

An Introduction to the Clocked Lambda Calculus

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Abstract

We give a brief introduction to the *clocked λ -calculus*, an extension of the classical λ -calculus with a unary symbol τ used to witness the β -steps. In contrast to the classical λ -calculus, this extension is infinitary strongly normalising and infinitary confluent. The infinitary normal forms are enriched Lévy–Longo Trees, which we call *clocked Lévy–Longo Trees*.

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1 The Clocked Lambda Calculus

The classical λ -calculus [1] is based on the β -rule

$$(\lambda x.M)N \rightarrow M[x := N]$$

This calculus is neither infinitary normalising

$$(\lambda x.xx)(\lambda x.xx) \rightarrow (\lambda x.xx)(\lambda x.xx) \rightarrow \dots,$$

nor infinitary confluent. To see this, let

$$Y_0 \equiv \lambda f.\omega_f \omega_f \qquad \omega_f \equiv \lambda x.f(xx)$$

be Curry’s fixed point combinator. The term $Y_0 I$ admits the infinite (strongly convergent) rewrite sequences

$$\begin{aligned} Y_0 I &\rightarrow_\beta (\lambda x.I(xx))(\lambda x.I(xx)) \rightarrow_\beta I^\omega \\ Y_0 I &\rightarrow_\beta (\lambda x.I(xx))(\lambda x.I(xx)) \rightarrow_\beta^2 \Omega = (\lambda x.xx)(\lambda x.xx) \end{aligned}$$

Here infinitary confluence fails: the terms I^ω and Ω have no common reduct since they reduce only to themselves (see [2] and [14, Chapter 12]). Even though infinitary confluence fails, the calculus has the property of infinitary unique normal forms. When considering the β - and η -rule together, even this property fails, see further [8, 4].

The *clocked λ -calculus* [10] consists of the following two rules:

$$\begin{aligned} (\lambda x.M)N &\rightarrow \tau(M[x := N]) \\ \tau(M)N &\rightarrow \tau(MN) \end{aligned}$$

Here every β -step produces a symbol τ as a witness of the step. The second rule is used to move the τ ’s out of the way of applications and hence potential β -redexes. We write $\rightarrow_{\odot}$ for the reduction relation of the clocked λ -calculus.

For a simple example, consider the following reduction:

$$III \rightarrow_{\odot} \tau(I)I \rightarrow_{\odot} \tau(II) \rightarrow_{\odot} \tau(\tau(I))$$

where $l = \lambda x.x$. Note that the second step moves the τ out of the way of a β -redex.

As a second example, let us consider Curry's fixed point combinator:

$$\begin{aligned} Y_0 f &\equiv (\lambda f. \omega_f \omega_f) f \rightarrow_{\beta} \tau(\omega_f \omega_f) \\ \omega_f \omega_f &\rightarrow_{\beta} \tau(f(\omega_f \omega_f)) \end{aligned}$$

Hence $Y_0 f$ rewrites to the infinite normal form

$$Y_0 f \rightarrow_{\beta}^{\infty} \tau(\tau(f(\tau(f(\tau(f(\dots)))))))$$

written without brackets as $\tau\tau z\tau z\tau z\dots$

The clocked λ -calculus enjoys the properties of infinitary confluence, infinitary strong normalization [12, 15, 5] and hence infinitary unique normal forms:

SN $^\infty$: all infinite rewrite sequences are strongly convergent;

CR $^\infty$: $\forall M, N_1, N_2 (N_1 \leftarrow_R M \rightarrow_R N_2 \implies N_1 \rightarrow_R \cdot \leftarrow_R N_2)$;

UN $^\infty$: $\forall M, N_1, N_2 (N_1 \leftarrow_R M \rightarrow_R N_2 \text{ and } N_1, N_2 \text{ normal forms} \implies N_1 \equiv N_2)$.

► **Lemma 1.** *The relation $\rightarrow_{\beta}^{\infty}$ has the properties CR $^\infty$, SN $^\infty$ and UN $^\infty$.*

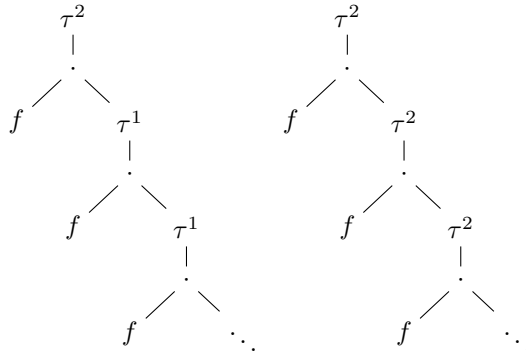
2 Clocked Lévy–Longo Trees

The unique infinitary normal forms with respect to $\rightarrow_{\beta}^{\infty}$ are *clocked Lévy–Longo Trees* [7, 9, 10], that is, Lévy–Longo Trees (a variant of Böhm Trees) enriched with symbols τ witnessing the β -steps performed in the reduction to the normal form. We write $\text{LLT}_{\beta}^{\infty}(M)$ for the unique infinite normal form of M .

