

A Solution to Endrullis-08 and Similar Problems*

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Abstract

We prove the termination of the string rewriting system Endrullis-08, the termination of infinitely many related systems, and even the termination of their union, an infinite string rewriting system.

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1 Endrullis's Problem

Jörg Endrullis raised the question whether the two-rule string rewriting system (SRS) $\{aabb \rightarrow bbba, ba \rightarrow aaaa\}$ terminates. This SRS was included in the Termination Competition 2006 as `SRS/Endrullis/08.srs` and was later renamed to `tpdb-8.0/TRS/Mixed_SRS/08.xml`. Chris Sternagel and René Thiemann have in 2011 succeeded in proving the termination of Endrullis-08.¹ They use the dependency pair framework with arctic matrices of dimension 3 and semantic labelling. A similar proof was done by APROVE in December of 2013.² This proof uses the dependency pair framework with semantic labelling. In this contribution, we give a different proof. In order to simplify the presentation, let us switch to the reflected image $\{bbaa \rightarrow abbb, ab \rightarrow aaaa\}$ of Endrullis's SRS.

2 The Termination Proof

Consider systems $\{bbaa \rightarrow abbb, ab \rightarrow a^i\}$ for various $i \in \mathbb{N}_0$. If $i \leq 2$ then we have termination, as shown by a length-then-lexicographic argument where $b > a$. If $i \in \{3, 5\}$ then there is a loop of length 9 starting from b^6a^2 . If $i \geq 7$ is odd then there is a loop of length $\frac{11+i}{2}$ starting from b^6a^2 . This leaves the case where $i \geq 4$ is even. It turns out that the proof can be done uniformly for all even $i \geq 4$. Not only can we prove that each $\{bbaa \rightarrow abbb, ab \rightarrow a^i\}$ terminates, but also that their union terminates:

► **Theorem 1.** *The infinite SRS $\{bbaa \rightarrow abbb\} \cup \{ab \rightarrow a^i \mid i \geq 2 \text{ is even}\}$ terminates.*

► **Corollary 2.** *The SRS $\{bbaa \rightarrow abbb, ab \rightarrow a^i\}$ terminates if, and only if, $i = 1$ or i is even.*

In the remainder of this contribution, we prove Theorem 1 by reducing the uniform termination problem of $E := \{bbaa \rightarrow abbb\} \cup \{ab \rightarrow a^i \mid i \geq 2 \text{ is even}\}$ in six steps (Lemmas 3, ..., 8).

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¹ http://cl-informatik.uibk.ac.at/software/ceta/experiments/semlab/with_semlab/___Users___rene___tpdb-8.0___TRS___Mixed_SRS___08.xml/proof.xml.xml

² <http://termcomp.uibk.ac.at/termcomp/competition/resultDetail.seam?resultId=480566&cid=68>

First Step: Interreduction

Interreduction means rewriting the right hand sides of a rewrite system. The set of descendants of the string ab^3 by the SRS $R := \{ab \rightarrow a^i \mid i \geq 2 \text{ is even}\}$ is $\{ab^3, a^i b^2, a^{i+1} b, a^i \mid i \geq 0 \text{ even}\}$. Define the SRS F by

$$F := \{b^2 a^2 \rightarrow r \mid r \in \{ab^3, a^i b^2, a^{i+1} b, a^i\}, i \geq 0 \text{ even}\}.$$

► **Lemma 3.** *If F terminates then E terminates.*

Proof. We have $\rightarrow_F \rightarrow_R \subseteq \rightarrow_F \cup \rightarrow_R \rightarrow_F$, where the non-overlapping case commutes, and where the overlapping case either degenerates to a rewrite step on the contractum, e.g. for even $i \geq 0$,

$$b^2 a^2 \rightarrow_F ab^3 \rightarrow_R a^i b^2 \text{ for which } b^2 a^2 \rightarrow_F a^i b^2,$$

or it commutes: for even $i \geq 2$ and even $j \geq 0$,

$$b^2 a^2 b \rightarrow_F a^i b \rightarrow_R a^{i+j-1} \text{ for which } b^2 a^2 b \rightarrow_R b^2 a^{j+1} \rightarrow_F a^{i+j-1}.$$

By a result of Doornbos and von Karger [2], if both R and F terminate and $\rightarrow_F \rightarrow_R \subseteq \rightarrow_R(\rightarrow_F \cup \rightarrow_R)^* \cup \rightarrow_F$, then $R \cup F$ terminates. By construction, $\rightarrow_E \subseteq \rightarrow_R \cup \rightarrow_F$, so if F terminates then E terminates. ◀

