

From Small Space to Small Width in Resolution

Yuval Filmus¹, Massimo Lauria², Mladen Mikša², Jakob Nordström², and
Marc Vinyals²

¹ Institute for Advanced Study, Princeton, NJ 08540, USA

² KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden

Abstract. In 2003, Atserias and Dalmau established that the space complexity of formulas in resolution is an upper bound on the width needed to refute them. Their proof is beautiful but somewhat mysterious in that it relies on tools from finite model theory. We give an alternative proof that works by simple syntactic manipulations of resolution refutations. As a by-product, we develop a “black-box” technique for proving space lower bounds.

Resolution is arguably the most well-studied proof system in propositional proof complexity, and has served as a natural starting point in the quest to prove lower bounds on *proof length/size* for increasingly stronger proof systems. It is also intimately connected to SAT solving, in that it lies at the foundation of state-of-the-art SAT solvers using so-called conflict-driven clause learning (CDCL). This connection has motivated the study of *proof space* as a second interesting complexity measure for resolution.

For both length and space, it turns out that a key role is played by the auxiliary measure of *width*, i.e., the size of a largest clause in the proof. Ben-Sasson and Wigderson [3] showed that there are short resolution refutations of a formula if and only if there are also (reasonably) narrow ones, and almost all known lower bounds on resolution length can be (re)derived using this connection.

Atserias and Dalmau [1] established that width also provides lower bounds on space, resolving a problem that had been open since the space complexity of propositional proofs started being studied in the late 1990s. This means that for space also, almost all known lower bounds can be rederived by using width lower bounds and appealing to [1]. This is not a two-way connection, however, in that formulas of almost worst-case space complexity may require only constant width as shown in [2].

The starting point of our work is the lower bound on space in terms of width in [1]. This is a very elegant but also magical proof in that it translates the whole problem to Ehrenfeucht–Fraïssé games in finite model theory, and shows that resolution space and width correspond to players’ strategies in such games. Unfortunately, this also means that one obtains essentially no insight into what is happening on the proof complexity side (other than that the bound on space in terms of width is true).

In this talk, we present a purely combinatorial proof in terms of simple syntactic manipulations of resolution refutations. To summarize in one sentence, we

study the conjunctions of clauses in memory at each time step in a small-space refutation, negate these conjunctions and then expand them to conjunctive normal form again, and finally argue that the new sets of clauses listed in reverse order (essentially) constitute a small-width refutation of the same formula.

This new, simple proof also allows us to obtain a new technique for proving space lower bounds. This approach is reminiscent of [3] in that one defines a static “progress measure” on refutations and argues that when a refutation has made substantial progress it must have high complexity with respect to the proof complexity measure under study. Previous lower bounds on space have been inherently adaptive and in that sense less explicit.

This results were published in [4], and a similar proof of the space-width relation has also been obtained independently by Razborov [5].

References

1. Atserias, A., Dalmau, V.: A combinatorial characterization of resolution width. In: Proceedings of the 18th IEEE Annual Conference on Computational Complexity (CCC '03). pp. 239–247 (Jul 2003)
2. Ben-Sasson, E., Nordström, J.: Short proofs may be spacious: An optimal separation of space and length in resolution. In: Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08). pp. 709–718 (Oct 2008)
3. Ben-Sasson, E., Wigderson, A.: Short proofs are narrow—resolution made simple. In: Proceedings of the 31st Annual ACM Symposium on Theory of Computing (STOC '99). pp. 517–526 (May 1999)
4. Filmus, Y., Lauria, M., Mikša, M., Nordström, J., Vinyals, M.: From small space to small width in resolution. In: Proceedings of the 31st Symposium on Theoretical Aspects of Computer Science (STACS '14). pp. 300–311 (Mar 2014)
5. Razborov, A.: Personal communication (2014)