

Quasipolynomial Size Frege Proofs of Frankl's Theorem on the Trace of Sets (Abstract)

James Aisenberg*

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112, USA
jaisenberg@math.ucsd.edu

Maria Luisa Bonet[†]

Lenguajes y Sistemas Informáticos
Universidad Politécnica de Cataluña
Barcelona, Spain
bonet@lsi.upc.edu

Sam Buss*

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112, USA
sbuss@math.ucsd.edu

We are interested in the exponential separation problem for Frege and extended Frege proof systems. There have been few candidate families of tautologies proposed to provide such a separation. Bonet, Buss and Pitassi [2] studied this question carefully, and found only two reasonable candidates other than partial consistency statements. These candidates were the matrix inversion principle $AB = I \rightarrow BA = I$ where A and B are square matrices over the field $\mathbb{F}_2$ and I is the identity matrix, and a theorem of Peter Frankl on the traces of finite sets. Recently Hrubes and Tzameret [4] showed that the matrix inversion principle (and other so-called hard matrix identities) have quasipolynomial size Frege proofs. We prove [1] that the tautologies based on Frankl's Theorem have quasipolynomial size Frege proofs. Thus, there are no remaining compelling combinatorial tautologies conjecturally separating Frege and extended Frege systems, except possibly Kneser-Lovász tautologies, see Istrate and Crăciun [5].

The Frankl's Theorem tautologies are based on:

Theorem 1 (Frankl [3]) *Let t be a positive integer and $m \leq n \frac{2^t-1}{t}$. Then for any $m \times n$ 0/1-matrix with distinct rows, there is a column such that if this column is deleted, the resulting $m \times (n-1)$ matrix will contain fewer than 2^{t-1} pairs of equal rows.*

The special case where $t = 1$, known as Bondy's Theorem, was shown by Bonet, Buss and Pitassi [2] to have polynomial size Frege proofs, and the $t = 2$

*Supported in part by NSF grants DMS-1101228 and CCF-1213151.

[†]Supported in part by grant TIN2010-20967-C04-02.

case, called Bollobas's Theorem, was shown by Nozaki, Arai, and Arai [6] to have polynomial size Frege proofs. We generalize this to all $t \geq 1$.

Theorem 2 *There are quasipolynomial size Frege proofs of the propositional translations of Frankl's Theorem.*

When t is fixed, the tautologies have Frege proofs that are constant depth, polynomial size reductions to the pigeonhole principle.

Theorem 3 *Fix $t > 0$. There are $\text{AC}^0\text{-Frege} + \text{PHP}_n^{n+1}$ proofs of the propositional translations of Frankl's Theorem which have polynomial size (in m, n) and in which all formulas have alternation depth $O(t) = O(1)$.*

The usual proof of Frankl's Theorem [3] proceeds in two stages: first it is shown that it suffices to consider hereditary matrices; second, a counting argument using the Kruskal-Katona Theorem establishes the theorem from the hereditary case. [2] showed that the second stage has polynomial size Frege proofs. Theorem 2 also uses a reduction to the hereditary case, but the construction is quite different from the one usually used for Frankl's Theorem.

For the proof Theorem 3, the reduction to the hereditary case is carried out with constant-depth, polynomial size Frege proofs. The second stage of the proof shows that a hereditary counterexample to Frankl's Theorem implies a violation of the pigeonhole principle. This is done with the aid of a version of the Kruskal-Katona Theorem.

References

- [1] J. AISENBERG, M. L. BONET, AND S. BUSS, *Quasipolynomial-size Frege proof of Frankl's theorem on the trace of finite sets*. In preparation, 2014.
- [2] M. L. BONET, S. R. BUSS, AND T. PITASSI, *Are there hard examples for Frege systems?*, in Feasible Mathematics II, P. Clote and J. Remmel, eds., Boston, 1995, Birkhäuser, pp. 30–56.
- [3] P. FRANKL, *On the trace of finite sets*, Journal of Combinatorial Theory, Series A, 34 (1983), pp. 41–45.
- [4] P. HRUBEŠ AND I. TZAMERET, *The proof complexity of polynomial identities*, in Proc. 24th IEEE Conf. on Computational Complexity (CCC), 2009, pp. 41–51.
- [5] G. ISTRATE AND A. CRĂCIUN, *Proof complexity and the Kneser-Lovász theorem (I)*. arXiv:1402.4338 [cs.CC], February 2014.
- [6] A. NOZAKI, T. ARAI, AND N. H. ARAI, *Polynomial-size Frege proofs of Bollobás' theorem on the trace of sets*, Proceedings of the Japan Academy, Series A. Math. Sci., 84 (2008), pp. 159–161.