

Turtle graphics of morphic streams

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May 21, 2014

1 Introduction and morphic streams

Streams are the simplest possible infinite objects: just infinite sequences. The simplest streams are periodic, up to an initial part. The one but simplest are morphic streams: fixed points of particular morphisms mapping single symbols to strings of symbols. A typical example is the fixed point starting in 0 of the morphism ϕ defined by $\phi(0) = 01, \phi(1) = 10$, yielding

$$\phi(0) = 01, \phi^2(0) = 0110, \phi^3(0) = 01101001, \phi^4(0) = 0110100110010110, \dots$$

As $\phi^n(0)$ is a prefix of $\phi^{n+1}(0)$ for every n , the limit of this process yields the unique fixed point of ϕ starting in 0: the Thue-Morse stream **Morse**. Alternatively, **Morse** is defined as the unique infinitary normal form of $0 : S$ with respect to the orthogonal TRS

$$\begin{aligned} S &\rightarrow 1 : \phi(S) \\ \phi(0 : x) &\rightarrow 0 : 1 : \phi(S) \\ \phi(1 : x) &\rightarrow 1 : 0 : \phi(x). \end{aligned}$$

More general, the pure morphic stream $f^\omega(a)$ can be defined as the unique infinitary normal form of $a : S$ with respect to the TRS

$$\begin{aligned} S &\rightarrow u : f(S) \\ f(a : x) &\rightarrow a : u : f(x) \\ f(b : x) &\rightarrow u_b : f(x) \quad \text{for all } b \neq a \end{aligned}$$

in which $f(a) = au$, $f(b) = u_b$. It is well-defined and effectively computable by this TRS since it is *productive*.

The simplest way to visualize a stream is by a turtle curve: for every alphabet symbol fix an angle, and then proceed the stream elements by drawing unit segments and turn the corresponding angle. In this note we investigate turtle curves of morphic streams. Arbitrarily choosing simple morphisms and angles typically yield turtle curves that show up a complete mess. However, sometimes the turtle curve is finite, that is, it consists of finitely many segments that are drawn over and over again, or shows up fractal or self-similar behaviour: it contains an upscaled copy of itself. In this note we give criteria for turtle curves to be finite or self-similar, and show several examples.

This note is an extended abstract of [3], in which full proofs of all theorems are given.

2 Turtle curves

We consider streams over a set A . Let for every $a \in A$ an angle $\alpha(a) \in \mathbf{R}$ be given. Then for a stream σ over A its *turtle curve* $C(\sigma, \alpha) \subseteq \mathbf{R}^2$ is obtained by connecting the successive points in its point set $P(\sigma, \alpha) \subseteq \mathbf{R}^2$ that is defined as follows. First we extend α to $\alpha : A^* \rightarrow \mathbf{R}$ by defining $\alpha(\epsilon) = 0$ and $\alpha(a_1, a_2, \dots, a_n) = \sum_{i=1}^n \alpha(a_i)$. For $u \in A^*$ its position $P(u, \alpha) \in \mathbf{R}^2$ is defined inductively by $P(\epsilon, \alpha) = (0, 0)$ and

$$P(ua, \alpha) = P(u, \alpha) + (\cos(\alpha(ua)), \sin(\alpha(ua)))$$

for $u \in A^*$ and $a \in A$. Now for a stream σ over A the set $P(\sigma, \alpha)$ is defined by

$$P(\sigma, \alpha) = \bigcup_{u \in \text{Pref}(\sigma)} \{P(u, \alpha)\}.$$

As a first example consider the stream σ for which $\sigma(i) = 0$ for all $i \in \mathbf{N}$, and $\alpha(0) = \pi/2$. Then the turtle curve consists of a square, and $P(\sigma, \alpha) = \{(0, 0), (0, 1), (-1, 1), (-1, 0)\}$ since $P(\epsilon, \alpha) = (0, 0)$, $P(0, \alpha) = (0, 1)$, $P(00, \alpha) = (-1, 1)$, $P(000, \alpha) = (-1, 0)$ and $P(0^i, \alpha) = P(0^{i-4}, \alpha)$ for $i \geq 4$.

