

A Model Theory for $\mathbf{R}$ -calculus without Cut

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Abstract. In this paper, we propose a model theory for $\mathbf{R}$ -calculus without cut. First, validity of $\mathbf{R}$ -transitions and $\mathbf{R}$ -transition sequences is defined using models of first-order languages. Then reducibility of $\mathbf{R}$ -transition sequences is defined and it is proved that every $\mathbf{R}$ -transition sequence, which transits to an $\mathbf{R}$ -termination, can be reduced to an irreducible $\mathbf{R}$ -transition sequence. Finally, the soundness and completeness of $\mathbf{R}$ -calculus is redefined. And $\mathbf{R}$ -calculus without cut is proved to still be sound and complete.

Keywords: revision calculus, model, validity, soundness, completeness

1 Introduction

In the process of scientific discovery, a scientific theory is always subject to change to incorporate new findings. Since these new findings may conflict with the scientific theory, some laws of the scientific theory may have to be discarded before the new findings can be consistently incorporated. The problem is how to perform such discarding rationally. To address this problem, a more general problem called belief revision is been studied, which is a central topic in knowledge representation and reasoning. Many studies focus on the the postulates that a rational revision function should satisfy. The first and well-known set of postulates is the AGM postulates [1, 2]. There are also other sets of postulates, such as the KM postulates [3] for revision of knowledge bases represented by sets of propositional formulas and the DP postulates [4] for iterated revision. There are also many studies about the formal constructions of revision functions, such as [5–8].

The $\mathbf{R}$ -calculus was first proposed in 1992 [9]. The basic idea of $\mathbf{R}$ -calculus is to build a inference system in the first-order language, instead of a revision function, to reason about revising a theory with respect to a set of formulas which may contradict with the theory by discarding as few as possible of formulas in the theory. In many studies of belief revision except a few [10], the theory (belief set, or belief base) is revised by a formula, while in $\mathbf{R}$ -calculus, the theory is revised by a consistent set of atomic formulas and negations of atomic formulas (i.e. the facts). Another difference is that the result of belief contraction has to be unique, while $\mathbf{R}$ -calculus is aimed to deduce all possible maximal contractions that are

subsets of the theory which are maximally consistent with the facts. Finally, the construction of many revision or contraction functions relay on the first-order deduction system, while the **R**-calculus without cut [11] is an inference system on logical connective symbols and quantifier symbols and does not depend on the first-order deduction system.

Although the semantics of formulas in **R**-calculus is the same as that of formulas in first-order language [12], the semantics of **R**-calculus, in particular, the semantics of **R**-transition and **R**-transition sequence is not clearly defined previously. In this paper, we intend to define the validity of **R**-transition and **R**-transition sequence, revisit the concepts of soundness and completeness of **R**-calculus and redefine them using the concept of validity of **R**-transition sequences. By doing so, we establish a model theory for **R**-calculus without cut.

The rest of the paper is organized as follows. In Section 2, we shall recall some concepts used in **R**-calculus and briefly reintroduce **R**-calculus without cut. In Section 3, validity of **R**-transitions and **R**-transition sequences shall be defined. Reducibility of **R**-transition sequences is defined and studied in Section 4. Based on validity and reducibility of **R**-transition sequences, the soundness and completeness of **R**-calculus are redefined. **R**-calculus without cut is proved to be sound and complete in the new definition.

2 **R**-calculus without cut

Let us first recall some concepts used in **R**-calculus [12, 13]. In what follows, let Γ be a finite formula set and Δ be a formal theory consisting of finitely many atomic formulas and negations of atomic formulas.

Definition 1 (R-refutation and R-contraction). *If Δ and Γ are inconsistent, then Δ is called an **R**-refutation of Γ .*

*A formal theory Λ is called an **R**-contraction of Γ with respect to an **R**-refutation Δ if Λ is the maximal subset of Γ that is consistent with Δ .*

Definition 2 (Model of R-refutation). *Let $\Delta = \{A_1, \dots, A_n\}$ and $\Gamma \models \neg A_1 \vee \dots \vee \neg A_n$.*

If there is a model $\mathbf{M}$ such that $\mathbf{M} \models \Delta$, then we call $\mathbf{M}$ the model of refutation by facts of Γ with respect to Δ . Let

$$\Gamma_{\mathbf{M}(\Delta)} = \{B \mid B \in \Gamma, \mathbf{M} \models B, \mathbf{M} \models \Delta\}.$$

