

Normal Forms and Infinity

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May 26, 2014

1 Introduction

Normal forms, and especially ground normal forms, are the things we consider as passive data in rewriting. Rewriting stops at normal forms, rewrite models have them as values.

In term rewriting we know what a normal form is — we can even define it more generally, at the level of Abstract Reduction Systems: a normal form is a value not in the domain of the reduction relation. Moreover, this concept remains unaffected regardless whether we look at a single-step relation $\rightarrow$ or its transitive closure $\rightarrow^+$.

The concept does not survive the transition into the world of pre-orders unscathed, since reflexivity ensures that there are no normal forms in the previous sense. In that context one can consider notions of quasi-normal-forms, values that either (i) only reduce to themselves, or values that (ii) can reduce to themselves from any value they reduce to.

How about infinitary rewriting? Is the question what a “proper” normal form is as unproblematic as the literature suggests? There are potential issues with terms of infinite size, and especially in connection with infinite reductions.

We will be discussing some of the issues related to that and derive the new concept of a pseudo-constructor which is better suited for co-inductive reasoning and model-based reasoning.

2 Terms of infinite size

One problem with normal forms of infinite size is that their characteristic is undecidable, and — not unrelated to that — that being a normal form is an intrinsically discontinuous property.

One way to observe a discontinuity is the following: take any infinite term t — it necessarily has an infinite set of positions. Now take any non-reflexive rewrite

rule $l \rightarrow r$; we have $t[l]_p \rightarrow t[r]_p$ for any position p in t . One can argue that one should have $t \rightarrow t$ in the limit for continuity reasons: we can take the set $A = \{t[r]_p \mid p \in \text{Pos}(t)\}$ of all contracta derived from t and form its topological closure $\text{Cl}(A)$ — which in this case simply adds t itself to the set. Then we form the inverse image of this set under the relation $\rightarrow$, i.e. we form the set $B = \{u \mid u \rightarrow s, s \in \text{Cl}(A)\}$. If the relation $\rightarrow$ were upper semi-continuous then its inverse image would preserve (topologically) closed sets, but B is not closed: it contains all $t[l]_p$ but not their limit t .

However, reflexivity of rewriting is something we could deal with separately anyway.

The clear discrimination provided by the yes/no answer to the question: “is t a normal form or not” is discontinuous in its own right. We could address that by providing a sliding scale of normalness, in which a term is assigned a numerical value for “how normal” it actually is. We will revisit that thought.

3 Normal Forms of Infinite Reductions

We derive infinite reduction relations from a single-step reduction $\rightarrow$ by various means [3, 2, 1], leading to slightly different infinite reduction relations. These have therefore (potentially) different normal forms.

Pretty much all these notions result in reflexive relations and therefore need some modification to avoid the complete absence of normal forms in the traditional ARS interpretation of the terminology.

That issue aside, the different notions of infinitary reduction fall broadly into two categories: the ones that “extend to the left”, and the ones which do not.

Infinitary reductions add topological limits to a reduction relation in some way, and originally these were only formed in the codomain of the relation. The most general of these is the relation $\rightarrow_p$ (from [2]): here, we have $t \rightarrow_p u \iff u \in \text{Cl}\{s \mid t \rightarrow^* s\}$ — and we can avoid the intrinsic reflexivity of this construction by replacing $\rightarrow^*$ by $\rightarrow^+$ in the definition. This relation, as well as its subrelations such as weak and strong reduction, have the same normal forms as the single-step relation, and this is therefore what is normally considered an infinitary normal form.

However, some constructions add limits at either end of the relation. This applies to the relation $\rightarrow_t$ from [2] (making the graph of the relation topologically closed), as well as the co-inductively defined relation $\twoheadrightarrow$ from [1]. Or we can define a relation $\twoheadrightarrow_b$ as the smallest that contains $\rightarrow$, is transitive, and where both $\twoheadrightarrow_b$ and $\leftarrow_b$ are pointwise closed. All these affect the notion of normal form.

Take the iTRS with the single ground rewrite rule $F(A) \rightarrow A$. For all notion of infinitary reduction that “only extend on the right” we have (for this system) that their transfinite reduction relation coincides with ordinary transitive closure, and normal forms are exactly those terms not containing $F(A)$ as a subterm. However, this is not true for reductions that “extend on the left”. In particular, we have $F^\infty \twoheadrightarrow A$ (as well as $F^\infty \rightarrow_t A$ and $F^\infty \twoheadrightarrow_b$

A): under the context $F(\cdot)$ we can co-inductively assume $F^\infty \twoheadrightarrow A$, giving us $F^\infty = F(F^\infty) \twoheadrightarrow F(A)$, $F(A) \twoheadrightarrow A$ holds because $\rightarrow$ is included in $\twoheadrightarrow$, and eventually we get $F^\infty \twoheadrightarrow A$ by transitivity. That means that F^∞ is no longer a normal form under the relation $\twoheadrightarrow$.

Relations that “extend to the left” are somewhat uncharted territory in infinitary rewriting, and require in several ways a re-think. For example, the notion of overlap needs to be re-thought as rules can overlap “in the limit”: add to our example another ground rewrite rule $F(B) \rightarrow B$. The resulting system is orthogonal, but it does not have unique normal forms under $\twoheadrightarrow$, as we have both $F^\infty \twoheadrightarrow A$ and $F^\infty \twoheadrightarrow B$.

4 How normal is my form?

In Constructor TRSs we typically only count constructor terms as “proper” normal forms. These can be computed gradually — from the top: once a constructor is at the stop it will stay there. Consequently we can also provide a continuous quantitative measure for “how normal” a term t is: set it to 2^{-p} where p is the length of the shortest position of t at which we find a non-constructor.

