

Parameter-free induction in bounded arithmetic

Emil Jeřábek

Institute of Mathematics of the Academy of Sciences
Žitná 25, 115 67 Praha 1, Czech Republic, email: jerabek@math.cas.cz

In the area of strong fragments of Peano arithmetic, it proved fruitful to study not just the usual induction fragments $I\Sigma_i$, but also fragments axiomatized by *parameter-free* induction schemata

$$(\varphi\text{-}IND^-) \quad \varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x)$$

where φ has no free variable other than x , and theories axiomatized using the closely related induction *inference rules*

$$(\varphi\text{-}IND^R) \quad \frac{\varphi(0, \vec{y}) \quad \varphi(x, \vec{y}) \rightarrow \varphi(x+1, \vec{y})}{\varphi(x, \vec{y})}.$$

See e.g. [6, 1, 2]; in particular, the last two papers detail the connection of induction rules and parameter-free schemata to reflection principles.

In contrast, induction rules and parameter-free schemata have been largely ignored in bounded arithmetic literature. Apart from IE_i^- briefly discussed in [5], the corresponding fragments of Buss's S_2 were only studied in [3, 4]. Many basic questions about these fragments were disregarded, in particular, neither paper even mentions Π_i^b -induction rules and parameter-free schemata, despite that they could be expected to behave rather differently from Σ_i^b rules by analogy with the case of strong fragments.

In this talk, we will try to systematically investigate parameter-free and inference rule versions of the theories T_2^i and S_2^i : that is, $\hat{\Sigma}_i^b\text{-}IND^-$, $\hat{\Pi}_i^b\text{-}IND^-$, $\hat{\Sigma}_i^b\text{-}IND^R$, $\hat{\Pi}_i^b\text{-}IND^R$, and the corresponding variants of $PIND$. (While there are many other ways how to make parameter-free schemata or rules out of the various induction or minimization schemata equivalent to S_2^i or T_2^i , most turn out to be equivalent to one of those above; in particular, parameter-free *rules* $\hat{\Sigma}_i^b/\hat{\Pi}_i^b\text{-}(P)IND^{R-}$ are equivalent to their versions with parameters.)

We will particularly focus on the following questions:

- Reductions between the various rules and schemata. (A rule R reduces to a rule R' if $T + R \subseteq T + R'$ for every theory $T \supseteq PV_1$.) See Fig. 1 for a summary.
- Conservativity results between the fragments, and the formula complexity of axioms.
- The number of instances (nesting) of R needed to generate the closure $T + R$, and finite axiomatizability of fragments.
- Reformulation of axioms and rules in terms of reflection principles for propositional proof systems.
- Conditional or relativized separations between the fragments.

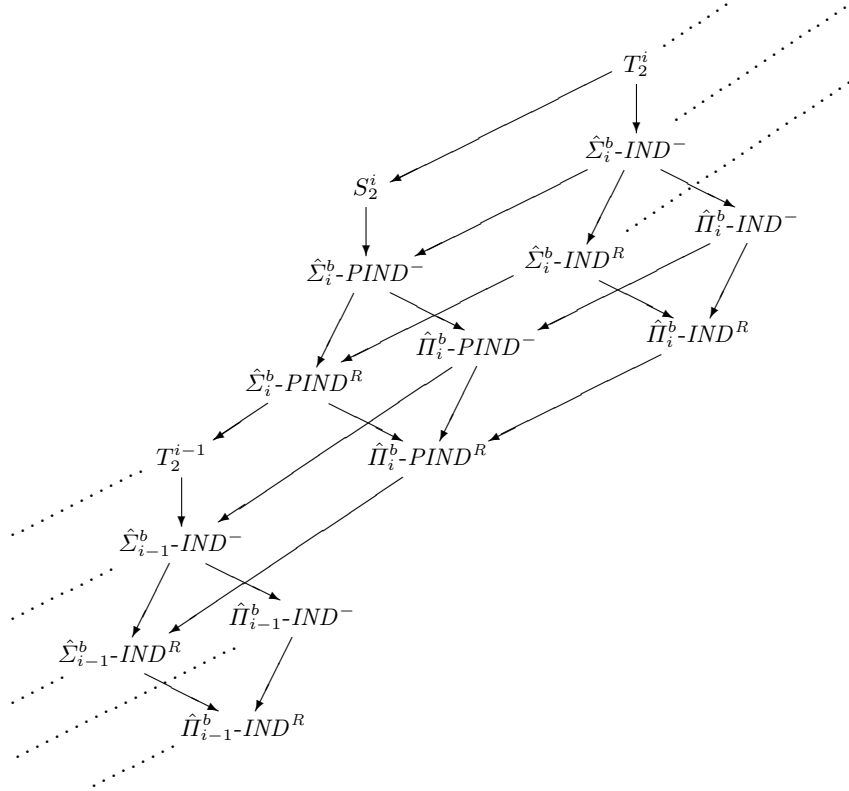


Fig. 1. Reductions between rules

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