*We call $\mathbf{M}$ a model of **R**-refutation of Γ with respect to Δ if $\Gamma_{\mathbf{M}(\Delta)}$ is maximal, i.e. there does not exist another model $\mathbf{M}'$ of refutation by facts with respect to Δ such that $\Gamma_{\mathbf{M}(\Delta)} \subset \Gamma_{\mathbf{M}'(\Delta)}$.*

Definition 3 (R-configuration, inconsistent R-configuration, and R-termination). *$\Delta \mid \Gamma$ is called an **R**-configuration. If Δ is an **R**-refutation of Γ , then $\Delta \mid \Gamma$ is an inconsistent **R**-configuration. If Δ is consistent with Γ , then $\Delta \mid \Gamma$ is an **R**-termination.*

Definition 4 (R-transition and R-transition sequence).

$$\Delta \mid \Gamma \Longrightarrow \Delta' \mid \Gamma'$$

is called an **R-transition**. It transforms the **R-configuration** $\Delta \mid \Gamma$ into the **R-configuration** $\Delta' \mid \Gamma'$.

In particular, the **R-transition**

$$\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma$$

denotes the transformation of the **R-configuration** $\Delta \mid A, \Gamma$ into $\Delta \mid \Gamma$. As a result, A in the formula set A, Γ on the right-hand side of $\mid$ is deleted.

We call

$$\Delta \mid \Gamma \Longrightarrow \cdots \Longrightarrow \Delta \mid \Gamma_n \Longrightarrow \cdots \Longrightarrow \Delta \mid \Gamma'$$

an **R-transition sequence** and denote it as

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$$

with $\Longrightarrow^*$ denoting finite or countably infinite transitions.

The **R-calculus** without cut is defined as follows.

Definition 5 (R-calculus without cut). **R-calculus without cut** consists of two sets of rules. One is the **R-axiom** and rules about logical connective and quantifier symbols in formulas that are to be deleted, including the **R- $\wedge$** rule, the **R- $\vee$** rule, the **R- $\rightarrow$** rule, the **R- $\neg$** rule, the **R- $\forall$** rule, and the **R- $\exists$** rule, denoted by **R-rules**. The other is the **R'-axiom** and rules about logical connective and quantifier symbols in formulas that will not be deleted, including the **R'- $\wedge$** rule, the **R'- $\vee$** rule, the **R'- $\rightarrow$** rule, the **R'- $\neg$** rule, the **R'- $\forall$** rule, and the **R'- $\exists$** rule, denoted by **R'-rules**. **R-rules** and **R'-rules** are defined in Table 1.

From the above definition, we know that every rule of **R-calculus** without cut is a decomposition rule about a logical connective symbol or a quantifier symbol. Thus the proof tree of **R-calculus** without cut can be defined inductively as follows.

Definition 6 (R-inference tree and R-proof tree). An **R-inference tree** $\mathcal{T}$ is inductively defined as follows:

- (1) A single node tree is an **R-inference tree**, if its node (i.e. its root) is an **R-transition**.
- (2) Suppose that $\mathcal{T}_1$ is an **R-inference tree** whose root is $\Delta \mid \Gamma_1 \Longrightarrow \Delta \mid \Gamma'_1$. If

$$\frac{\Delta \mid \Gamma_1 \Longrightarrow \Delta \mid \Gamma'_1}{\Delta \mid \Gamma \Longrightarrow \Delta \mid \Gamma'}$$

is an instance of a rule of **R-calculus** without cut, then

$$\frac{\mathcal{T}_1}{\Delta \mid \Gamma \Longrightarrow \Delta \mid \Gamma'}$$

is an **R-inference tree**.