In arbitrary term-rewriting we may or may not have constructors, but we can describe terms that *behave like* constructors: a *pseudo-constructor* is a finite linear term t , such that (i) does not unify with any left-hand side, and (ii) all its subterms are either pseudo-constructors or variables.

This definition ensures that a pseudo-constructor is a normal form, but also that when we substitute terms for its variables then all redexes will remain below the substitution barrier, even after rewriting. The reason for the linearity constraint is the following: although $F(x, x)$ is non-unifiable with $F(A, B)$ there could still be a substitution instance of $F(x, x)$ which rewrites to $F(A, B)$.

Infinitary pseudo-constructors can then be defined co-inductively: an *infinitary pseudo-constructor* is either a variable, or a substitution instance of a finite pseudo-constructor in which all substitutes are infinitary pseudo-constructors.

Are infinitary pseudo-constructors normal forms of our extend-to-the-left relations? They are for $\rightarrow_b$, however both $\rightarrow_t$ and $\twoheadrightarrow$ are automatically reflexive on infinite terms. The topologically closed relation $\rightarrow_t$ is reflexive on infinite terms, provided the rewrite system contains an (arbitrary) rule $l \rightarrow r$: let t be an infinite term, the set of pairs $\{(t[l]_p, t[r]_p) \mid p \in \text{Pos}(t)\}$ is a subset of $\rightarrow$, and the closure of this set will contain (t, t) . Regarding $\twoheadrightarrow$, it is also reflexive on infinite terms, even if we use transitive closure (rather than reflexive-transitive closure) in its definition, and even if the rewrite system is empty: let $F(t_1, \dots, t_n)$ be an infinite term, such that t_i is infinite; we coinductively assume $t_i \twoheadrightarrow t_i$ and get by congruence $F(t_1, \dots, t_n) \twoheadrightarrow F(t_1, \dots, t_n)$. However, infinite pseudo-constructors are normal forms of the irreflexive kernels of these relations.

Finite ground normal forms are automatically pseudo-constructors. Infinite ground normal forms (of the single-step relation) may not be pseudo-constructors though: in particular, the term F^∞ is not an infinitary pseudo-

constructor w.r.t. the rule $F(A) \rightarrow A$. More generally, whenever we have a term with an infinite path made solely from defined symbols then the infinite term cannot be a pseudo-constructor.

5 What can we do with pseudo-constructors? Translate them!

If a TRS is orthogonal or almost orthogonal, we can turn pseudo-constructors into constructors, and the system as a whole into a constructor TRS: by definition, the strict subterms of the left-hand sides of rules of such systems are necessarily pseudo-constructors.

So, we can apply the following transformation:

1. Double up the signature — provide for each symbol F of the old signature two new ones, one defined (F_d), one constructor (F_c).
2. Define two operations on terms: the functions $\lceil _ \rceil, \lfloor _ \rfloor$ replace all function symbols in a term by their defined and constructor versions, respectively.
3. Replace each rule $F(p_1, \dots, p_n) \rightarrow t$ of the old system with the new rule: $F_d(\lfloor p_1 \rfloor, \dots, \lfloor p_n \rfloor) \rightarrow \lceil t \rceil$.
4. For each finite (and minimal w.r.t. subsumption) pseudo-constructor $t = G(t_1, \dots, t_k)$ add the rule $G_d(\lfloor t_1 \rfloor, \dots, \lfloor t_k \rfloor) \rightarrow \lfloor t \rfloor$.

The minimality requirement in the last point is not needed for soundness, it is there to avoid redundancies in the system — and keep it finite, if both signature and rewrite rule set are finite. The resulting system will be almost orthogonal: the constructor formation rules can overlap, but only trivially.

It is not hard to see that the $\lceil _ \rceil$ function preserves and reflects semantic equality. Moreover, if t is a ground normal form of the old system then $\lceil t \rceil$ will rewrite to $\lfloor t \rfloor$ in the new system.

6 Semantic Bonus

With a slight tweak on the standard definitions, infinitary terms (and their substitutions) over any signature form a CPO (and even a Scott-Ershov domain), with (the tweaked) subsumption relation as its underlying order.

The tweak is simply to (i) assume a well-ordering on the variable set, and (ii) to work with so-called *strict* substitutions which, whenever they substitute variables for variables, are not allowed to replace a variable by a bigger variable. If we now require for the tweaked subsumption relation $\sigma \sqsubseteq \theta \iff \exists \tau. \tau \circ \sigma = \theta$, that the witness substitution τ is strict then $\sqsubseteq$ gives a CPO structure on strict substitutions, with the identity substitution as bottom element.

This provides us with a way of forming models based on normal forms: use as carrier set the set of open (pseudo-) constructor terms. Its CPO structure

allows us to find fixpoints of continuous (or increasing) functions, giving us a way of solving recurrences in the model.

References

- [1] Jörg Endrullis, Helle Hvid Hansen, Dimitri Hendriks, Andrew Polonsky, and Alexandra Silva. A coinductive treatment of infinitary rewriting. In *Workshop on Infinitary Rewriting*, page 8, 2013.
- [2] Stefan Kahrs. Infinitary rewriting: Foundations revisited. In Christopher Lynch, editor, *Proceedings of the 21st International Conference on Rewriting Techniques and Applications*, volume 6 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 161–176, Dagstuhl, Germany, 2010. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [3] Richard Kennaway and Fer-Jan de Vries. *Term Rewriting Systems*, chapter Infinitary Rewriting, pages 668–711. Cambridge University Press, 2003.