Second Step: Encompassment

We encompass each rule of F in a left context of the form $ba^m, m \in \mathbb{N}_0$ and a right context of the form $b^n a, n \in \mathbb{N}_0$. This yields the SRS

$$G := \{ba^m b^2 a^2 b^n a \rightarrow r \mid r \in \{ba^{m+1} b^{n+3} a, ba^{m+i} b^{n+2} a, ba^{m+i+1} b^{n+1} a, ba^{m+i} b^n a\}, \\ i \geq 2 \text{ is even}, m, n \in \mathbb{N}_0\}.$$

► **Lemma 4.** *If G terminates then F terminates.*

Proof. For all $s \in \{a, b\}^*$, we have: G admits a non-terminating derivation starting from bsa if, and only if, F admits a non-terminating derivation starting from s . ◀

Third Step: Semantic Labelling

Let the interpretation $[s] : \mathcal{D} \rightarrow \mathcal{D}$ of a string s on the domain $\mathcal{D} = \{0, 1\} \times \{0, 1, 2\}$ be defined by

$$[a](x, y) = \begin{cases} (x, 2), & \text{if } y = 1, \\ (x, 1), & \text{else;} \end{cases} \quad [b](x) = \begin{cases} (0, 0), & \text{if } x = 1 \text{ and } y = 0, \\ (1, 0), & \text{else.} \end{cases}$$

For all $n > 0$, we get

$$[b^n a](x, y) = \begin{cases} (0, 0), & \text{if } n \text{ is even,} \\ (1, 0), & \text{else.} \end{cases}$$

For all $m > 0, n > 0$ we get $[a^m b^n a](x, y) = (x', y')$ where

$$x' = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{else} \end{cases} \quad \text{and} \quad y' = \begin{cases} 2, & \text{if } m \text{ is even,} \\ 1, & \text{else.} \end{cases}$$

$$\begin{aligned}
\text{Case 1: } m \text{ odd, } n \text{ odd: } & \ell_1 = b_{01}a^m b b_{12}a^2 b^{n-1} b_{xy'}a, & r_1 &= b_{02}a^{m+1} b^{n+2} b_{xy'}a, \\
& r'_1 = b_{11}a^{m+i} b^{n+1} b_{xy'}a, & r''_1 &= b_{02}a^{m+i+1} b^n b_{xy'}a, & r'''_1 &= b_{11}a^{m+i} b^{n-1} b_{xy'}a. \\
\text{Case 2: } m = 0, n \text{ odd: } & \ell_2 = b^2 b_{12}a^2 b^{n-1} b_{xy'}a, & r_2 &= b_{01}a b^{n+2} b_{xy'}a, \\
& r'_2 = b_{12}a^i b^{n+1} b_{xy'}a, & r''_2 &= b_{01}a^{i+1} b^n b_{xy'}a, & r'''_2 &= b_{12}a^i b^{n-1} b_{xy'}a. \\
\text{Case 3: } m > 0 \text{ even, } n \text{ odd: } & \ell_3 = b_{02}a^m b b_{12}a^2 b^{n-1} b_{xy'}a, & r_3 &= b_{01}a^{m+1} b^{n+2} b_{xy'}a, \\
& r'_3 = b_{12}a^{m+i} b^{n+1} b_{xy'}a, & r''_3 &= b_{01}a^{m+i+1} b^n b_{xy'}a, & r'''_3 &= b_{12}a^{m+i} b^{n-1} b_{xy'}a. \\
\text{Case 4: } m \text{ odd, } n = 0: & \ell_4 = b_{01}a^m b b_{xy'}a^3, & r_4 &= b_{12}a^{m+1} b^2 b_{xy'}a, \\
& r'_4 = b_{01}a^{m+i} b b_{xy'}a, & r''_4 &= b_{12}a^{m+i+1} b_{xy'}a, & r'''_4 &= b_{xy''}a^{m+i+1}. \\
\text{Case 5: } m \text{ odd, } n > 0 \text{ even: } & \ell_5 = b_{01}a^m b b_{02}a^2 b^{n-1} b_{xy'}a, & r_5 &= b_{12}a^{m+1} b^{n+2} b_{xy'}a, \\
& r'_5 = b_{01}a^{m+i} b^{n+1} b_{xy'}a, & r''_5 &= b_{12}a^{m+i+1} b^n b_{xy'}a, & r'''_5 &= b_{01}a^{m+i} b^{n-1} b_{xy'}a. \\
\text{Case 6: } m = 0, n = 0: & \ell_6 = b^2 b_{xy'}a^3, & r_6 &= b_{11}a b^2 b_{xy'}a, \\
& r'_6 = b_{02}a^i b b_{xy'}a, & r''_6 &= b_{11}a^{i+1} b_{xy'}a, & r'''_6 &= b_{xy'}a^{i+1}. \\
\text{Case 7: } m = 0, n > 0 \text{ even: } & \ell_7 = b^2 b_{02}a^2 b^{n-1} b_{xy'}a, & r_7 &= b_{11}a b^{n+2} b_{xy'}a, \\
& r'_7 = b_{02}a^i b^{n+1} b_{xy'}a, & r''_7 &= b_{11}a^{i+1} b^n b_{xy'}a, & r'''_7 &= b_{02}a^i b^{n-1} b_{xy'}a. \\
\text{Case 8: } m > 0 \text{ even, } n = 0: & \ell_8 = b_{02}a^m b b_{xy'}a^3, & r_8 &= b_{11}a^{m+1} b^2 b_{xy'}a, \\
& r'_8 = b_{02}a^{m+i} b b_{xy'}a, & r''_8 &= b_{11}a^{m+i+1} b_{xy'}a, & r'''_8 &= b_{xy'}a^{m+i+1}. \\
\text{Case 9: } m > 0 \text{ even, } n > 0 \text{ even: } & \ell_9 = b_{02}a^m b b_{02}a^2 b^{n-1} b_{xy'}a, & r_9 &= b_{11}a^{m+1} b^{n+2} b_{xy'}a, \\
& r'_9 = b_{02}a^{m+i} b^{n+1} b_{xy'}a, & r''_9 &= b_{11}a^{m+i+1} b^n b_{xy'}a, & r'''_9 &= b_{02}a^{m+i} b^{n-1} b_{xy'}a.
\end{aligned}$$

