

Dynamical Systems, Attractors, and Well-Behaved Infinitary Rewriting

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The basic notion of interest in dynamical systems is to study the long-term behaviour of iterations of continuous maps on compact (or, merely complete) metric spaces.

At RTA 2012 Andersen and Simonsen introduced the study of term rewriting systems as dynamical systems. One of the notions introduced there was to study the evolution of *sets*, A , of terms under the map F induced by a TRS in the following way: For each set A , $F(A)$ is the set of terms t such that either there is a term $s \in A$ with $s \rightarrow t$, or $t \in A$ is a normal form. To obtain complete metric spaces, it is in general necessary to employ sets of infinite terms. Andersen and Simonsen showed that, for finite signatures, when equipping the set of compact sets of terms with the Hausdorff metric, any TRS with a finite number of rules induces a continuous map F . However, they did not consider the connections between long-term iteration of maps F (that is, the behaviour of $F^n(A)$ as n goes to infinity), and they did not consider the possible connections with standard notions from infinitary rewriting.

In this preliminary work, we investigate the connections between (a) the notions of strong convergence, top-termination, and normalization from infinitary rewriting, and (b) the limiting behaviour of $F^n(A)$ for various sets of terms A . As expected by the use of the Hausdorff metric, there are strong connections to the study of attractors in fractal geometry, indeed methods similar to those employed for iterated function systems in that setting can be fruitfully employed.

The talk will touch upon all of the above subjects. The present note sets the stage and presents a few results for the reader to get a feel for the flavour of the research direction.

1 The evolution of (compact) sets of infinite terms under rewriting

Let Σ be a first-order signature. We consider only the set of finite and infinite ground terms $T^\infty(\Sigma, \emptyset)$. If s is a finite or infinite term and n a non-negative integer, $s|_{\leq n}$ denotes the linear term identical to s up to, and including, depth n (where subterms at positions longer than n are replaced by distinct variables).

Our main object of study is *sets* of subsets of $T^\infty(\Sigma, \emptyset)$ and how they change under the usual one-step reduction when all possible redexes are contracted. This is the motivation for the following definition from [1]:

Definition 1. Let R be a TRS. We then define the map $F_R : \mathcal{H}(T^\infty(\Sigma, \emptyset)) \longrightarrow \mathcal{H}(T^\infty(\Sigma, \emptyset))$ by

$$F_R(A) = \{t \in T^\infty(\Sigma, \emptyset) : s \in A \wedge s \rightarrow_R t\} \cup \{t \in A : t \text{ is a normal form}\}$$

For a set of terms A , $F_R(A)$ consists of (i) all reducts of elements of A , and (ii) all normal forms in A . The reason for including the normal forms in A is to ensure that F_R is well-defined (as, otherwise, any set consisting solely of normal forms would be mapped into the empty set).

The natural metric on the set of *compact* subsets of $T^\infty(\Sigma, \emptyset)$ is the Hausdorff metric:

Definition 2. Let (X, d) be a metric space and let $\mathcal{H}(X)$ be the set of non-empty compact subsets of X . The *Hausdorff metric* on $\mathcal{H}(X)$ is then the metric d_H defined by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$$

If (X, d) is an ultrametric space, then so is $(\mathcal{H}(X), d_H)$, if (X, d) is complete, so is $(\mathcal{H}(X), d_H)$. The Hausdorff metric is well-studied in fractal geometry where it is the “natural” metric on compact subsets of $\mathbb{R}^d$ [2].