For the same stream $\sigma = 0^\omega$ we obtain

- if $\alpha(0) = 0$ then $P(\sigma, \alpha) = \{(i, 0) \mid i \in \mathbf{N}\}$,
- if $\alpha(0) = \pi$ for $n \geq 3$ then $P(\sigma, \alpha) = \{(0, 0), (-1, 0)\}$,
- if $\alpha(0) = 2\pi/n$ for $n \geq 3$ then $P(\sigma, \alpha)$ consists of the nodes of a regular n -gon,
- if $\alpha(0) = x\pi$ for an irrational number x , then $P(\sigma, \alpha)$ consists of infinitely many points on a circle.

By definition the distance between two consecutive elements of $P(\sigma, \alpha)$ is 1, so $P(\sigma, \alpha)$ contains at least two points for every σ, α . The above examples already show that it can be finite or infinite; if it is finite then $|P(\sigma, \alpha)|$ may be any number ≥ 2 , and if it is infinite then it may be either bounded or unbounded.

2.1 Finite turtle curves

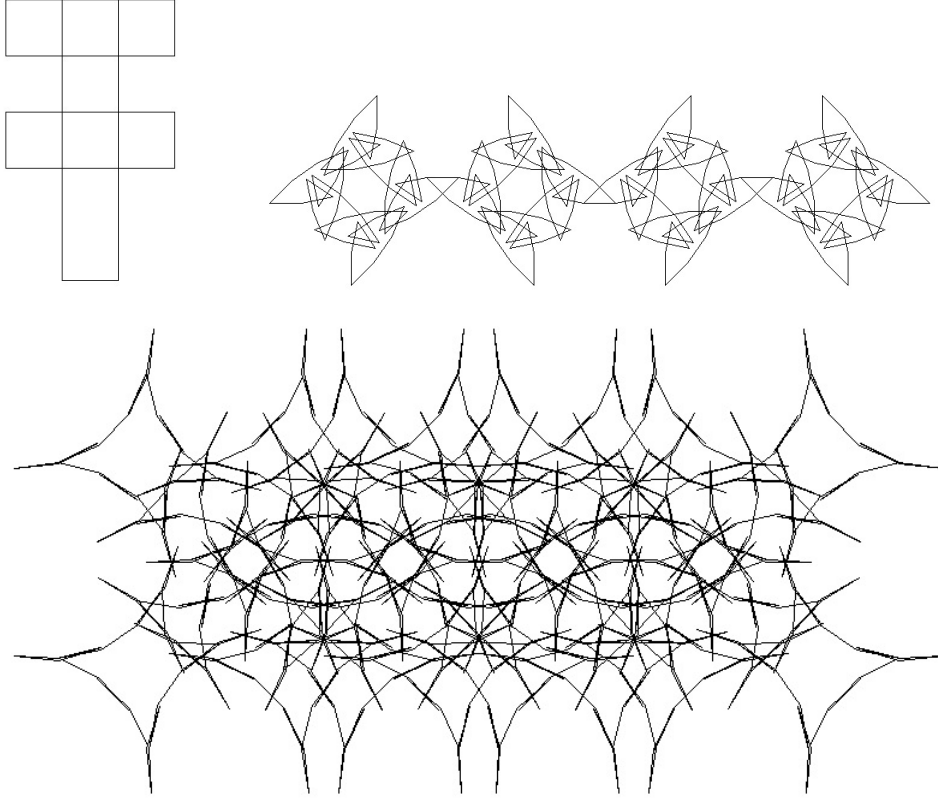
For a set $L \subseteq A^+$ of non-empty strings we define L^ω to consist of all streams that can be written as $u_1 u_2 u_3 \dots$ for $u_i \in L$ for all i .