Consider the well-known fixed point combinators of Curry and Turing, Y_0 and Y_1 :

$$\begin{aligned} Y_0 &\equiv \lambda f. \omega_f \omega_f & Y_1 &\equiv \eta \eta \\ \omega_f &\equiv \lambda x. f(xx) & \eta &\equiv \lambda x f. f(xxf) \end{aligned}$$

Figure 1 displays the clocked Lévy–Longo Trees of $Y_0 f$ (left) and $Y_1 f$ (right) where we write $\tau^n(t)$ for $\tau(\underbrace{\tau(\dots(\tau(t))\dots)}_{n\text{-times}})$. For $Y_0 f$ we have seen the reduction to the infinite normal form



■ **Figure 1** Clocked Lévy–Longo Trees $\text{LLT}_{\beta}^{\infty}(Y_0 f)$ and $\text{LLT}_{\beta}^{\infty}(Y_1 f)$ of $Y_0 f$ and $Y_1 f$, respectively.

above, and a similar computation leads to the clocked Lévy–Longo Tree of $Y_1 f$. The τ 's in the clocked Lévy–Longo Tree witness the number of head reduction steps needed to normalise the corresponding subterm to weak head normal form.

3 Discriminating Lambda Terms

For more details on the results in this section, we refer to [10, 9].

We define $\rightarrow_\tau$ by the rule

$$\tau(M) \rightarrow M$$

and use $=_\tau$ to denote the equivalence closure of $\rightarrow_\tau$. For $M, N \in \text{Ter}^\infty(\lambda\tau)$, we define

- (i) $M \succeq_{\text{LLT}} N$, M is globally improved by N iff $\text{LLT}_{\text{LLT}}(M) \twoheadrightarrow_\tau \text{LLT}_{\text{LLT}}(N)$;
- (ii) $M =_{\text{LLT}} N$, M eventually matches N iff $\text{LLT}_{\text{LLT}}(M) =_\tau \text{LLT}_{\text{LLT}}(N)$.

For example, Y_0f globally improves Y_1f ($Y_0f \preceq_{\text{LLT}} Y_1f$) as can be seen from the clocked Lévy–Longo Trees of Y_0f and Y_1f in Figure 1.

► **Theorem 2.** *Clocks are accelerated under reduction, that is, if $M \twoheadrightarrow N$, then the reduct N improves M globally, that is, $\text{LLT}_{\text{LLT}}(M) \twoheadrightarrow_\tau \text{LLT}_{\text{LLT}}(N)$.*

As a consequence we obtain the following method for discriminating λ -terms:

► **Theorem 3.** *Let M and N be λ -terms. If N cannot be improved globally by any reduct of M , then $M \neq_\beta N$.*

In [9] we use this theorem to answer the following question of Selinger and Plotkin [13]: *Is there a fixed point combinator Y such that*

$$A_Y \equiv Y(\lambda z.fzz) =_\beta Y(\lambda x.Y(\lambda y.fxy)) \equiv B_Y$$

or in other notation:

$$\mu z.fzz =_\beta \mu x.\mu y.fxy,$$

with the usual definition $\mu x.M(x) = Y(\lambda x.M(x))$. The terms A_Y and B_Y have the same Böhm Trees, namely the solution of $T = fTT$. Clocked Lévy–Longo Trees can be employed to show that such fixed point combinators do not exist, see [9].

For a large class of λ -terms the clocks are invariant under reduction, that is, the clocked Lévy–Longo Trees coincide up to insertion and removal of a finite number of τ 's.

► **Definition 4** (Simple terms). A redex $(\lambda x.M)N$ is called:

- (i) *linear* if x has at most one occurrence in M ;
- (ii) *call-by-value* if N is a normal form; and
- (iii) *simple* if it is linear or call-by-value.