For the standard metric on infinite terms, there is a simple characterization of the Hausdorff metric:

Theorem 1 (See e.g. Ch. 1 of [3]). *Let x be a variable and define, for any non-negative integer n and any $s \in T^\infty(\Sigma, \emptyset)$, the term $\alpha_n(s)$ as the term s where we have replaced any subterm of s at positions p with $|p| > n$ by x . For $A \in \mathcal{H}(T^\infty(\Sigma, \emptyset))$, define $\alpha_n(A) = \{\alpha_n(s) : s \in A\}$.*

The Hausdorff metric on $T^\infty(\Sigma, \emptyset)$ is then given by

$$d_H(A, B) = 2^{-\inf\{n : \alpha_n(A) \neq \alpha_n(B)\}}$$

The map F_R has the nice property of being continuous for iTRSs with finitely many rules:

Theorem 2 ([1]). *If R has finitely many rules, F_R is continuous.*

In general, merely considering the one-step rewrite relation results in very restricted results. In the remainder of this note, we overload $\rightarrow$ to mean *any* appropriate relation induced by the rewriting relation. For example, $\rightarrow$ can be the relation on terms induced by an outermost strategy, a Gross-Knuth strategy on non-collapsing orthogonal iTRSs, and so forth. Note that Theorem 2 continues to hold as long as $\rightarrow$ is a *restriction* of the rewriting relation on pairs of terms (e.g. is the relation induced by a one-step outermost strategy).

2 Connections to fractal geometry: iterated function systems and contractions

The Hausdorff metric is well-studied [2] in fractal geometry where it is (one of the) way(s) to give a formal definition of the notion of *attractor*.

Definition 3. A *contraction* on a metric space (X, d) is a map $f : X \rightarrow X$ such that $\forall x, y \in X. d(x, y) > d(f(x), f(y))$. An *iterated function system* (IFS) on (X, d) is a finite set of contractions on a complete metric space (X, d) .

It is easy to see that all contractions are continuous, and a straightforward application of Banach’s fixed point theorem implies that:

Lemma 3. *Let (X, d) be a complete metric space and $\{f_1, \dots, f_k\}$ an IFS on (X, d) . Then For any compact, non-empty $A \subseteq X$, the map $H : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$ defined by $H(A) = \bigcup_{1 \leq i \leq k} f_i(A)$ satisfies that $S = \lim_{n \rightarrow \infty} H^n(A)$ is compact and satisfies $H(S) = S$, and S is the unique compact subset of X with the property $H(S) = S$.*

The set S of Lemma 3 is called the *attractor* of the IFS.

For the specific case $X = \mathbb{R}^m$ with $m > 0$ and d the standard Euclidean metric on $\mathbb{R}$, Lemma 3 guarantees existence and uniqueness of the well-known attractors in fractal geometry, e.g. the Cantor Set, the Menger Sponge, the Fractal Flames, etc. The “practical” application of the lemma is that one can construct these sets by giving suitable IFSs and simply iterating them starting from any non-empty compact set: With successive iterations of the IFS, better-and-better approximations of the attractors are obtained.

There is at least one straightforward way to obtain contractions F_R , namely by considering iTRSs that strictly decrease distance between terms in each step, as shown below. Say that for iTRS R , the relation $\rightarrow$ is *contractive* if $s, s', t, t' \in T^\infty(\Sigma, \emptyset)$, $s \rightarrow t$, $s' \rightarrow t'$ implies that $d(t, t') < d(s, s')$.

The astute reader will observe that it is very hard to construct an iTRS that is contractive if $\rightarrow$ is taken as the full, ordinary rewriting relation (i.e. if no reduction strategy is enforced). This is due to the fact that $s \rightarrow t$ and $s' \rightarrow t'$ may occur due to contraction of redexes *deeper* than the

least depth at which s and s' differ. However, with appropriate changes to the notion of rewriting (e.g., taking $\rightarrow$ as a complete development of the set of all outermost redexes in an orthogonal iTRS), contractive iTRSs.