R-axiom: $A, \Delta \mid \neg A, \Gamma \Rightarrow A, \Delta \mid \Gamma \quad \neg A, \Delta \mid A, \Gamma \Rightarrow \neg A, \Delta \mid \Gamma$	R'-axiom: $\Delta \mid A, \neg A, \Gamma \Rightarrow \Delta \mid A, \Gamma \quad \Delta \mid A, \neg A, \Gamma \Rightarrow \Delta \mid \neg A, \Gamma$
R-$\wedge$ rule: $\frac{\Delta \mid A, B, \Gamma \Rightarrow \Delta \mid B, \Gamma \quad \Delta \mid A, B, \Gamma \Rightarrow \Delta \mid A, \Gamma}{\Delta \mid A \wedge B, \Gamma \Rightarrow \Delta \mid \Gamma} \quad \frac{\Delta \mid A, B, \Gamma \Rightarrow \Delta \mid A, \Gamma \quad \Delta \mid A, B, \Gamma \Rightarrow \Delta \mid B, \Gamma}{\Delta \mid A \wedge B, \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\wedge$ rule: $\frac{\Delta \mid C, A, B, \Gamma \Rightarrow \Delta \mid A, B, \Gamma}{\Delta \mid C, A \wedge B, \Gamma \Rightarrow \Delta \mid A \wedge B, \Gamma}$
R-$\vee$ rule: $\frac{\Delta \mid A, \Gamma \Rightarrow \Delta \mid \Gamma \quad \Delta \mid B, \Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid A \vee B, \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\vee$ rule: $\frac{\Delta \mid C, A, \Gamma \Rightarrow \Delta \mid A, \Gamma \quad \Delta \mid C, B, \Gamma \Rightarrow \Delta \mid C, \Gamma}{\Delta \mid C, A \vee B, \Gamma \Rightarrow \Delta \mid A \vee B, \Gamma}$
R-$\rightarrow$ rule: $\frac{\Delta \mid \neg A, \Gamma \Rightarrow \Delta \mid \Gamma \quad \Delta \mid B, \Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid A \rightarrow B, \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\rightarrow$ rule: $\frac{\Delta \mid C, \neg A, \Gamma \Rightarrow \Delta \mid \neg A, \Gamma \quad \Delta \mid C, B, \Gamma \Rightarrow \Delta \mid C, \Gamma}{\Delta \mid C, A \rightarrow B, \Gamma \Rightarrow \Delta \mid A \rightarrow B, \Gamma}$
R-$\forall$ rule: $\frac{\Delta \mid A[t/x], \forall x A(x), \Gamma \Rightarrow \Delta \mid \forall x A(x), \Gamma}{\Delta \mid \forall x A(x), \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\forall$ rule: $\frac{\Delta \mid C, A[t/x], \forall x A(x), \Gamma \Rightarrow \Delta \mid A[t/x], \forall x A(x), \Gamma}{\Delta \mid C, \forall x A(x), \Gamma \Rightarrow \Delta \mid \forall x A(x), \Gamma}$
where t is a term.	
R-$\exists$ rule: $\frac{\Delta \mid A[y/x], \Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid \exists x A(x), \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\exists$ rule: $\frac{\Delta \mid C, A[y/x], \Gamma \Rightarrow \Delta \mid A[y/x], \Gamma}{\Delta \mid C, \exists x A(x), \Gamma \Rightarrow \Delta \mid \exists x A(x), \Gamma}$
where y is either x or an eigen-variable, i.e. y is different from all the variables in the denominator of the rule.	
R-$\neg$ rule: $\frac{\Delta \mid A', \Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid A, \Gamma \Rightarrow \Delta \mid \Gamma}$	R'-$\neg$ rule: $\frac{\Delta \mid C, A', \Gamma \Rightarrow \Delta \mid A', \Gamma}{\Delta \mid C, A, \Gamma \Rightarrow \Delta \mid A, \Gamma}$
where A and A' are specified by the following table:	
$\frac{A \mid \neg(D \wedge E) \mid \neg(D \vee E) \mid \neg \neg D \mid \neg(D \rightarrow E) \mid \neg \forall x D(x) \mid \neg \exists x D(x)}{A' \mid \neg D \vee \neg E \mid \neg D \wedge \neg E \mid D \mid D \wedge \neg E \mid \exists x \neg D(x) \mid \forall x \neg D(x)}$	

Table 1. **R**-calculus without cut

(3) Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are **R**-inference trees whose roots are $\Delta \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma'_1$ and $\Delta \mid \Gamma_2 \Rightarrow \Delta \mid \Gamma'_2$ respectively. If

$$\frac{\Delta \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma'_1 \quad \Delta \mid \Gamma_2 \Rightarrow \Delta \mid \Gamma'_2}{\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'}$$

is an instance of a rule of **R**-calculus without cut, then

$$\frac{\mathcal{T}_1 \quad \mathcal{T}_2}{\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'}$$

is an **R**-inference tree.

