

Computing with Gödel's completeness theorem

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Let us consider classical first-order logic over a language of function symbols taken in $\mathcal{F}un$ with arity $\#f$ for $f \in \mathcal{F}un$ and predicate symbols in $\mathcal{P}red$ with arity $\#P$ for $P \in \mathcal{P}red$. We consider the formulae built from the negative connectives and quantifiers $\neg$, $\bot$ and $\forall$, with $\neg A$ defined as $A \rightarrow \bot$ (the dot on the symbols indicates that these are symbols of an object language). We let x range over variables and t over the set of terms, written $\mathcal{T}$.

Let a model $\mathcal{M}$ be a triple $(\mathcal{D}, \mathcal{F}, \mathcal{P})$ made of a domain $\mathcal{D}$, an interpretation of function symbols $\mathcal{F}$ such that for all $f \in \mathcal{F}un$, $\mathcal{F}(f) \in \mathcal{D}^{\#f} \rightarrow \mathcal{D}$ and an interpretation of predicate symbols $\mathcal{P}$ such that for all $P \in \mathcal{P}red$, $\mathcal{M}_{\mathcal{P}}(P) \subset \mathcal{D}^{\#P}$.

The interpretation $\llbracket x \rrbracket_{\mathcal{M}}^{\sigma}$ of a term under an assignment σ of free variables to $\mathcal{D}$ is standard. The truth of a formula is given by:

$$\begin{aligned} \llbracket P(t_1, \dots, t_{a_P}) \rrbracket_{\mathcal{M}}^{\sigma} &\triangleq (\llbracket t_1 \rrbracket_{\mathcal{M}}^{\sigma}, \dots, \llbracket t_{a_P} \rrbracket_{\mathcal{M}}^{\sigma}) \in \mathcal{M}_{\mathcal{P}}(P) \\ \llbracket \bot \rrbracket_{\mathcal{M}}^{\sigma} &\triangleq \bot \\ \llbracket A \rightarrow B \rrbracket_{\mathcal{M}}^{\sigma} &\triangleq \llbracket A \rrbracket_{\mathcal{M}}^{\sigma} \rightarrow \llbracket B \rrbracket_{\mathcal{M}}^{\sigma} \\ \llbracket \forall x A \rrbracket_{\mathcal{M}}^{\sigma} &\triangleq \forall v \in \mathcal{M}_D \llbracket A \rrbracket_{\mathcal{M}}^{\sigma \cup [x \leftarrow v]} \end{aligned}$$

We say that the model is classical, written $Classic(\mathcal{M})$, if $\llbracket \neg \neg A \rightarrow A \rrbracket_{\mathcal{M}}^{\sigma}$ for all formulae A . In a classical metalanguage, any model is classical. However, we shall look at a constructive proof of completeness, so for a model to be sound with respect to classical reasoning in the object language, we need to enforce explicitly that it satisfies double-negation elimination.

Let us consider the following formulation of Gödel's completeness theorem:

$$(\mathbf{Compl}) \quad (\forall \mathcal{M} \forall \sigma (Classic(\mathcal{M}) \rightarrow \llbracket A \rrbracket_{\mathcal{M}}^{\sigma})) \rightarrow \vdash A$$

It is known from Gödel, Kreisel, McCarty that **Compl** intuitionistically implies Markov's Principle (MP). Conversely, MP is enough to intuitionistically prove **Compl** in second order arithmetic. We shall use the computational interpretation of MP as an exception mechanism from the first author.

We shall consider Henkin's proof of the completeness theorem and give a simple executable variant of it.

Let $\lceil A \rceil$ and ϕ form a Gödel's numbering of formulas such that $\lceil \phi(n) \rceil = n$. Let x_n be a variable fresh in $\phi(0), \dots, \phi(n)$. Henkin axioms at step n are defined by taking Θ_0 to be empty and Θ_{n+1} to be Θ_n unless $\phi(n) = \forall x A$ in which case it is $A[x_n/x] \rightarrow \forall x A, \Theta_n$.

