

Constructive completeness and exploding models

Christian Espíndola - Henrik Forssell

It has been known since 1957 that completeness theorems for intuitionistic or classical theories imply non constructive principles, when Gödel derived Markov's principle from the assumption of completeness in a metatheory that included the Axiom of Dependent Choice and a form of Herbrand's theorem. This motivated the question of what the correct notion of constructive model is. In a classical metatheory, one works with Tarskian models for classical logic and Kripke and Beth models for intuitionistic logic. If we wish to work in a strictly constructive metatheory, however, these semantics cannot be proven to be complete, and one has to appeal to more general semantics like locale and topoi models, which seem more suitable. In 1976, Veldman proved that if one restricts to a countable signature, it is possible to prove the completeness of intuitionistic logic with respect to a modified notion of Kripke semantics which is classically equivalent to the usual one, while intuitionistically lies very close to it. As he explained, to make the completeness theorem constructive, one needs to redefine the notion of Kripke models and allow for the so called exploding nodes, in which $\perp$ (and hence every formula) is forced. In that way one sacrifices the decidability of a node being exploding, but the resulting completeness theorem is intuitionistically provable. More generally, some intuitionistic completeness theorems for classical logic have been proposed later, working with a notion of Tarskian model which is allowed to be exploding in the aforementioned sense. One of the main purposes of this article is to present a unifying categorical approach that proves constructive completeness theorems for classical and intuitionistic first order logic, as well as for the coherent fragment. We will see that Veldman's modified Kripke semantics is actually a particular case of sheaf semantics, which confirms the adequacy of sheaf models in constructive mathematics. On the other hand, our proof, as that of Veldman's, makes essential use of the Fan theorem (and we show it is in fact equivalent to it), which rises the question of how far one can get in proving completeness constructively without its use. We present here a fragment for which not only the appeal to the Fan theorem is eliminated, but also the restrictions on the decidability of the theories and the size of the signature can be disregarded. This latter restriction is a fundamental constraint in Veldman's proof, since as we will show, we will not be able to prove completeness theorem with respect to modified Kripke semantics in full generality (that is, without assuming countability of the signature) when working in intuitionistic set theory.

When we work in a constructive metatheory such as intuitionistic Zermelo Fraenkel set theory, IZF, sheaf semantics is as candidate for the "correct" constructive notion of model. For example, it was shown by Palmgren that there is a conservative model of any first-order theory in the category of coherent sheaves on its syntactic category. In a category theoretic formulation, if $\mathcal{C}$ is a small Heyting category, then the Yoneda embedding

$$y : \mathcal{C} \longrightarrow \text{Sh}(\mathcal{C}, K),$$

where K is the coherent coverage (finite covering families), is a conservative Heyting functor. Much of the complexity of the sheaf category can be simplified if one considers instead an embedding into a presheaf category, which essentially the same as working with Kripke semantics. However, passing from sheaves to presheaves is done at the expenses that the proof becomes not constructive.

Joyal's completeness theorem provides this embedding in the form of an evaluation functor:

$$ev : \mathcal{C} \longrightarrow \mathbf{Set}^{\text{Mod}_c(\mathcal{C})^{\text{op}}},$$

where $\text{Mod}_c(\mathcal{C})$ is the category of coherent models of $\mathcal{C}$, and this new functor ev factors through y . Thus we have a diagram:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y} & \mathbf{Sh}(\mathcal{C}, K) \\
& \searrow ev & \swarrow \\
& \mathbf{Set}^{\mathbf{Mod}_c(\mathcal{C})^{\text{op}}} &
\end{array}$$

where all functors are Heyting. As we shall prove, when the signature is countable and the theory is semi-decidable, one can make the completeness proof constructive by replacing the presheaf category with a sheaf model $\mathbf{Sh}(\mathbf{Mod}_c^E(\mathcal{C}), E)$, where the underlying category of the site differs from $\mathbf{Mod}_c(\mathcal{C})$ just in that it also includes exploding models, and where E is an appropriate “exploding” topology. Classically, $\mathbf{Sh}(\mathbf{Mod}_c^E(\mathcal{C}), E)$ and $\mathbf{Set}^{\mathbf{Mod}_c(\mathcal{C})^{\text{op}}}$ will be equivalent categories, so that ev still factors through y :

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y} & \mathbf{Sh}(\mathcal{C}, K) \\
& \searrow ev & \swarrow \\
& \mathbf{Sh}(\mathbf{Mod}_c^E(\mathcal{C}), E) &
\end{array}$$

Constructively, though, the topos $\mathbf{Sh}(\mathbf{Mod}_c^E(\mathcal{C}), E)$ will be essentially the same as Veldman’s universal modified Kripke model.

We cannot relax the hypothesis on the size of the signature while remaining constructive. Indeed, we have that the completeness theorem for modified Kripke semantics for full first order logic is unprovable in IZF. Indeed, since classically modified Kripke semantics is just the usual Kripke semantics, the completeness of the former (MKC) is classically equivalent to the completeness of the latter (KC). On the other hand, adding to IZF the Principle of Excluded Middle (PEM) one gets ZF. Hence, if MKC was derivable in IZF, we would have $\text{ZF} = \text{IZF} + \text{PEM} = \text{IZF} + \text{MKC} + \text{PEM} = \text{ZF} + \text{KC}$, which is absurd since $\text{ZF} + \text{KC}$ implies the Boolean Prime Ideal theorem, which is known to be independent of ZF.

However, if we consider the disjunction-free fragment $\mathcal{L}_\vee$ of first-order logic and work with syntactic categories of theories over that fragment, we can still give a constructive version of the diagram above. Let $\mathcal{C}$ be a small $\mathcal{L}_\vee$ -free category, i.e., a Heyting category where subobjects lattices do not necessarily have unions (such a category supports hence the logic of the disjunction-free fragment). A functor preserving this structure is called a $\mathcal{L}_\vee$ -functor. We have the Yoneda embedding:

$$y : \mathcal{C} \longrightarrow \mathbf{Sh}(\mathcal{C}, \text{reg}_\perp),$$

where the $\text{reg}_\perp$ topology is generated by finite covering families with at most one element. Then, analogously to the Heyting case, one can see that the Yoneda embedding is a $\mathcal{L}_\vee$ -functor. We shall see constructively that there is a similar embedding:

$$ev : \mathcal{C} \longrightarrow \mathbf{Sh}(\mathbf{Mod}_{r\perp}^E(\mathcal{C}), E)$$

which is a $\mathcal{L}_\vee$ -functor (where now one considers the category of $\mathcal{L}_\vee$ -models), and that, classically, $\mathbf{Sh}(\mathbf{Mod}_{r\perp}^E(\mathcal{C}), E)$ is equivalent to $\mathbf{Set}^{\mathbf{Mod}_{r\perp}(\mathcal{C})^{\text{op}}}$. In other words, we have a diagram:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y} & \mathbf{Sh}(\mathcal{C}, \text{reg}_\perp) \\
& \searrow ev & \swarrow \\
& \mathbf{Sh}(\mathbf{Mod}_{r\perp}^E(\mathcal{C}), E) &
\end{array}$$

where the functors involved are all $\mathcal{L}_\vee$ -functors. This answers the question posed originally.