

Work in Progress : Algebraic Abstract Reduction Systems*

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Abstract

Several extensions of the λ -calculus allow to rewrite a term into a linear combination of terms taking coefficients in $\mathbb{R}^+$. The goal of this work is to define a notion of algebraic rewrite system abstracting away these calculi. In particular, we look for a notion of normalization induced by a rewriting compatible with the semi-ring structure and the usual topology on non-negative reals. The goal is to achieve a kind of Newman's lemma: strong normalization together with local confluence implies confluence. We state the formal definitions and show how a (naïve) expected implication fails short with several non-trivial examples.

1 Introduction

Several extensions of the λ -calculus add to regular λ -terms a structure of module over non-negative real numbers (e.g. [3, 4, 1, 7, 6]): this makes linear combinations of terms into regular terms. The β -reduction is adjusted to provide a reasonable equational theory. In particular, if M reduces to N , then $\alpha \cdot M + P$ reduces to $\alpha \cdot N + P$.

Because of the structure of module, the corresponding rewrite-system may suffer a serious drawback: any term reducing at least once is non converging. For example, if again $M \rightarrow N$, the sequence $M \rightarrow \frac{1}{2}M + \frac{1}{2}N \rightarrow \frac{1}{4}M + \frac{3}{4}N \rightarrow \frac{1}{8}M + \frac{7}{8}N \rightarrow \dots$ is infinite. This is due to the fact that the rewrite system does not take into account the additional structure of module.

The literature proposes various approaches to answer the problem. Vaux [7] suggests to ask for *finitely splittable scalars*. Although this solution does reestablish a sound notion of normalization, it does not answer the problem for non-negative reals. Arrighi and Dowek [1] take another route, by *orienting* the module equations, effectively making the module structure at the same level as the other λ -term constructs.

In this short report, we attack the problem from a different angle: we propose to change the notion of normalization, taking into account the topology over real numbers and being able to speak about a *infinite sequence of steps converging to a finite value*. The (yet unachieved) goal is to provide a consistent theory for a notion of algebraic abstract rewrite system for modules over non-negative real numbers, where

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notions of strong normalization, local confluence and confluence can be defined, in order to retrieve Newman's lemma, stating that confluence can be derived from strong normalization and weak confluence. Our attempts recall in the spirit Kennaway's approach [5], mixing standard rewriting theory with notions from topology allowing to speak about convergence at the infinite. However, the topology that we consider in this report is the one induced by the module structure $(\mathbb{R}^+)$ while Kennaway works with topologies having more "rewriting" flavours.

The report is organized as follows. First we develop the definition of algebraic abstract rewrite system that we wish to work with. Then we present the extension of the existing concepts to this new setting, and show how most of the expected results fail by providing counter-examples.

2 Rewrite Systems on Modules

Let X be a set, we denote by $(\mathbb{R}^+)^X$ the module of vectors indexed by the elements in X and taking scalars in the rig (a.k.a. semi-ring) of non-negative reals $\mathbb{R}^+$. The elements of X are called *simple terms* and are denoted by x, y, z , while the vectors in $(\mathbb{R}^+)^X$ are called *terms* and are denoted by S, R, T . The scalars are denoted by p, q, r . We denote by $S(x)$ the scalar of the simple term x in S . The *support* $|S|$ of S is the set of those simple terms having a non-zero scalar in S .

2.1 Algebraic Abstract Rewrite Systems

Definition 1. Given a binary relation $\mathcal{R} \subseteq X \times (\mathbb{R}^+)^X$, an *elementary reduction step* of $\mathcal{R}$, ers for short, is a quadruple (x, S', p, T) , where $(x, S') \in \mathcal{R}$ and $p \in \mathbb{R}^+ \setminus \{0\}$, $T \in (\mathbb{R}^+)^X$. We call x the *redex* and p the *weight* of the ers. We may write an ers (x, S', p, T) simply as $S \xrightarrow{\mathcal{R}} U$, whenever $S = px + T$, $U = pS' + T$. Also, we may omit to write the relation $\mathcal{R}$ and the weight p when uninteresting. We write $S \xrightarrow{p} U$, whenever either $p = 0$ and $S = U$, or there is a finite sequence of ers $S_0 \xrightarrow{p_0} \dots \xrightarrow{p_{n-1}} S_n$ such that $S = S_0$, $U = S_n$ and $p = p_0 + \dots + p_{n-1}$.

