs with *cardinality constraints* and *choice rules* (CC, without classic negation $\neg$) (Simons, Niemelä, and Soeninen 2002), (*simple*) *definite causal theories*⁴ (S/DT) (Giunchiglia et al. 2004) and

⁴A theory is simple if each rule body is a conjunction of literals.

two-valued programs (TV) (Lifschitz 2012) are as expressive as PF and NP-complete for consistency checking. But they have a non-trivial succinctness picture, see Fig. 1.

Besides the theoretical interests, succinctness also tells us something like “which for what is the best” in choosing KR formalisms for a given application. E.g., one should choose ASP instead of SAT (or DT) if the application involves reasoning about PATH or Transitive Closure⁵, because the former provides compact representations to avoid unnecessary overload. Recall that from the complexity viewpoint, even *one* extra variable may *double* the search space for intractable problems.

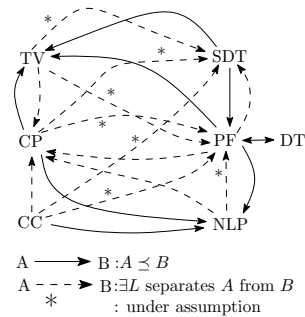


Figure 1: Succinctness Pic.

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References

- Abiteboul, S.; Hull, R.; and Vianu, V. 1995. *Foundations of Databases*. Addison-Wesley.
- Arora, S., and Barak, B. 2009. *Computational Complexity: A Modern Approach*. Cambridge University Press.
- Biere, A.; Heule, M.; van Maaren, H.; and Walsh, T., eds. 2009. *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press.
- Brewka, G.; Eiter, T.; and Truszczyński, M. 2011. Answer set programming at a glance. *Communications of the ACM* 54(12):92–103.
- Coste-Marquis, S.; Lang, J.; Liberatore, P.; and Marquis, P. 2004. Expressive power and succinctness of propositional languages for preference representation. In *Proceedings of the Ninth International Conference on Principles of Knowledge Representation and Reasoning*, 203–212. AAAI Press.
- Dantsin, E.; Eiter, T.; Gottlob, G.; and Voronkov, A. 2001. Complexity and expressive power of logic programming. *ACM Computing Surveys* 33(3):374–425.

⁵An NL-complete problem. It is believed that $NL \not\subseteq NC^1/Poly$.

- Erdem, E., and Lifschitz, V. 2003. Tight logic programs. *Theory and Practice of Logic Programming* 3(4):499–518.
- French, T.; van der Hoek, W.; Iliev, P.; and Kooi, B. P. 2013. On the succinctness of some modal logics. *Artificial Intelligence* 197:56–85.
- Gebser, M.; Schaub, T.; and Thiele, S. 2007. Gringo: A new grounder for answer set programming. In *Proceedings of the 9th International Conference of Logic Programming and Nonmonotonic Reasoning (LNCS4483)*, 266–271. Springer.
- Giunchiglia, E.; Lee, J.; Lifschitz, V.; McCain, N.; Turner, H.; and Lifschitz, J. L. V. 2004. Nonmonotonic causal theories. *Artificial Intelligence* 153(1-2):49–104.
- Gogic, G.; Kautz, H. A.; Papadimitriou, C. H.; and Selman, B. 1995. The comparative linguistics of knowledge representation. In *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence*, 862–869. Morgan Kaufmann Publishers Inc.
- Grohe, M., and Schweikardt, N. 2005. The succinctness of first-order logic on linear orders. *Logical Methods in Computer Science* 1(1).
- Jukna, S. 2012. *Boolean Function Complexity - Advances and Frontiers*. Springer.
- Lee, J.; Lifschitz, V.; and Yang, F. 2013. Action language BC: Preliminary report. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence*, 983–989. AAAI Press.
- Lee, J. 2005. A model-theoretic counterpart of loop formulas. In *Proceedings of the 19th international joint conference on Artificial intelligence*, 503–508. Morgan Kaufmann Publishers Inc.
- Lifschitz, V., and Razborov, A. 2006. Why are there so many loop formulas? *ACM Transactions on Computational Logic* 27(2):261–268.
- Lifschitz, V. 1999. Nested expressions in logic programs. *Annals of Mathematics and Artificial Intelligence* 25:369–389.
- Lifschitz, V. 2008. What is answer set programming? In *Proceedings of the AAAI Conference on Artificial Intelligence*, 1594–1597. MIT Press.
- Lifschitz, V. 2012. Two-valued logic programs. In *Technical Communications of the 28th International Conference on Logic Programming*, 259–266. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik.
- Lin, F., and Zhao, Y. 2004. ASSAT: computing answer sets of a logic program by SAT solvers. *Artificial Intelligence* 157(1-2):115–137.
- Simons, P.; Niemelä, I.; and Soininen, T. 2002. Extending and implementing the stable model semantics. *Artificial Intelligence* 138(1-2):181–234.
- You, J.-H.; Yuan, L.-Y.; and Zhang, M. 2003. On the equivalence between answer sets and models of completion for nested logic programs. In *Proceedings of the 18th international joint conference on Artificial intelligence*, 859–864. Morgan Kaufmann Publishers Inc.