Theorem 1 *Let $\sigma \in \{u_1, \dots, u_n\}^\omega$ for which $\alpha(u_i)$ is a multiple of 2π and $P(u_i, \alpha) = (0, 0)$ for $i = 1, \dots, n$. Then $P(\sigma, \alpha)$ is finite and $|P(\sigma, \alpha)| \leq \sum_{i=1}^n |u_i|$.*

The following theorem is obtained by applying Theorem 1 to Morse.

Theorem 2 *Let $\alpha(0) + \alpha(1) = k\pi/2^n$ for k odd. Then $P(\text{Morse}, \alpha)$ is finite and $|P(\text{Morse}, \alpha)| \leq 2^{n+4}$.*

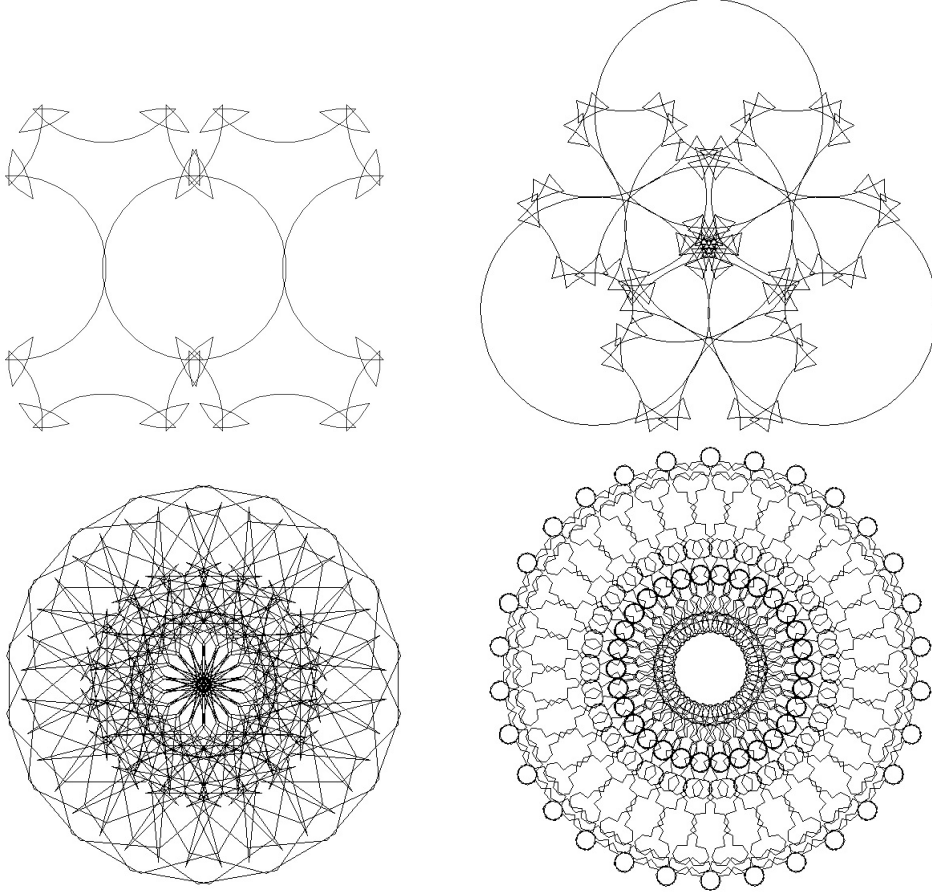
The following three pictures show finite turtle curves of Morse for which Theorem 2 applies. The parameters of the first one on the left are $\alpha(0) = 0$ and $\alpha(1) = \pi/2$; of the second one right from it $\alpha(0) = \pi/16$ and $\alpha(1) = 3\pi/4$; and of the third one below $\alpha(0) = \pi/8$ and $\alpha(1) = 63\pi/64$. The corresponding sets $P(\sigma, \alpha)$ consist of 20, 250 and 1018 points, respectively, all being close to the bounds 32, 256 and 1024 given by Theorem 2.