Lemma 2. The relation $\rightarrow_{\mathcal{R}}$ is congruent with respect to the module structure. That is, if $S \rightarrow_{\mathcal{R}} U$ then for all T and for all scalars p , we have $pS + T \rightarrow_{\mathcal{R}} pU + T$. $\square$

Definition 3. Let β be an ordinal greater than 0, a family $\mathfrak{a} = \{S_\alpha \xrightarrow{p_\alpha} S_{\alpha+1}\}_{\alpha < \beta}$ of ers is a *reduction sequence* whenever for all limit ordinals $\lambda < \beta$, $S_\lambda = \lim_{\alpha < \lambda} S_\alpha$, where $\lim_{\alpha < \lambda} S_\alpha$ is computed point-wise, i.e. $(\lim_{\alpha < \lambda} S_\alpha)(x) = \lim_{\alpha < \lambda} S_\alpha(x)$. The *length* of $\mathfrak{a}$ is β , and S_0 is its *starting element*.

The sequence $\mathfrak{a}$ is *algebraically converging* to R with weight p (or simply *converging*) whenever (1) $R = \lim_{\alpha < \beta} S_{\alpha+1}$ and (2) $p = \sum_{\alpha < \beta} p_\alpha$. We write $S \xrightarrow{\mathfrak{a}}_\beta R$ whenever there is an algebraically converging sequence from S to R with weight p and length β . We omit the weight and length when uninteresting.

We are now ready to define the notion of abstract system we shall use in this paper.

Definition 4. An *algebraic abstract rewriting system*, shortened as aARS, is a tuple $(X, A, \mathcal{R})$ such that $\mathcal{R} \subseteq X \times (\mathbb{R}^+)^X$ and such that A is a submodule of $(\mathbb{R}^+)^X$ closed under $\Rightarrow_{\mathcal{R}}$. That is, for every $S \in A$ and $p \in \mathbb{R}^+$ if $S \xrightarrow{p}_{\mathcal{R}} U$, then $U \in A$.

Example 5. Let $X = \{\star\}$, $A = \{p \star \mid p \in \mathbb{R}^+\}$ and consider the relation $\mathcal{R} = \{(\star, \frac{1}{2}\star)\}$. This forms an aARS. A sequence of this aARS can reduce to 0: the sequence $\{\frac{1}{2^n}\star \xrightarrow{\frac{1}{2^n}} \frac{1}{2^{n+1}}\star\}_{n < \omega}$ converges to 0 with weight equal to 2. A sequence might also reduce to some arbitrary other element $p \star$ of A : consider for example $\{p \star + \frac{1}{2^n}\star \xrightarrow{\frac{1}{2^n}} p \star + \frac{1}{2^{n+1}}\star\}_{n < \omega}$. An algebraically converging sequence can also contain more than ω terms. For example, the sequence $\{\frac{1}{2}\star + \frac{1}{2^k}\star \xrightarrow{\frac{1}{2^k}} \frac{1}{2}\star + \frac{1}{2^{k+1}}\star\}_{0 < k < \omega} \cup \{\frac{1}{2^k}\star \xrightarrow{\frac{1}{2^k}} \frac{1}{2^{k+1}}\star\}_{0 < k < \omega}$ contains 2ω elements and has weight 2.

