

Narrow Proofs May Be Maximally Long (Abstract*)

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Abstract. We prove that there are 3-CNF formulas over n variables that can be refuted in resolution in width w but require resolution proofs of size $n^{\Omega(w)}$. This shows that the simple counting argument that any formula refutable in width w must have a proof in size $n^{O(w)}$ is essentially tight. Moreover, our lower bounds can be generalized to polynomial calculus resolution (PCR) and Sherali-Adams, implying that the corresponding size upper bounds in terms of degree and rank are tight as well. Our results do not extend all the way to Lasserre, however—the formulas we study have Lasserre proofs of constant rank and size polynomial in both n and w .

The main contribution of this paper is showing that the upper bounds on proof size in terms of width for resolution, degree for PCR, and rank for Sherali-Adams are essentially tight (up to constant factors in the exponent). Moreover, an interesting feature of our result is that we can actually use the *same formula family* to prove tightness simultaneously for all the proof systems. What this means is that we obtain upper bounds on size in resolution that tightly match lower bounds in the much stronger systems PCR and Sherali-Adams (which are in turn tight for these systems since resolution width is an upper bound on both PCR degree and Sherali-Adams rank). The formal statement of this result is as follows.

Theorem 1. *Let $w = w(n)$ be such that $w = O(n^c)$ for some positive constant $c < 1/4$. Then there are 3-CNF formulas $F_{n,w}$ with $O(wn)$ clauses over $O(n)$ variables such that the following holds.*

1. $F_{n,w}$ has a resolution refutation in simultaneous size $n^{O(w)}$, width $O(w)$ and space $O(w)$.
2. Any refutation of $F_{n,w}$ in resolution, PCR, or Sherali-Adams must have size $n^{\Omega(w)}$.

For resolution this actually shows something slightly stronger than that the counting upper bound on size in terms of width is tight. Namely, since the

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formulas in Theorem 1 have the same asymptotic upper bound on space as on width, it follows that even for formulas of space complexity $O(w)$ —which is a more stringent requirement than width complexity $O(w)$ —it is still impossible to obtain any size upper bound better than $n^{O(w)}$ in general.

Theorem 1 has an interesting consequence for the analysis of CDCL solver performance, which we state as a formal corollary. By way of background, it was shown in [1] that if a CNF formula F over n variables has a resolution refutation in width w , then with high probability any CDCL solver will only need time $n^{O(w)}$ to decide that F is indeed unsatisfiable. An obvious question is whether this result is tight. Theorem 1 shows that the answer is “yes,” since no CDCL solver can run faster than the shortest resolution proof it can possibly find.

Corollary 1. *There are formulas F over n variables refutable in resolution in width w for which any resolution-based CDCL solver cannot run faster than $n^{\Omega(w)}$, and hence the result in [1] is optimal up to constants in the exponent.*

Another interesting aspect of our lower bound for resolution is in the context of Berkholz’s EXPTIME-completeness result for deciding resolution width [2]. What Berkholz showed is that given a formula F over n variables and a parameter w , it cannot be decided in time less than $n^{(w-3)/12}$ whether F has a resolution refutation in width w or not. Optimizing the constants in Theorem 1, we can show that there are 4-CNF formulas refutable in width w for which no resolution refutation can be shorter than $n^{w/2-o(1)}$. It is worth noting that this bound is stronger than that in [2], although it of course applies only for the more restricted setting where the algorithm has to output a width- w resolution refutation rather than for the general decision problem. Still, we believe this sheds interesting light on Berkholz’s result.

References

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