Next we give a theorem by which finiteness of $P(\sigma, \alpha)$ can be concluded similar to Theorem 1, but with a modified condition on u_1 .

Theorem 3 *Let $\sigma \in \{u_1, \dots, u_n\}^\omega$ for which $\alpha(u_i)$ is a multiple of 2π and $P(u_i, \alpha) = (0, 0)$ for $i = 2, \dots, n$, and $\alpha(u_1) = q\pi$ for q being rational and not a multiple of 2. Then $P(\sigma, \alpha)$ is finite.*

We give a few examples of finite turtle curves of the stream $\sigma = \phi^\omega(1)$ for ϕ defined by $\phi(0) = 00$ and $\phi(1) = 101$, and $\alpha(0) = k\pi/2^n$ for some k, n . Then $\phi^{n+1}(0) = 0^{2^{n+1}}$, by which the turtle curve of $\phi^{n+1}(0)$ is a regular 2^{n+1} -gon or 2^{n+1} -star, yielding $\alpha(\phi^{n+1}(0)) = 2k\pi$ and $P(\phi^{n+1}(0), \alpha) = (0, 0)$. Further observe that $\sigma \in \{\phi^{n+1}(1), \phi^{n+1}(0)\}^\omega$, by which Theorem 3 applies for proving finiteness of $P(\sigma, \alpha)$ if $\alpha(\phi^{n+1}(1))$ satisfies the rationality condition. The first picture is obtained by choosing $\alpha(0) = \pi/16, \alpha(1) = 3\pi/4$, the second picture is obtained by choosing $\alpha(0) = \pi/32, \alpha(1) = -2\pi/3$, the third picture is obtained by choosing $\alpha(0) = \pi/4, \alpha(1) = -17\pi/18$, and the last picture is obtained by choosing $\alpha(0) = 5\pi/16, \alpha(1) = -29\pi/60$.



2.2 Self-similar turtle curves

Roughly speaking, a set $P \subseteq \mathbf{R}^2$ is called *fractal* or *self-similar* if it contains a copy of itself when zooming in or out. As in our turtle curves all drawing steps have unit length, it is most natural to focus on zooming out, leading to the following definition. We use the notation $c(x, y) = (cx, cy)$ for scalar vector multiplication and $R_\theta(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$ for rotation over θ .

Definition 4 A set $P \subseteq \mathbf{R}^2$ is called self-similar if there exists $(x_0, y_0) \in \mathbf{R}^2$, a rotation angle θ and a scaling factor $c > 1$ such that

$$(x_0, y_0) + cR_\theta(x, y) \in P$$

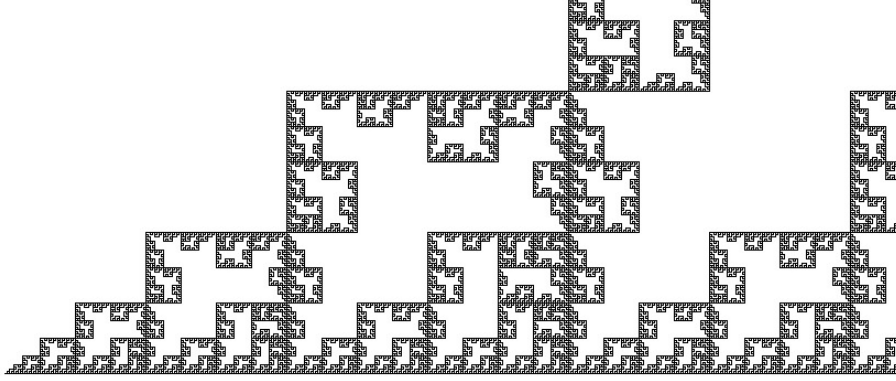
for all $(x, y) \in P$.