A λ -term M is *simple* if (a) it has no weak head normal form, or the head reduction to whnf contracts only simple redexes and is of one of the following forms: (b) $M \twoheadrightarrow_h \lambda x.M'$ with M' a simple term, or (c) $M \twoheadrightarrow_h yM_1 \dots M_m$ with $M_1, \dots, M_m$ simple terms.

Note that this definition is inherently coinductive; this is similar to the definition of Böhm Trees in [1]. The infinitary rewrite relation itself can also be defined coinductively, see further [3, 11, 6].

► **Theorem 5.** *Let N be a reduct of a simple term M . Then N eventually matches M (i.e., $\text{LLT}_{\text{LLT}}(M) =_\tau \text{LLT}_{\text{LLT}}(N)$).*

For simple terms, the discrimination method can be simplified as follows:

► **Theorem 6.** *If simple terms M, N do not eventually match ($\text{LLT}_{\odot}(M) \neq_{\tau} \text{LLT}_{\odot}(N)$), then they are not β -convertible, that is, $M \neq_{\beta} N$.*

► **Example 7.** We show that the fixed point combinators $Y_0, Y_1, Y_2, \dots$ of the Böhm sequence are all inconvertible. For $n \geq 1$, define

$$Y_n = \eta\eta\delta^{\sim n-1}$$

where

$$\begin{aligned} MN^{\sim 0} &= M \\ MN^{\sim n+1} &= MNN^{\sim n} \end{aligned}$$

The clocked Lévy–Longo Trees of Y_0x and Y_1x are shown in Figure 1. We now determine the clocked Lévy–Longo Trees of Y_nx for $n \geq 2$:

$$\begin{aligned} Y_n &\equiv \eta\eta\delta^{\sim n-1}x \\ &\rightarrow_{\odot} \tau(\lambda f.f(\eta\eta f))\delta^{\sim n-1}x \\ &\rightarrow_{\odot}^* \tau((\lambda f.f(\eta\eta f))\delta)\delta^{\sim n-2}x \\ &\rightarrow_{\odot} \tau(\tau(\delta(\eta\eta\delta))\delta^{\sim n-2}x) \\ &\rightarrow_{\odot}^* \tau^2(\delta(\eta\eta\delta)\delta^{\sim n-2}x) \\ &\rightarrow_{\odot}^* \tau^4(\delta(\eta\eta\delta\delta)\delta^{\sim n-3}x) \\ &\vdots \\ &\rightarrow_{\odot}^* \tau^{2n-2}(\delta(\eta\eta\delta^{\sim n-1})x) \\ &\rightarrow_{\odot}^* \tau^{2n}(x(\eta\eta\delta^{\sim n-1}x)) \end{aligned}$$

None of these steps duplicate a redex, hence Y_n is a simple term. We have

$$\text{LLT}_{\odot}(Y_nx) \equiv \tau^{2n}(x \text{LLT}_{\odot}(Y_nx))$$

Observe that all of the clocked Lévy–Longo Trees $\text{LLT}_{\odot}(Y_nx)$ differ in an infinite number of τ 's. By Theorem 6 it follows that all terms in the Böhm sequence are inconvertible.

4 Atomic Clocked Lambda Calculus

The clocked λ -calculus can be enhanced to not only recording whether head reduction steps have taken place, but also where they took place. We use $\{\lambda, L, R, \tau\}^*$ for the positions.

The *atomic clocked λ -calculus* consists of the rules

$$\begin{aligned} (\lambda x.M)N &\rightarrow \tau_{\epsilon}(M[x := N]) \\ \tau_p(M)N &\rightarrow \tau_{Lp}(MN) \end{aligned}$$

The atomic clocks further strengthen the discrimination power of method Lévy–Longo Trees.

Let $S = \lambda abc.ac(bc)$. For $k, n_1, \dots, n_k \in \mathbb{N}$ define a fixed point combinator $Y^{\langle n_1, \dots, n_k \rangle}$ by

$$Y^{\langle n_1, \dots, n_k \rangle} = G_{n_k}[\dots G_{n_1}[Y_0] \dots]$$

where $G_n = \square(SS)S^{\sim n}I$.

As fixed point combinators, they all have the same Lévy–Longo Tree $\lambda x.x(x(x(\dots)))$. However, using atomic clocked Lévy–Longo Trees we have shown in [9] that all these fixed point combinators are different, all of them are inconvertible: $\vec{n} \neq \vec{m}$ implies $Y^{\vec{n}} \neq_{\beta} Y^{\vec{m}}$.

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