Nonetheless, the number of steps of any rewrite sequence is countable:

Proposition 6 (From [5]). *If $\{S_\alpha \xrightarrow{p_\alpha} S_{\alpha+1}\}_{\alpha < \beta}$ is an algebraically converging sequence of ers, then β is a countable ordinal.* $\square$

The relation $\Rightarrow_\omega$ cannot in general be represented by a sequence of finite steps of $\rightarrow$, as the sequence $\star \xrightarrow{2}_\omega 0$ of Example 5. Nonetheless, we strongly conjecture that it is possible to perform any rewriting in only ω steps:

Claim 7 (Compression). *For any transfinite ordinal β , if $S \xrightarrow{p}_\beta U$ then $S \xrightarrow{p}_\omega U$.*

2.2 Transfinite Counterparts of Existing Concepts

Following Kennaway [5], we define several transfinite equivalent notions of local confluence (or *weak Church-Rosser*), confluence (or *Church-Rosser*) and strong normalization. S is a *normal form* (shortened as *nf*) whenever for every $(x, S') \in \mathcal{R}$, $x \notin |S|$. The term S has a normal form whenever there is a normal form U such that $S \Rightarrow U$. Given an algebraic rewriting system, we define the following properties.

WCR	$\forall S, S_1, S_2$, if $S_1 \leftarrow S \rightarrow S_2$, then $\exists S_3$ such that $S_1 \rightarrow S_3 \leftarrow S_2$
WCR$^\infty$	$\forall S, S_1, S_2$, if $S_1 \leftarrow S \rightarrow S_2$, then $\exists S_3$ such that $S_1 \Rightarrow S_3 \Leftarrow S_2$.
CR	$\forall S, S_1, S_2$, if $S_1 \leftarrow S \rightarrow S_2$, then $\exists S_3$ such that $S_1 \rightarrow S_3 \leftarrow S_2$.
CR$^\infty$	$\forall S, S_1, S_2$, if $S_1 \leftarrow S \rightarrow S_2$, then $\exists S_3$ such that $S_1 \Rightarrow S_3 \Leftarrow S_2$.
SN	every reduction sequence is finite.
SN$^\infty$	every reduction sequence is algebraically converging (Definition 3).

Lemma 8. *An aARS $(\mathcal{T}, \mathcal{R})$ is **WCR** (resp. **WCR $^\infty$**) iff for every $(x, S'), (x, S'') \in \mathcal{R}$, there is a term S''' such that $S' \rightarrow S''' \leftarrow S''$ (resp. $S' \Rightarrow S''' \Leftarrow S''$). $\square$*

Lemma 9 (Diamond property). *An aARS $(\mathcal{T}, \mathcal{R})$ is **CR** if for every $(x, S'), (x, S'') \in \mathcal{R}$, there is a term S''' such that $S' \xrightarrow{p}_\omega S''' \xrightarrow{q}_\omega S''$. $\square$*

3 Unfinished Quest for Newman's Lemma

In this section, we show how several intuitions one might have with respect to aARS fail short. First, we show that **CR $^\infty$** and **CR** are not comparable.

Example 10 (WCR + CR $^\infty \not\Rightarrow$ CR). The following aARS is an algebraic variant of a well-known counter-example to **WCR** $\Rightarrow$ **CR**. Consider the set of simple terms $\{a, b, c, d, \star\}$ and the following rewriting rules: the two critical pairs $a \leftarrow b \rightarrow c$ and

$$\begin{array}{ccccccccccc}
\star + a_0 & \xrightarrow{1} & \frac{3\star + a_1}{2} & \xrightarrow{\frac{1}{2}} & \frac{7\star + a_2}{4} & \xrightarrow{\frac{1}{4}} & \frac{15\star + a_3}{8} & \xrightarrow{\frac{1}{8}} & \dots & \xrightarrow{3} & 2\star \\
\swarrow_1 & & \swarrow_1 & & \swarrow_{\frac{1}{2}} & & \swarrow_{\frac{1}{4}} & & & & \\
\Box & \xrightarrow{1} & b_0 & \xrightarrow{1} & \frac{\star + b_1}{2} & \xrightarrow{\frac{1}{2}} & \frac{3\star + b_2}{4} & \xrightarrow{\frac{1}{4}} & \dots & \xrightarrow{3} & \star
\end{array}$$

Figure 1: Counter-example to the implication $\mathbf{CR} + \mathbf{SN}^\infty \implies \mathbf{CR}^\infty$.