If the root of $\mathcal{T}$ is $\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$, then we say that $\mathcal{T}$ is an **R**-inference tree of $\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$.

If $\mathcal{T}$ is a finite **R**-inference tree of $\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$ and its leaf nodes are all instances of the **R**-axiom or **R'**-axiom, then $\mathcal{T}$ is called an **R**-proof tree of $\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$.

Using the concept of proof tree, the provability of **R**-transition and **R**-transition sequence is defined as follows.

Definition 7 (Provability). We say that an **R**-transition is provable if it has an **R**-proof tree. Otherwise, we say that it is unprovable.

We say that an **R**-transition sequence is provable if every **R**-transition in the **R**-transition sequence is provable.

It has been proved that by adding a set Θ which is consistent with Δ to the left-hand side of and a set Λ to the right-hand side of the two **R**-configurations in a provable **R**-transition, the new **R**-transition is also provable [11].

Lemma 1 (Addition rule). *Let Θ be a set of atomic formulas and the negations of atomic formulas, and Θ is consistent with Δ . Let Λ be an arbitrary set of formulas. The following addition rule holds:*

$$\frac{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta, \Theta \mid A, \Gamma, \Lambda \Longrightarrow \Delta, \Theta \mid \Gamma, \Lambda}.$$

3 Validity of **R**-transitions and **R**-transition sequences

The philosophy behind **R**-calculus is occam's razor. It requires that, in **R**-calculus, we should only delete those formulas which cause inconsistencies between the current theory Γ and the facts Δ such that the remaining formulas of Γ is a maximal subset of Γ which is consistent with Δ . That is, validity of an **R**-transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is that the deleting of A is necessary, i.e. A causes an inconsistency between Δ and Γ and the inconsistency can be resolved by deleting A . In another words, validity of an **R**-transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is that the formula A , which is to be deleted, together with Δ and some formulas in Γ should form an unsatisfiable set which can be satisfiable again by deleting A . To define it formally, let us introduce the concept of minimal unsatisfiable subset first.

Definition 8 (Minimal unsatisfiable subset). *Let $\Delta \mid \Gamma$ be an **R**-configuration and Λ be a subset of Γ . Λ is a minimal unsatisfiable subset of Γ with respect to Δ , if there does not exist a model $\mathbf{M}$ which satisfies Δ such that $\mathbf{M} \models \Lambda$, and for any proper subset Λ' of Λ , there exists a model $\mathbf{M}$ which satisfies Δ such that $\mathbf{M} \models \Lambda'$.*

Using minimal unsatisfiable subset, validity of **R**-transitions is defined as follows.

Definition 9 (Validity of **R-transitions).** *An **R**-transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is valid, if there exists a minimal unsatisfiable subset Λ of $\Gamma \cup \{A\}$ with respect to Δ such that $A \in \Lambda$.*

It can be proved that any valid **R**-transition is provable.

Theorem 1. *If an **R**-transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is valid, then it is provable.*

Proof. If the **R**-transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is valid, then there exists a minimal unsatisfiable subset Λ of $\Gamma \cup \{A\}$ with respect to Δ such that $A \in \Lambda$. Let $\Gamma' = \Lambda - \{A\}$. According to Definition 8, $\Gamma' \cup \Delta$ is satisfiable, but $\Gamma' \cup \{A\} \cup \Delta$ is unsatisfiable. Thus Γ' is consistent with Δ and Δ, A, Γ' is inconsistent. So Γ' is an **R**-contraction of $\Gamma' \cup \{A\}$ with respect to Δ . According to Lemma 3 in [11], $\Delta \mid \Gamma', A \Longrightarrow \Delta \mid \Gamma'$ is provable. Because $\Gamma' \subset \Gamma$, according to Lemma 1, $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ is provable as well.

It is natural to ask whether the converse of the above theorem is true. The answer is negative, i.e. a provable **R**-transition is not necessarily valid. Let us consider the following example.