We now obtain the following results:

Proposition 4. *Let R be an iTRS having no normal forms such that $\rightarrow$ is contractive. Then, F_R is a contraction.*

Proof. By Theorem 1, $d_H(A, B) = 2^{-\inf\{n : \alpha_n(A) \neq \alpha_n(B)\}}$. $F_R(A)$ and $F_R(B)$ are the set of terms containing all one-step $\rightarrow$ -reducts of terms. As there are no normal forms, every term of A and B contains a redex, whence every element of $F_R(A)$ and $F_R(B)$ is a reduct of a term in A , respectively in B . Consider arbitrary $t_A \in F_R(A)$ and $t_B \in F_R(B)$. Then there are $s_A \in A$ and $s_B \in B$ with $s_A \rightarrow t_A$ and $s_B \rightarrow t_B$. By assumption, we have $d(t_A, t_B) < d(s_A, s_B)$. As the reducts were arbitrary, this implies that for any pair of terms in $F_R(A)$, and $F_R(B)$, the least depth at which they differ is at least one larger than the least depth that any of their origins in A and B differ. Hence, $\inf\{n : \alpha_n(F_R(A)) \neq \alpha_n(F_R(B))\} > \inf\{n : \alpha_n(A) \neq \alpha_n(B)\}$, whence $d_H(F_R(A), F_R(B)) < d_H(A, B)$, proving that F_R is a contraction. $\square$

Lemma 3 *also* shows that contractive iTRSs are hard to come by: Any set of normal forms will be a fixed point of F_R , hence the existence of just two distinct normal forms s and t entails that there will be at least two distinct fixed points of F_R , namely $\{s\}$ and $\{t\}$, whence F_R cannot be a contraction. Note that appealing to Lemma 3 (which allows for a finite number of distinct contractions) does not ameliorate this problem

However, for iTRSs without any normal forms, examples do exist of contractions. We give a simple example.

Example 1. Let $\Sigma = \{f/2, a/0\}$ and $R = \{f(x, y) \rightarrow f(f(x, x), f(y, y)), a \rightarrow f(a, a)\}$. Clearly, R has no normal forms. Let $\rightarrow$ be the relation that performs a complete development of all the (necessarily finite) number of redexes at a minimal depth of a term s containing redexes. Then, F_R is a contraction: each position in a term holds either f or a , so if two terms differ at position p , one must have f , the other a , at p . Rewriting either of results is a term having symbol f at p . As all redexes at minimal depth are contracted, any differences at this depth are erased after one $\rightarrow$ -step.

For any compact (e.g. any finite) set A of ground terms, $\lim_{n \rightarrow \infty} F^n(A) = \{s\}$ where s is the term satisfying $s = f(s, s)$. Note that the set of finite and infinite sequences of positions in s , equipped with the usual metric, is homeomorphic to the Cantor set. Similar constructions can be used for generalizations of the Cantor set, e.g. the Sierpinski carpet or the Menger sponge.

3 Going forward

Many questions are left unexplored in both this note and the accompanying talk. There may clearly be differences between weak and strong termination and the action of F_R in these different regimes (especially terms with one-step cycles, see e.g. [4]). There are clearly more interesting connections between (methods from) fractal geometry and notions of strong convergence in infinitary rewriting (also, possible connections could be made by constructing contractions by considering termination, akin to automated productivity analysis using (finitary) termination tools) [5]. Furthermore, the evolution of sets of terms under $\rightarrow$ may be studied by other means than contractions: For example by considering the lcomplete lattice of subsets of $T^\infty(\Sigma, \emptyset)$ ordered by $\subseteq$ in the case where F_R is monotone—classic applications of the Knaster-Tarski fixed point theorem may potentially yield useful insights.

References

- [1] S. B. Andersen and J. G. Simonsen. Term rewriting systems as topological dynamical systems. In A. Tiwari, editor, *RTA*, volume 15 of *LIPICs*, pages 53–68. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2012.
- [2] M. Barnsley. *Fractals Everywhere*. Morgan Kaufmann, 1993.
- [3] J. de Bakker and E. de Vink. *Control Flow Semantics*. The MIT Press, 1996.
- [4] J. G. Simonsen. Weak convergence and uniform normalization in infinitary rewriting. In *Proceedings of the International Conference on Rewriting Techniques and Applications (RTA '10)*, volume 6 of *LIPICs*, pages 311–324. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010.
- [5] H. Zantema. Well-definedness of streams by transformation and termination. *Logical Methods in Computer Science*, 6(3), 2010.