■ **Table 1** The constituents of the SRS H

The interpretation is a model: Every left hand side and every right hand side have a prefix of the form $b^n a$ where n is odd. Hence $[\ell](x, y) = (1, 0) = [r](x, y)$ holds for every $(x, y) \in \mathcal{D}$ and rule $\ell \rightarrow r$.

We apply a Semantic Labelling [3] to G . We label only b symbols, and we label b by (x, y) only if $y \neq 0$. In effect this means that in each sequence of successive b symbols, only the rightmost b symbol receives a label. The symbol b labelled by (x, y) is denoted b_{xy} . Let H denote the resulting labelled SRS.

In order to compute the labels, we perform a case analysis on whether $m = 0$, $m > 0$ is even, or m is odd, and whether $n = 0$, $n > 0$ is even, or n is odd. For each case c , we get a family $H_c := \{\ell_c \rightarrow r_c, \ell_c \rightarrow r'_c, \ell_c \rightarrow r''_c, \ell_c \rightarrow r'''_c\}$ of labelled rules, indexed by even $i \geq 2$, $m, n \in \mathbb{N}_0$, $x \in \{0, 1\}$, $y \in \{0, 1, 2\}$. Table 1 shows the labelled strings $\ell_c, r_c, r'_c, r''_c, r'''_c$ in the nine cases c . We use the abbreviations $(x, y') = [a](x, y)$, $(x, y'') = [a](x, y')$. Note that $y', y'' \in \{1, 2\}$ and so that $[a](x, y'') = (x, y')$ and $[b](x, y') = [b](x, y'') = (1, 0)$. Note also that $y = 0$ and $y = 2$ yield the same labelled rule because the rightmost symbol in each left and right hand side is a .

Zantema's Theorem yields:

► **Lemma 5.** *If H terminates then G terminates.*

Fourth Step: Interpretation

In the remaining steps, we prove termination of the labelled SRS, H . We do so incrementally, using the concept of Relative Termination [1]. R is called *terminating relative to S* if there is no infinite $R \cup S$ derivation that contains infinitely many R steps. If S terminates and R terminates relative to S then $R \cup S$ terminates. Termination of R relative to S can be proven by a quasiorder $\succsim$ such that $\rightarrow_R \subseteq >$ and $\rightarrow_S \subseteq \sim$.

For the fourth step let $H' := H_1 \cup \{\ell_4 \rightarrow r_4''' \mid (x, y') \in \{(1, 2), (0, 1)\}\}$.

► **Lemma 6.** H' terminates relative to $H \setminus H'$.

Proof. We use an interpretation that counts the number of b_{12} and b_{01} symbols. All rules in H' decrease this number by 2, and all rules in $H \setminus H'$ keep it constant. ◀

Fifth Step: Tuple Representation

Termination of $H \setminus H'$ remains to be proved. The rules in $H \setminus H'$ keep the number of b_{12} and b_{01} symbols constant. This suggests one to apply some lexicographic order on a tuple representation of the rules, where the symbols b_{12} and b_{01} act as separators.