Example 1. Consider **R**-transition $\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C$. According to the **R**- $\vee$ rule and the **R'**-axiom, we have the following proof tree.

$$\frac{\frac{\mid \neg A, \neg C, \neg B \wedge C, A \Longrightarrow \mid \neg A, \neg C, \neg B \wedge C}{\mid \neg A \wedge \neg C, \neg B \wedge C, A \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C} \quad \frac{\mid \neg A \wedge \neg C, \neg B, C, B \Longrightarrow \mid \neg A \wedge \neg C, \neg B, C}{\mid \neg A \wedge \neg C, \neg B \wedge C, B \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C}}{\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C}$$

The **R**-transition is provable. But the only minimal unsatisfiable subset of $\{\neg A \wedge \neg C, \neg B \wedge C, A \vee B\}$ with respect to the empty set is $\{\neg A \wedge \neg C, \neg B \wedge C\}$ which does not contain $A \vee B$. By Definition 9, The **R**-transition is not valid.

For **R**-transitions $\mid \neg A \wedge \neg C, \neg B \wedge C, A \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C$ and $\mid \neg A \wedge \neg C, \neg B \wedge C, B \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C$, since $\{\neg A \wedge \neg C, A\}$ is a minimal unsatisfiable subset of $\{\neg A \wedge \neg C, \neg B \wedge C, A\}$ with respect to the empty set which contains A and $\{\neg B \wedge C, B\}$ is a minimal unsatisfiable subset of $\{\neg A \wedge \neg C, \neg B \wedge C, B\}$ with respect to the empty set which contains B , by Definition 9, both **R**-transitions are valid. This indicates that the **R**- $\vee$ rule does not preserve validity.

The goal of **R**-calculus is to deduce all **R**-contractions. This usually involve deleting of multiple formulas which is not possible with a single **R**-transition and hence need to be accomplished by an **R**-transition sequence. Thus we should also define validity of **R**-transition sequences. A valid **R**-transition sequence should guarantee that every formula, which is deleted by the **R**-transition sequence, is necessary, i.e. the **R**-transition sequence can never reach an **R**-termination whose right-hand side is an **R**-contraction without deleting any of these formula.

Definition 10 (Validity of R-transition sequences). *An **R**-transition sequence $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$ is valid, if every **R**-transition in it is valid and there exists a model **M** of **R**-refutation such that Γ' is the set of all the formulas in Γ which are true under **M**.*

From the previous studies of **R**-calculus and **R**-calculus without cut, we have already known that for a provable **R**-transition sequence $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$, Γ' is not guarantee be an **R**-contraction of Γ with respect to Δ . This means that not all provable **R**-transition sequences are valid. Therefore, we need to find out which kind of **R**-transition sequences is valid. Let us first study some properties of **R**-transition sequences.

4 Reducibility of R-transition Sequences

In order to figure out which kind of **R**-transition sequences is valid, we shall first record the set of formulas which are used in the deletion of formula in an **R**-transition. By doing so, over deleting of formulas may be avoid. Let us first define the set of necessary antecedents of **R**-proof tree.

Definition 11 (Set of necessary antecedents of $\mathbf{R}$ -proof tree).

The set of necessary antecedents Λ of an $\mathbf{R}$ -proof tree $\mathcal{T}$ is structural inductively defined as follows.

- (1) Single node tree. If it is an instance of $\mathbf{R}$ -axiom, then its set of necessary antecedents is the empty set; if it is an instance of $\mathbf{R}'$ -axiom of the form $\Delta \mid \Gamma, A, \neg A \Longrightarrow \Delta \mid \Gamma, \neg A$, then its set of necessary antecedents is $\{\neg A\}$; if it is an instance of $\mathbf{R}'$ -axiom of the form $\Delta \mid \Gamma, A, \neg A \Longrightarrow \Delta \mid \Gamma, A$, then its set of necessary antecedents is $\{A\}$.
- (2) Suppose that $\mathcal{T}$ is an $\mathbf{R}$ -proof tree of the form

$$\frac{\mathcal{T}_1}{\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma} \quad (a)$$

and the set of necessary antecedents of $\mathcal{T}_1$ is Λ_1 .

If (a) is an instance of an $\mathbf{R}$ -rule, the set of necessary antecedents Λ of $\mathcal{T}$ is obtained by removing all sub-formulas of A in Λ_1 .

If (a) is an instance of an $\mathbf{R}'$ -rule, suppose $B \in \Gamma$ is the formula decomposed by this rule, and let Λ'_1 be the set obtained by removing all sub-formula of A in Λ_1 , the set of necessary antecedents of $\mathcal{T}$ is $\Lambda'_1 \cup \{B\}$.