$b \leftarrow c \rightarrow d$, and the two rules $a \rightarrow \frac{1}{2}\star + \frac{1}{2}a$ and $d \rightarrow \frac{1}{2}\star + \frac{1}{2}d$. This aARS is **WCR**, $\mathbf{CR}^\infty$ (every simple term algebraically converges to $\star$) but not **CR**. Notice also that it is not $\mathbf{SN}^\infty$ (hence not **SN**) because of the loop $b \rightarrow c \rightarrow b$.

Example 11 (**CR** does not imply $\mathbf{CR}^\infty$). An example is given by the following aARS. The set of simple terms is $X := \{\star\} \cup \overline{\mathbb{N}}$, the module is defined by $\mathcal{T} := \{v \in (\mathbb{R}^+)^X ; \sum_{n=0}^\infty v(\overline{n}) < \infty\}$ and $\mathcal{R} := \{(\overline{n}, q\star + \frac{1}{2}\overline{n+1}) ; n \in \mathbb{N}, q \in \mathbb{Q}^+\}$. This aARS is **CR**, using Lemma 9 and considering the set of critical pairs $q_1\star + \frac{1}{2}\overline{n+1} \xleftarrow{1} \overline{n} \xrightarrow{1} q_2\star + \frac{1}{2}\overline{n+1}$ (with $q_1 > q_2$), closing to $q_1\star + \frac{1}{4}\overline{n+2}$. On the other hand, the system is not $\mathbf{CR}^\infty$, since $\overline{0} \xRightarrow{2} n\star$ for every $n \in \mathbb{N}$.

3.1 First Attempt

The expected instance of Newman's lemma for aARS would be that $\mathbf{SN}^\infty$ together with $\mathbf{WCR}^\infty$ implies $\mathbf{CR}^\infty$. Unfortunately, this is not correct:

Example 12 ($\mathbf{SN}^\infty + \mathbf{WCR}^\infty \not\Rightarrow \mathbf{CR}^\infty$). Consider the aARS $((\mathbb{R}^+)^X, \mathcal{R})$ defined by $X = \{\star, \Box, a\}$ and $\mathcal{R} := \{(a, \frac{1}{2}(\star + a)), (a, \frac{1}{2}(\Box + a))\}$. This aARS fails $\mathbf{CR}^\infty$ since the critical pair $\Box \xleftarrow{2} a \xRightarrow{2} \star$ cannot be closed. However, it is $\mathbf{SN}^\infty$: every linear combination of simple terms eventually converges to $p\star + q\Box$, for $p, q \in \mathbb{R}^+$. It is also $\mathbf{WCR}^\infty$, by Lemma 8 and since the critical pair $\frac{\star+a}{2} \leftarrow a \rightarrow \frac{\Box+a}{2}$ closes to $\frac{\star+\Box}{2}$.

One could be tempted to study a weaker version of the Newman's lemma, relating **WCR** (instead of $\mathbf{WCR}^\infty$) with $\mathbf{CR}^\infty$ in presence of $\mathbf{SN}^\infty$. Unfortunately, even the implication $\mathbf{CR} + \mathbf{SN}^\infty \implies \mathbf{CR}^\infty$ is false.

Example 13 ($\mathbf{CR} + \mathbf{SN}^\infty \not\Rightarrow \mathbf{CR}^\infty$). Let $X := \{\star, \Box\} \cup \{a_n, b_n ; n \in \mathbb{N}\}$, $\mathcal{T} := \{v \in (\mathbb{R}^+)^X ; \sum_{x \in |v|} v(x) < \infty\}$ and the rewrite rules: $b_0 \leftarrow \Box \rightarrow \star + a_0$, and for all $n \geq 0$, $a_n \rightarrow \frac{1}{2}(\star + a_{n+1})$, $b_n \rightarrow \frac{1}{2}(\star + b_{n+1})$ and $b_n \rightarrow 2^n\star + \frac{1}{2}(\star + a_{n+1})$. This aARS is **CR**, as the critical pairs (recall Lemma 9): $\star + a_0 \leftarrow \Box \rightarrow b_0$ close to $\frac{3}{2}\star + \frac{1}{2}a_1$ in one step, and $2^n\star + \frac{1}{2}\star + \frac{1}{2}a_{n+1} \leftarrow b_n \rightarrow \frac{1}{2}\star + \frac{1}{2}b_{n+1}$, close $\frac{3}{4}\star + 2^n\star + \frac{1}{4}a_{n+2}$ in one step. However, it is not $\mathbf{CR}^\infty$ since $\star \xleftarrow{3} \Box \xRightarrow{3} 2\star$ cannot close (see Fig. 1).