A self-similar set containing two distinct elements is always unbounded and hence infinite, since for every $(x, y) \neq (x', y') \in P$ two points $(x_0, y_0) + cR_\theta(x, y)$ and $(x_0, y_0) + cR_\theta(x', y')$ in P can be obtained of which the distance is increased by a factor $c > 1$.

Theorem 5 Let $\sigma \in A^\omega$ satisfy $\sigma = \phi(\sigma)$ for some $\phi : A \rightarrow A^+$ satisfying $\alpha(\phi(a)) = \alpha(a)$ and $P(\phi(a), \alpha) = cR_\theta(P(a, \alpha))$ for all $a \in A$, for some scaling factor $c > 1$ and some rotation angle θ .

Then $P(\sigma, \alpha)$ is self-similar.

As a first application of a self-similar turtle curve we show a fragment of $C(\sigma, \alpha)$ for $\sigma = \phi^\omega(1)$, for $\phi(0) = 111100$, $\phi(1) = 10$, and $\alpha(0) = 0$, $\alpha(1) = \pi/2$:

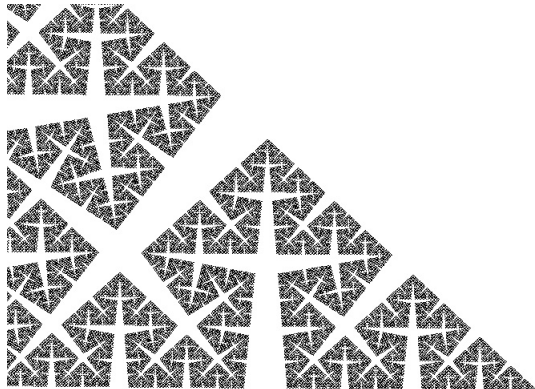
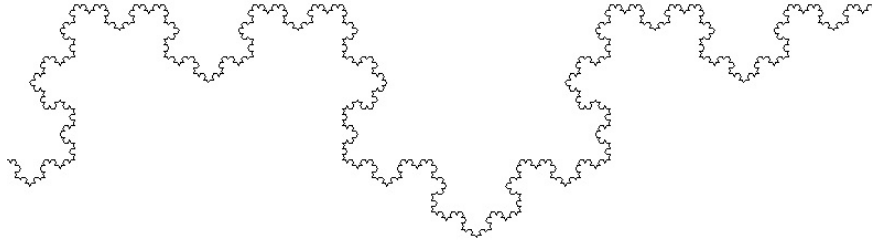


Indeed, Theorem 5 applies, since $\alpha(111100) = \alpha(0) = 0$, $\alpha(10) = \alpha(1) = \pi/2$, $P(111100, \alpha) = (2, 0) = 2P(1, \alpha)$ and $P(10, \alpha) = (0, 2) = 2P(1, \alpha)$, satisfying the requirements for $c = 2$ and $\theta = 0$.



As a second example we consider the well-known Koch curve: start by a single segment, and replace it by the four segments shown on the left.

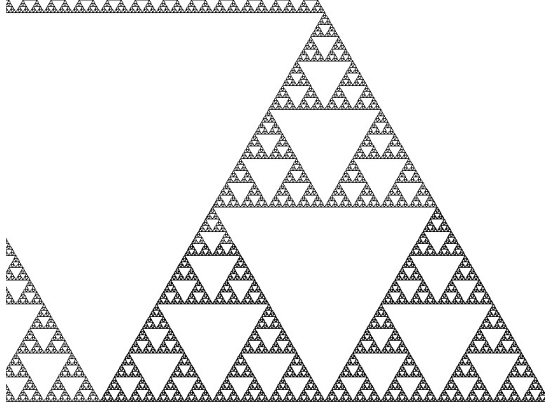
The standard Koch curve is obtained by repeating this for every created new segment, where the length of every new segment is one third of the length of its ancestor, and then take the limit. Apart from the scaling, this can be described in turtle graphics by first doing a unit step in the same direction as before, next turn $\pi/3$ to the right, then do a unit step, next turn $2\pi/3$ to the left and do a unit step, and finally turn $\pi/3$ to the right and do a unit step. Taking the limit to the infinite rather than to the finite, this is described as $C(\text{Koch}, \alpha)$ for $\text{Koch} = \phi^\omega(0)$, $\phi(0) = 0010$, $\phi(1) = 1010$, $\alpha(0) = \pi/3$, $\alpha(1) = -2\pi/3$, to which Theorem 5 applies for $c = 3$ and $\theta = 0$. Hence we call $C(\text{Koch}, \alpha)$ the *Koch turtle curve*. The following is a fragment.