- (3) Suppose that $\mathcal{T}$ is an $\mathbf{R}$ -proof tree of the form

$$\frac{\mathcal{T}_1 \quad \mathcal{T}_2}{\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma} \quad (b)$$

and the sets of necessary antecedents of $\mathcal{T}_1$ and $\mathcal{T}_2$ are Λ_1 and Λ_2 respectively.

If (b) is an instance of an $\mathbf{R}$ -rule, and let Λ'_1 and Λ'_2 be the sets obtained by removing all sub-formulas of A , the set of necessary antecedents Λ of $\mathcal{T}$ is $\Lambda'_1 \cup \Lambda'_2$.

If (b) is an instance of an $\mathbf{R}'$ -rule, suppose $B \in \Gamma$ is the formula decomposed by this rule, and let Λ'_1 and Λ'_2 be the sets obtained by removing all sub-formulas of A , the set of necessary antecedents Λ of $\mathcal{T}$ is $\Lambda'_1 \cup \Lambda'_2 \cup \{B\}$.

Then necessary antecedents of an $\mathbf{R}$ -transition is defined as follows.

Definition 12 (Necessary antecedent of $\mathbf{R}$ -transitions).

Let $\mathcal{T}$ be an $\mathbf{R}$ -proof tree of the $\mathbf{R}$ -transition $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ and Λ be the set of necessary antecedents of $\mathcal{T}$. The set $\Theta = \Lambda - \Delta$ is called a set of necessary antecedents of $\Delta \mid \Gamma, A \Longrightarrow \Delta \mid \Gamma$ with respect to $\mathcal{T}$. In this case, the $\mathbf{R}$ -transition can be also written as

$$\Delta \mid \Gamma, A \xRightarrow{\Theta} \Delta \mid \Gamma.$$

By definition, a set of necessary antecedents is the set of formulas which are needed for deleting a formula in an $\mathbf{R}$ -transition. Thus the following lemma holds.

Lemma 2. If $\Delta \mid \Gamma, A \xRightarrow{\Theta} \Delta \mid \Gamma$ is provable, then $\Delta \mid \Theta, A \Longrightarrow \Delta \mid \Theta$ is also provable.

Proof. This lemma can be proved by structural induction on **R**-proof tree.

Let us consider the following Example.

Example 2. Consider **R**-transition $\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg B \wedge C, A \vee B$. According to the **R**- $\wedge$ rule, the **R'**- $\wedge$ rule, and the **R'**-axiom, we have the following proof tree.

$$\frac{\frac{\mid \neg A, \neg C, \neg B, C, A \vee B \Longrightarrow \mid \neg A, \neg B, C, A \vee B}{\mid \neg A, \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg A, \neg B \wedge C, A \vee B}}{\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg B \wedge C, A \vee B}$$

Thus, the **R**-transition is provable and valid. By definition, the **R**-transition sequence consist of **R**-transition $\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg B \wedge C, A \vee B$ is also provable and valid.

But from the **R**-transition $\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow \mid \neg A \wedge \neg C, \neg B \wedge C$ which is considered in Example 1, we can only obtain provable **R**-transitions sequence like $\mid \neg A \wedge \neg C, \neg B \wedge C, A \vee B \Longrightarrow^* \mid \neg B \wedge C$ which is not valid by definition. The only difference between these two **R**-transition sequences is that the former deletes $\neg A \wedge \neg C$ first but the latter deletes $A \vee B$ first and then deletes $\neg A \wedge \neg C$.

From the above example, we know that in **R**-transition sequences, the deleting order of formulas effects the result of the transition and by switch the deleting order of two formulas, we can transform an **R**-transition sequence to another and change the result of the transition. This can be formally defined as follows.

Definition 13 (Reducibility of **R-transition sequences).** Suppose that the **R**-transition sequence $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow^* \Delta \mid \Gamma$ is provable, i.e. the following **R**-transitions are provable.

$$\begin{aligned} \Delta \mid \Gamma, A_1, \dots, A_m &\xrightarrow{\Theta_1} \Delta \mid \Gamma, A_2, \dots, A_m, \\ \Delta \mid \Gamma, A_2, \dots, A_m &\xrightarrow{\Theta_2} \Delta \mid \Gamma, A_3, \dots, A_m, \\ &\vdots \\ \Delta \mid \Gamma, A_m &\xrightarrow{\Theta_m} \Delta \mid \Gamma. \end{aligned}$$

We say that $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow^* \Delta \mid \Gamma$ is reducible, if

- (1) there exists i and j such that $j < i$ and $A_i \in \Theta_j$, and
- (2) $\Delta \mid \Gamma, A_j, A_{i+1}, \dots, A_m \Longrightarrow \Delta \mid \Gamma, A_{i+1}, \dots, A_m$ is unprovable.

