

Effective Bounds on the Podelski-Rybalchenko Termination Theorem

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We report here on current work towards an effective proof (with explicit bounds) of Podelski and Rybalchenko Termination Theorem [7]. Our long-term project is to obtain a priori-bounds for the termination of computer programs, and compare these with bounds obtained via other intuitionistic proofs of the Termination Theorem.

Theorem 1 (Termination Theorem). *A program P is terminating iff there exists a well-founded transition invariant for P .*

The authors proved this result by using Ramsey Theorem, which is a purely classical results [1]. In [9] and [2] two proofs of an intuitionistic versions of this theorem have been proposed, where the notion of a program being terminating is replaced by intuitionistically weaker (but classically equivalent) notion. Let us call this “inductively terminating”.

Theorem 2. *A program P is (inductively) terminating iff there exists a (inductively) well-founded transition invariant for P .*

In this result both the hypothesis and the thesis are intuitionistically weaker than the ones in Theorem 1, since, intuitionistically, classical well-foundedness is strictly stronger than intuitionistic well-foundedness. Coquand’s proof is based on almost-full relations (see also [4]), while the second one is based on H -well-founded relations.

In this work we want to intuitionistically prove Podelski and Rybalchenko Termination Theorem considering the classical definition of well-foundedness.

In the result by Podelski they define a program to be terminating iff

$$R \cap (\text{Acc} \times \text{Acc}) \text{ is well-founded,}$$

where R is the transition relation of P and Acc is the set of accessible states. In this work instead we consider the following definition, which is classically equivalent to the Podelski’s one. Assume I is any set, R is a binary relation over I , and S (the set of initial states) is included in Acc which is included in I (the set of states). We call a computation over I any sequence $\alpha : \mathbb{N} \rightarrow I$, and for all $n \in \mathbb{N}$ we write α_n for $\alpha(n)$ in I .

Definition 1. *P is terminating iff $\forall \alpha (\alpha_0 \in S \implies \exists n \neg (\alpha_n R \alpha_{n+1}))$.*

Lemma 1. *Classically, P is terminating by Podelski definition iff P is terminating by definition 1.*

So we can prove the following result:

Theorem 3. *Given a program P , with transition relation R , if*

$$\begin{aligned} &\exists n \exists T_1, \dots, T_n \exists \omega_1, \dots, \omega_n (T_1 \cup \dots \cup T_n \supseteq R^+ \cap (\text{Acc} \times \text{Acc}) \\ &\quad \wedge \forall i \in [1, n] (\forall \alpha \exists j < \omega_i(\alpha) \neg(\alpha_j T_i \alpha_{j+1}))) \end{aligned}$$

then there exists Φ such that

$$\forall \alpha (\alpha_0 \in S \implies \exists m < \Phi(\bar{T}, \bar{\omega}, \alpha) \neg(\alpha_m R \alpha_{m+1})).$$

The functional Φ is definable in $\mathsf{T} + \text{Bar}$ recursion. Thanks to the results of Schwichtenberg [8] and Kohlenbach [5] that give some closure properties for bar recursion, we can study the complexity of the bound Φ relative to the complexity of the relations T_i and the functions ω_i .

The basic ideas behind the proof are those presented in [6] for the classical Ramsey theorem. Our proof of Theorem 3 requires the Transitive Ramsey Theorem, a corollary of the Ramsey Theorem which can be stated as follows:

Theorem 4 (Transitive Ramsey Theorem). *For any $c : \mathbb{N}^2 \rightarrow n$ there exists infinite homogeneous chain.*

The structure of the proof is the following. Given a computation α , we consider all initial finite subsequences of α . We associate to each initial finite subsequence of α a finite tree which follows the idea of the Blackwell proof of Erdős-Szekerkes Theorem (see [3]). Then from each of these trees we obtain a set of monochromatic sequences whose length is bounded by the ω_i . Therefore if α is infinite by applying bar recursion we obtain a contradiction, which gives us the bound we use to define Φ .

References

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