3.2 Bounded Reductions

A potential problem in Example 13 comes from the fact that simple terms reduce to linear combinations with potentially unbounded coefficients. Unfortunately, adding boundedness is still not enough to get $\mathbf{CR}^\infty$, as shown by the next Example 15.

Definition 14. A relation $\mathcal{R} \subseteq X \times (\mathbb{R}^+)^X$ is *bounded* whenever there is a number $r \in \mathbb{R}^+$ such that for all $(x, S) \in \mathcal{R}$, $\sum_{x \in X} S(x) \leq r$.

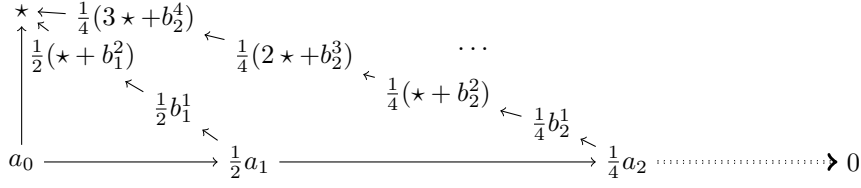


Figure 2: Counter-example to $\mathbf{CR} + \mathbf{SN}^\infty + \mathbf{Bound} \Rightarrow \mathbf{CR}^\infty$.

Example 15 ($\mathbf{CR} + \mathbf{SN}^\infty + \mathbf{Bound} \not\Rightarrow \mathbf{CR}^\infty$). Let X be the set of simple terms $\{\star\} \cup \bigcup_{n=0}^\infty \{a_n, b_n^1, \dots, b_n^{2^n}\}$, A be the set of linear combinations of the form $p \star + \sum_{n=0}^\infty \left(\alpha_n a_n + \sum_{k=1}^{2^n} \beta_{n,k} b_n^k \right)$ such that $\lim_{n \rightarrow \infty} 2^n \left(\alpha_n + \sum_{k=1}^{2^n} \beta_{n,k} \right) < \infty$, and $\mathcal{R}$ be given by the critical pairs $\star \leftarrow a_0 \rightarrow \frac{1}{2}a_1$ and $b_{n+1}^1 \leftarrow a_{n+1} \rightarrow \frac{1}{2}a_{n+1}$, and the rules $b_{n+1}^i \rightarrow \star + b_{n+1}^{i+1}$ and $b_{n+1}^{2^{n+1}} \rightarrow \star$ when $i < 2^{n+1}$. This aARS is $\mathbf{SN}^\infty$: every term eventually converges to some finite amount of $\star$. Although one can easily check that the system is $\mathbf{CR}$, it is not $\mathbf{CR}^\infty$ as shown in Figure 2.

In fact, the above example shows that the notion of boundedness is not a good one: every non-bounded aARS can be simulated by a bounded one, adding a suitable number of fresh simple elements and rewriting rules.

4 Conclusion

The definition of aARS as we set it up is appealing, and does fit well with our understanding of a rewrite system on a free module. Nonetheless, we are at loss to capture a relationship between concepts of confluence and strong normalization.

We foresee two approaches to solve the problem. The first would be to set up a Lefschetz topology over the modules of terms, in the spirit of finiteness spaces [2]. Another approach would be to not consider a variant of Newman's lemma, but instead retrieve confluence using parallel reductions, as done for example in Vaux [7].

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