In this case, $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow^* \Delta \mid \Gamma$ is reduced to $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow^* \Delta \mid \Gamma, A_j$. Otherwise, it is irreducible. A reducible **R**-transition sequence is denoted as $\Longrightarrow_r^*$, an irreducible **R**-transition sequence is denoted as $\Longrightarrow_i^*$.

Based on the above definition, we can prove that every provable **R**-transition sequence, which transits an inconsistent **R**-configuration to an **R**-termination, can be reduced to an irreducible **R**-transition sequence

Theorem 2 (Reducibility). *If $\mathbf{R}$ -transition sequence $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_1$ is provable and $\Delta \mid \Gamma_1$ is an $\mathbf{R}$ -termination, then it can be reduced to an irreducible $\mathbf{R}$ -transition sequence $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma_2$ such that $\Gamma_2 \supseteq \Gamma_1$.*

Proof. Suppose that $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_1$ is a reducible $\mathbf{R}$ -transition sequence. Let us denote $\Gamma - \Gamma_1$ as $\{A_1, \dots, A_m\}$. According to Definition 13, there exists i and j such that $j < i$, $A_i \in \Theta_j$ and $\Delta \mid \Gamma_1, A_j, A_{i+1}, \dots, A_m \Longrightarrow \Delta \mid \Gamma_1, A_{i+1}, \dots, A_m$ is unprovable. By switching the deleting order of A_i and A_j , $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_1$ can be reduced to $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_1, A_j$. $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_1, A_j$ is provable and $\Delta \mid \Gamma_1, A_j$ is an $\mathbf{R}$ -termination. Since there is only finitely many $\mathbf{R}$ -transitions in the $\mathbf{R}$ -transition sequence, by repeating this process, an irreducible $\mathbf{R}$ -transition sequence $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma_2$ can be obtained. The above process guarantees that $\Gamma_2 \supseteq \Gamma_1$.

We prove next that an irreducible $\mathbf{R}$ -transition sequence, which transits an inconsistent $\mathbf{R}$ -configuration to an $\mathbf{R}$ -termination, always guarantees that the right-hand side of the $\mathbf{R}$ -termination is an $\mathbf{R}$ -contraction. Let us prove the following auxiliary lemma first.

Lemma 3. *If $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow_i^* \Delta \mid \Gamma$, then for $1 \leq i \leq m$, $\Delta \mid \Gamma, A_i \Longrightarrow \Delta \mid \Gamma$ is provable.*

Proof. Since $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow_i^* \Delta \mid \Gamma$ is irreducible, by Definition 13, for any i and j such that $i < j$, either $A_i \notin \Theta_j$ or $\Delta \mid \Gamma, A_j, A_{i+1}, \dots, A_m \Longrightarrow \Delta \mid \Gamma, A_{i+1}, \dots, A_m$ is provable.

If there exists j such that $i < j$, $A_j \in \Theta_i$ and $\Delta \mid \Gamma, A_i, A_{j+1}, \dots, A_m \Longrightarrow \Delta \mid \Gamma, A_{j+1}, \dots, A_m$ is provable, then we switch A_i and A_j . Since $\Delta \mid \Gamma, A_1, \dots, A_m \Longrightarrow_i^* \Delta \mid \Gamma$ is irreducible, this does not effect provability of the $\mathbf{R}$ -transition sequence. Since the $\mathbf{R}$ -transition sequence is finite, we can repeat this process until for any j such that $i < j$, we have $A_j \notin \Theta_i$. Then, according to Lemma 2, $\Delta \mid \Theta_i, A_i \Longrightarrow \Delta \mid \Theta_i$ is provable and $\Theta_i \subseteq \Gamma$. Thus, by Lemma 1, $\Delta \mid \Gamma, A_i \Longrightarrow \Delta \mid \Gamma$ is provable.

