

Proof-search in natural deduction calculi for IPL

Mauro Ferrari¹, Camillo Fiorentini²

¹ DiSTA, Univ. degli Studi dell'Insubria, Via Mazzini, 5, 21100, Varese, Italy

² DI, Univ. degli Studi di Milano, Via Comelico, 39, 20135 Milano, Italy

We address the problem of proof-search in the natural deduction calculus **Ni** for intuitionistic propositional logic (IPL). Our aim is to improve the usual *näive* proof-search procedure where introduction rules are applied upwards and elimination rules downwards. In particular, we introduce **Nbu**, a sequent-style variant of **Ni**, and we show that its derivations can be reduced in θ -NF, a normal form (stronger than the usual one) where applications of $\vee E$ and $\rightarrow I$ satisfy additional conditions involving an *evaluation relation*. This limits the application of such rules in proof-search and restricts the search space.

A simple strategy [3] to build a natural deduction derivation of $[\Gamma \vdash H]$ is to use introduction rules (I-rules) reasoning bottom-up and elimination rules (E-rules) top-down, in order to “close the gap” between Γ (the assumptions) and H (the conclusion). Derivations built in this way are *normal* according to the standard definition [4]. This approach has been formalized [2] by introducing two kinds of judgments, we denote them by $[\Gamma \vdash H \uparrow]$ and $[\Gamma \vdash H \downarrow]$. A calculus to build derivations in normal form can be obtained by the rules in Fig. 1, erasing the labels from the $\uparrow$ -arrows and dropping the side condition from the rule $\vee E$.

A derivation $\mathcal{D}$ with root sequent $[\Gamma \vdash H \downarrow]$ represents a normal derivation of $[\Gamma \vdash H]$ having the rule Id, $\wedge E$ or $\rightarrow E$ as root rule. The rule $\Downarrow$ (coercion), is a sort of structural rule which “coerces” deductions in normal form. Actually, since $\Downarrow$ is restricted to atoms, derivations are in *long normal form* [4].

A strategy to search for a normal derivation of $[\Gamma \vdash H]$ can be sketched as follows. From $[\Gamma \vdash H \uparrow]$, we start an $\uparrow$ -expansion applying bottom-up the I-rules; meanwhile, for every $G \in \Gamma$, from $[\Gamma \vdash G \downarrow]$ we start a $\downarrow$ -expansion applying downwards the rules $\wedge E$ and $\rightarrow E$. At each step, we get “open proof-trees” having $\uparrow$ -sequents as leaves. To successfully build a derivation in normal form, we have to glue such trees using the rules $\Downarrow$, $\perp E$ or $\vee E$. In general, the two expansion steps must be interleaved, hence the search space is huge. E.g., when in a $\downarrow$ -expansion we apply the rule $\rightarrow E$ to a tree with root $[\Gamma \vdash A \rightarrow B \downarrow]$, we get $[\Gamma \vdash B \downarrow]$ as new root and $\sigma_A = [\Gamma \vdash A \uparrow]$ as a new leaf. Thus, to turn the tree into a derivation, we must enter a new $\uparrow$ -expansion step to get a derivation of σ_A . During an $\uparrow$ -expansion, contexts can increase. E.g., suppose that we have to $\uparrow$ -expand the sequent $[\Gamma \vdash A \rightarrow B \uparrow]$. Then, we have to apply one of the rules $\rightarrow I_1$ and $\rightarrow I_2$. In the former case the context Γ does not change; in the latter, A is added to Γ and we must run a new $\downarrow$ -expansion from the enlarged context A, Γ . A similar case concerns the application of $\vee E$.