Instead of the tuple representation, we prefer a modified tuple representation: The Modified Tuple Representation $T_f(s)$ of $s \in \Sigma^*$ through the interpretation f w.r.t. a non-empty set $\Delta \subseteq \Sigma$ of separator symbols is defined by

$$T_f(s) = (f(s_0t_0), f(s_1t_1), \dots, f(s_{k-1}t_{k-1}), f(s_k)),$$

provided that $k \geq 0$, $s = s_0t_0s_1t_1 \dots s_{k-1}t_{k-1}s_k$, $s_i \in (\Sigma \setminus \Delta)^*$ for all $i \in \{0, \dots, k\}$, and $t_i \in \Delta$ for all $i \in \{0, \dots, k-1\}$. The point of this modification is to keep the separators t_i accessible to the interpretation f .

For the fifth step let

$$\begin{aligned} H'' := & H_2 \cup H_3 \cup \{\ell_4 \rightarrow r_4''' \mid (x, y') \in \{(1, 1), (0, 2)\}\} \cup \\ & \{\ell_4 \rightarrow r_4, \ell_4 \rightarrow r_4'', \ell_5 \rightarrow r_5 \mid (x, y') \in \mathcal{D}\} \cup \\ & \{\ell_c \rightarrow r_c'', \ell_c \rightarrow r_c''' \mid 5 \leq c \leq 9, (x, y') \in \mathcal{D}\} . \end{aligned}$$

► **Lemma 7.** H'' terminates relative to $H \setminus (H' \cup H'')$.

Proof. For our purposes, we define the interpretation f by $f(s) = |s|_{b_*} - |s|_{b_{11}} + |s|_{b_{01}}$. Here $|s|_{b_*}$ means the number of b symbols, without a label or with whatever label, in the string s . Note that $|s|_{b_*} - |s|_{b_{11}} \geq 0$ holds, whence $f(s) \in \mathbb{N}_0$, for all strings s .

We compare the Modified Tuple Representations through f lexicographically from left to right, denoted by $>_{\text{lex}}$. For instance, rule $\ell_4 \rightarrow r_4'''$ in the case $x = 1, y' = 1$ satisfies

$$T_f(\ell_4) = (f(b_{01}), f(a^m b b_{11} a^3)) = (2, 1) >_{\text{lex}} (1, 0) = (f(b_{12}), f(a^{m+i+1})) = T_f(r_4''') .$$

For rule $\ell_4 \rightarrow r_4'''$ in the case $x = 0, y' = 2$ we get

$$T_f(\ell_4) = (f(b_{01}), f(a^m b b_{02} a^3)) = (2, 2) >_{\text{lex}} (2, 0) = (f(b_{01}), f(a^{m+i+1})) = T_f(r_4''') .$$

Thus we have solved the two remaining instances of rule $\ell_4 \rightarrow r_4'''$.

The remaining rules have a suffix in $b_{xy'}a^+$ both at the left hand side and at the right hand side. For the remainder of this proof, we can simplify our argument by dropping this suffix since our order is closed under right contexts and is insensitive to the number of a symbols.

Case c	2	3	4	5	6	7	8	9
$T_f(\ell_c)$	$(3, n-1)$	$(3, n-1)$	$(2, 1)$	$(2, n+1)$	(2)	$(n+2)$	(2)	$(n+2)$
$T_f(r_c)$	$(2, n+2)$	$(2, n+2)$	$(1, 2)$	$(1, n+2)$	(2)	$(n+2)$	(2)	$(n+2)$
$T_f(r'_c)$	$(1, n+1)$	$(1, n+1)$	$(2, 1)$	$(2, n+1)$	(2)	$(n+2)$	(2)	$(n+2)$
$T_f(r''_c)$	$(2, n)$	$(2, n)$	$(1, 0)$	$(1, n)$	(0)	(n)	(0)	(n)
$T_f(r'''_c)$	$(1, n-1)$	$(1, n-1)$	-	$(2, n-1)$	(0)	(n)	(0)	(n)

■ **Table 2** The Modified Tuple Representations through f

Table 2 summarizes the $T_f(s)$ for the left hand sides and the various corresponding right hand sides of the rewrite rules. Each right hand side tuple that equals its left hand side tuple is boxed. Check that a right hand side tuple that is not boxed is less w.r.t. $>_{\text{lex}}$ than its left hand side tuple. ◀

Sixth Step: Recursive Path Order

► **Lemma 8.** $H \setminus (H' \cup H'')$ terminates.

Proof. The SRS $H \setminus (H' \cup H'')$ is ordered by a Recursive Path Order, where all symbols in $\{b, b_{01}, b_{02}, b_{12}\}$ are equivalent in precedence, and each symbol from this set is greater in precedence than any of b_{11} and a . ◀

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