Let Δ be a context. We shall say that a context Γ of formulae of Gödel's numbers less than n is a subset of the equiconsistent extension of Δ at rank n , written $\Gamma \subset \hat{\Delta}^n$, in the following cases:

$$\begin{array}{c}
\frac{}{\Delta \subset \hat{\Delta}^0} I_0 \quad \frac{\Gamma \subset \hat{\Delta}^n}{\Gamma \subset \hat{\Delta}^{n+1}} I_S \quad \frac{\Gamma \subset \hat{\Delta}^n \quad \forall \Gamma' \subset \hat{\Delta}^n (\Theta_n, \Gamma', \phi(n) \vdash \perp) \longrightarrow (\Delta \vdash \perp)}{\Gamma, \phi(n) \subset \hat{\Delta}^{n+1}} I_n
\end{array}$$

Under the assumption that Δ is consistent, the statement $\Gamma \subset \hat{\Delta}^n$ reads as Γ is a subset of the approximation at step n of the construction of a consistent completion of Δ . Note that the saturation of the model by Henkin's axioms is done on the fly every time a universally quantified formula is considered: we simply use fresh variables rather than new constants, what avoids needing countably many steps of completion.

We write $\Delta \vdash^* A$ to mean that there exist n and Γ such that $\Gamma \subset \hat{\Delta}^n$ and $\Gamma \vdash A$. Under the assumption that Δ is consistent, this reads as A holds in the consistent completion of Δ .

Note that a Γ such that $\Gamma \subset \hat{\Delta}^n$ is made of formulae ordered along their Gödel number. We write $\Gamma_1 \cup \Gamma_2$ for the union of two such contexts into a context of formulae still ordered along their Gödel number. The following lemmas hold by induction on ranks:

Lemma 1 (Injecting). $\Delta \subset \hat{\Delta}^n$

Lemma 2 (Flushing). $\Gamma \subset \hat{\Delta}^n \wedge (\Theta_n, \Gamma \vdash \perp) \longrightarrow (\Delta \vdash \perp)$

Lemma 3 (Joining). $\Gamma_1 \subset \hat{\Delta}^{n_1} \wedge \Gamma_2 \subset \hat{\Delta}^{n_2} \longrightarrow \Gamma_1 \cup \Gamma_2 \subset \hat{\Delta}^{\max(n_1, n_2)}$

Let A_0 be a valid formula. We shall prove $\vdash A_0$ by proving $(\neg A_0 \vdash \perp) \longrightarrow \perp$ thanks to Markov's principle in the metalanguage and classical reasoning in the object language. We define a syntactic model $\mathcal{M}_0$ by $\mathcal{D} \triangleq \mathcal{T}$, $\mathcal{F}(f)(t_1, \dots, t_{a_f}) \triangleq f(t_1, \dots, t_{a_f})$ and $\mathcal{P}(P)(t_1, \dots, t_{a_P}) \triangleq (\neg A_0 \vdash^* P(t_1, \dots, t_{a_P}))$. On the assumption that $(\neg A_0 \vdash \perp) \longrightarrow \perp$, we can now prove the necessary and sufficient conditions of the following main lemma by induction on A . Computationally, this respectively corresponds to reify/reflect programs as found in Normalisation by Evaluation or Type-Directed Partial Evaluation.

Lemma 4 (Main). $\llbracket A \rrbracket_{\mathcal{M}_0}^\sigma \longleftrightarrow (\neg A_0 \vdash^* \sigma(A))$.

For instance, the reification $\downarrow$ of a proof of $m \in \llbracket A \rrbracket_{\mathcal{M}_0}^\sigma$ into some proof of $\neg A_0 \vdash^* \sigma(A)$ in the case $A = B \rightarrow C$, whose Gödel's number is n , is given by

$$\begin{aligned}
& (n, (\neg A_0, B \rightarrow C), I_n(\mathbf{Injecting}_n, \\
& \quad (\Gamma, f, p) \mapsto \text{dest } \downarrow_C (m(\uparrow_B (n, \Gamma, f, \pi_1^{\rightarrow} p))) \text{ as } (n', \Gamma', f', p') \\
& \quad \text{in Flushing}_{\Gamma \cup \Gamma'}^{\Gamma \cup \Gamma'} (\mathbf{Joining}_{nn'}^{\Gamma \Gamma'} (f, f'), \mathbf{app}(\pi_2^{\rightarrow} p, p'))), \mathbf{ax})
\end{aligned}$$

where $\uparrow$ is reflection, **dest** is $\exists$ -elimination in the metalanguage and, in the object language, **app** is modus ponens, **ax** is the axiom rule, $\pi_1^{\rightarrow}$ infers $\Gamma \vdash B$ from $\Gamma, B \rightarrow C \vdash \perp$ and $\pi_2^{\rightarrow}$ infers $\Gamma \vdash \neg C$ from $\Gamma, B \rightarrow C \vdash \perp$.

Completeness is then direct using reflection on $\neg A_0$ and reification on $\perp$. For any proof of the validity of A_0 , it computes a proof of $\vdash A_0$.