By choosing the same stream, but choosing another value for $\alpha(0)$, and choosing $\alpha(1) = -2\alpha(0)$, still Theorem 5 applies for $\theta = 0$, yielding another scaling factors c . For instance, taking $\alpha(0) = 17\pi/36$ and $\alpha(1) = -17\pi/18$ yields $c = 2 + 2 \arccos 17\pi/36$, and the picture on the left as a fragment of $C(\sigma, \alpha)$, sometimes called *Cesaro fractal*.

In earlier work, e.g. [2], it has been shown that particular turtle curves for the Thue-Morse stream Morse converge to the Koch curve in the Hausdorff metric. We go a step further: we observe that the point set of the Koch curve is a subset of a turtle curve for the Thue-Morse stream. More precisely, we have the following theorem in which we use the notation $\sigma \downarrow n$ for the prefix of σ of length n .

Theorem 6 *Let $\delta(0) = \pi/3$ and $\delta(1) = \pi$, then connecting the points $(P(\text{Morse} \downarrow_{4k+3}, \delta) + P(\text{Morse} \downarrow_{4k+4}, \delta))/2$ on $C(\text{Morse}, \delta)$ for $k = 0, 1, 2, \dots$ exactly yields the Koch turtle curve, scaled up by a factor $3/2$.*



As a last application of Theorem 5 on the left we see a fragment of the well-known Sierpinski triangle, obtained as $C(\sigma, \alpha)$ for $\sigma = \phi^\omega(1)$, $\phi(0) = 00$, $\phi(1) = 11110$, $\alpha(0) = 0$, $\alpha(1) = 2\pi/3$, for which Theorem 5 applies for $c = 2$ and $\theta = 0$.

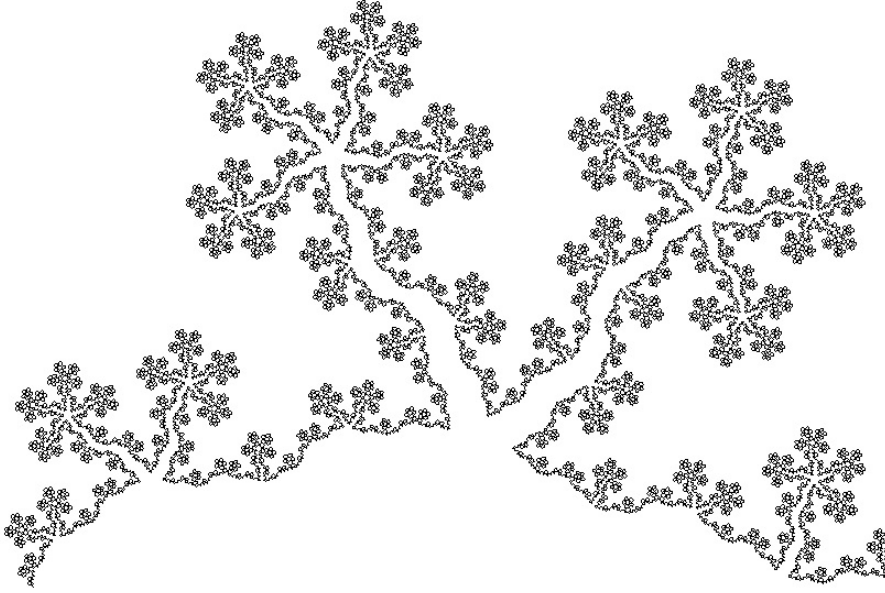
Next we give a variant of Theorem 5 for proving turtle curves to be self-similar, yielding a number of surprising examples.