Exploiting ideas from [1], we have introduced the calculus **Nbu** of Fig. 1 which allows us to limit the applications of $\rightarrow I_2$ and $\vee E$ during $\uparrow$ -expansion. **Nbu** acts on sequents where the arrow $\uparrow$ is decorated by one of the labels b

$$\begin{array}{c}
\frac{}{[A, \Gamma \vdash A \downarrow]} \text{Id} \quad \frac{[\Gamma \vdash p \downarrow]}{[\Gamma \vdash p \uparrow^l]} \Downarrow \quad p \in \mathcal{V} \quad \frac{[\Gamma \vdash \perp \downarrow]}{[\Gamma \vdash F \uparrow^l]} \perp E \quad F \in \mathcal{V} \cup \{\perp\} \\
\\
\frac{[\Gamma \vdash A \uparrow^l] \quad [\Gamma \vdash B \uparrow^l]}{[\Gamma \vdash A \wedge B \uparrow^l]} \wedge I \quad \frac{[\Gamma \vdash A_0 \wedge A_1 \downarrow]}{[\Gamma \vdash A_k \downarrow]} \wedge E \quad k \in \{0, 1\} \\
\\
\frac{[\Gamma \vdash A_k \uparrow^b]}{[\Gamma \vdash A_0 \vee A_1 \uparrow^l]} \vee I \quad k \in \{0, 1\} \quad \frac{[\Gamma \vdash A \vee B \downarrow] \quad [A, \Gamma \vdash D \uparrow^u] \quad [B, \Gamma \vdash D \uparrow^u]}{[\Gamma \vdash D \uparrow^u]} \vee E \\
\\
\frac{[\Gamma \vdash B \uparrow^l]}{[\Gamma \vdash A \rightarrow B \uparrow^l]} \rightarrow I_1 \quad \frac{[A, \Gamma \vdash B \uparrow^u]}{[\Gamma \vdash A \rightarrow B \uparrow^l]} \rightarrow I_2 \quad \frac{[\Gamma \vdash A \rightarrow B \downarrow] \quad [\Gamma \vdash A \uparrow^b]}{[\Gamma \vdash B \downarrow]} \rightarrow E
\end{array}$$

In $\vee E$, $D \in \mathcal{V} \cup \{\perp\}$ or $D = D_0 \vee D_1$

Fig. 1. The natural deduction calculus **Nbu** ($l \in \{b, u\}$).

(*blocked*) and u (*unblocked*). Note that the conclusion of $\vee E$ must be an $\uparrow^u$ -sequent and the formula on the right must be an atom or a disjunction; the right premise of $\rightarrow E$ is an $\uparrow^b$ -sequent. Proof-search starts now from an $\uparrow^u$ -sequent. In $\uparrow$ -expansions, the transition from $\uparrow^u$ -sequents to $\uparrow^b$ -sequents is marked by an application of rule $\vee I$. In expanding an $\uparrow^b$ -sequent, the label turns to u only if $\rightarrow I_2$ is applied. We point out that $\uparrow^b$ -arrows introduce a sort of *focus* on the right formula; indeed, proceeding bottom-up, in an $\uparrow$ -expansion of $[\Gamma \vdash H \uparrow^b]$ with H non-atomic, we are forced to apply an introduction rule to eliminate the main connective of H (applications of $\vee E$ are forbidden). Derivations in the calculus of [1] can be translated into **Nbu** and this proves its completeness.

To better control proof-search, we have introduced a notion of *θ -normal form* (θ -NF), based on an *evaluation relation* $[1] \models_\theta$ between sets of formulas and formulas. A derivation is in θ -NF iff: (1) rule $\rightarrow I_1$ is only applied if $\Gamma \models_\theta A$; (2) rule $\rightarrow I_2$ is only applied if $\Gamma \not\models_\theta A$; (3) $\vee E$ is only applied if $\Gamma \not\models_\theta A$ and $\Gamma \not\models_\theta B$. If $\sigma = [\Gamma \vdash H \uparrow^u]$ is provable in **Nbu**, then there exists a derivation of σ in θ -NF. Accordingly, we can limit ourselves to search for derivations in θ -NF, and this yields strong limitations on applicability of rules $\rightarrow I_2$ and $\vee E$. For more details we refer to <http://homes.di.unimi.it/~fiorentini/psc.pdf>.

References

1. M. Ferrari, C. Fiorentini, and G. Fiorino. A terminating evaluation-driven variant of G3i. In D. Galmiche and D. Larchey-Wendling, editors, *TABLEAUX 2013*, volume 8123 of *LNCS*, pages 104–118. Springer, 2013.
2. F. Pfenning. Automated theorem proving. Lecture notes, 2004.
3. W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). *Studia Logica*, 60(1):67–106, 1998.
4. A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2000.