Theorem 7 *Let $\sigma \in A^\omega$ satisfy $\sigma = \phi(\sigma)$ for $\phi : A \rightarrow A^+$, and there is $(x, y) \in \mathbf{R}^2$ with $(x+1)^2 + y^2 > 1$ such that for all $a \in A$ the following holds:*

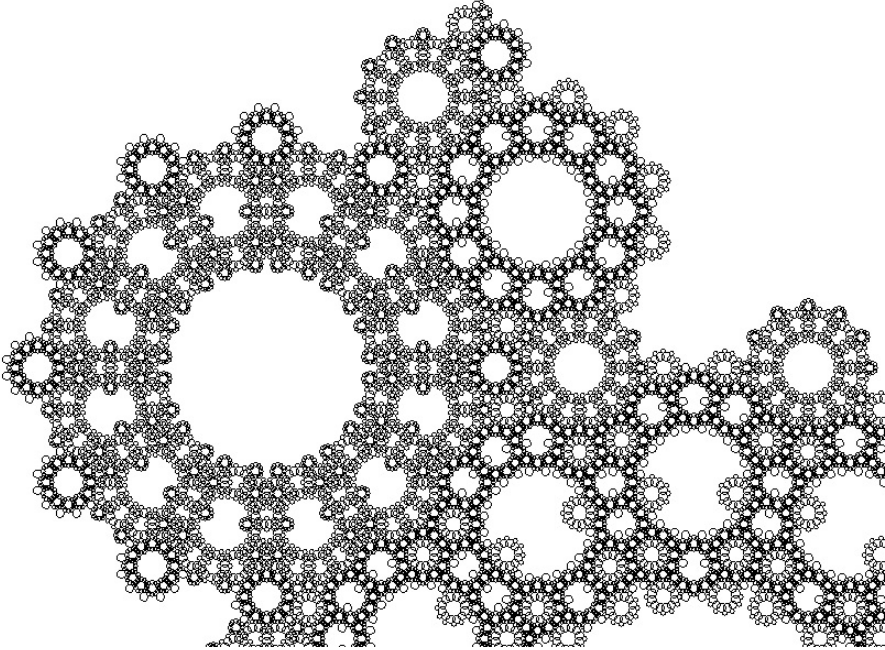
- $\alpha(\phi(a)) = \alpha(a)$, and
- $P(u_a, \alpha) = (x, y)$ for u_a obtained from $\phi(a)$ by removing its last element.

Then $P(\sigma, \alpha)$ is self-similar.

In case $\phi(0) = u0$, $\phi(1) = u1$, for some string u starting in 0, for $\sigma = \phi^\omega(0)$ Theorem 7 applies if $\alpha(u) = 0$ and $P(u, \alpha) = (x, y)$ satisfies $(x+1)^2 + y^2 > 1$. This is the case for $u = 00110$ and $\alpha(0) = 7\pi/18$, $\alpha(1) = -7\pi/12$, yielding a self-similar turtle curve with the following initial fragment (starting at the bottom left):



By choosing $\phi(0) = 01111111 = 01^7$, $\phi(1) = 00$, $\alpha(0) = \pi/6$, $\alpha(1) = -\pi/3$ yields a self-similar turtle curve for $\phi^\omega(0)$, since Theorem 7 applies for ϕ^2 . The following picture shows an initial fragment, starting at the top:



References

- [1] H. von Koch. Une methode geometrique elementaire pour l'etude de certaines questions de la theorie des courbes planes. *Acta Mathematica*, 30:145–174, 1906.
- [2] J. Ma and J. Holdener. When Thue-Morse meets Koch. *Fractals*, 13:191–206, 2005.
- [3] H. Zantema. Turtle graphics of morphic streams. To appear.