Lemma 4. *If $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma'$ and $\Delta \mid \Gamma'$ is an $\mathbf{R}$ -termination, then Γ' is an $\mathbf{R}$ -contraction of Γ with respect to Δ , and vice versa.*

Proof. Sufficiency. Proof by contradiction. Suppose that Γ' is not an $\mathbf{R}$ -contraction of Γ with respect to Δ . Since $\Delta \mid \Gamma'$ is an $\mathbf{R}$ -termination, Γ' is consistent with Δ . Thus, there exists a $A_i \in \Gamma - \Gamma'$ such that $\Gamma' \cup A_i$ is consistent with Δ . Because that $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma'$, by Lemma 3, $\Delta \mid \Gamma', A_i \Longrightarrow \Delta \mid \Gamma$ is provable. Suppose that its set of necessary antecedents is Θ , then $\Theta \subseteq \Gamma'$. By Lemma 2, $\Delta \mid \Theta, A_i \Longrightarrow \Delta \mid \Theta$ is provable. So $\Delta \mid \Theta, A_i$ is an inconsistent $\mathbf{R}$ -configuration, i.e. $\Theta \cup \{A_i\}$ is inconsistent with Δ . This leads to contradiction that Δ is consistent with Γ' .

Necessity. Since Γ' is an $\mathbf{R}$ -contraction of Γ with respect to Δ , $\Delta \mid \Gamma'$ is an $\mathbf{R}$ -termination. By the reachability of $\mathbf{R}$ -calculus without cut, there exists an provable $\mathbf{R}$ -transition sequence $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$. By Definition 13, it is irreducible. Otherwise, it can be proved that there exists a subset Γ_1 of Γ such

that Γ_1 is consistent with Δ and $\Gamma_1 \supset \Gamma'$. This leads to the contradiction that Γ' is an **R**-contraction.

5 Soundness and Completeness

Based on validity and irreducibility of **R**-transition sequences, we can redefine the soundness and completeness of **R**-calculus. The soundness of **R**-calculus is that “if an **R**-transition sequence is irreducible and its right-hand side is an **R**-termination, then it is valid.” The completeness of **R**-calculus is that “if an **R**-transition sequence is valid, then it is irreducible and its right-hand side is an **R**-termination.” We shall prove in the following that **R**-calculus without cut is sound and complete.

Theorem 3 (Soundness). *If an **R**-transition sequence is irreducible and its right-hand side is an **R**-termination, then it is valid.*

Proof. Let $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma'$ be an irreducible **R**-transition sequence and $\Delta \mid \Gamma'$ be an **R**-termination. By Lemma 4, Γ' is an **R**-contraction of Γ with respect to Δ . Thus, there exists a model **M** of **R**-refutation such that Γ' is the set of all the formulas in Γ which are true under **M**. Suppose that $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma'$ is invalid, then at least one **R**-transition in it is invalid. Without loss of generality, suppose that $\Delta \mid \Gamma', A_i, A_{i+1}, \dots, A_m \Longrightarrow \Delta \mid \Gamma', A_{i+1}, \dots, A_m$ is invalid. By Definition 9, A_i is not in any minimal unsatisfiable subset. According to Definition 8, A_i is true in any model **M** which satisfies Δ . Hence, A_i should be contained in any **R**-contraction of Γ with respect to Δ . This contradicts the fact that Γ' is an **R**-contraction of Γ with respect to Δ and $A_i \notin \Gamma'$. Thus, $\Delta \mid \Gamma \Longrightarrow_i^* \Delta \mid \Gamma'$ is valid.

Theorem 4 (Completeness). *If an **R**-transition sequence is valid, then it is irreducible and its right-hand side is an **R**-termination.*

Proof. Suppose that the **R**-transition sequence is $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$. Since the **R**-transition sequence is valid, by Definition 10, every **R**-transition in it is valid. According to Theorem 1, every **R**-transition in this **R**-transition sequence is provable. Thus, this **R**-transition sequence is provable. According to Definition 10, there exists a model **M** of **R**-refutation such that Γ' is the set of formulas in Γ which are true under **M**. Hence, Γ' is an **R**-contraction of Γ with respect to Δ . So $\Delta \mid \Gamma'$ is an **R**-termination. Suppose that $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$ is reducible. By Definition 13, there exists a subset Γ_1 of Γ such that Γ_1 is consistent with Δ and $\Gamma_1 \supset \Gamma'$. This contradicts the fact that Γ' is an **R**-contraction. Thus, $\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$ is irreducible and $\Delta \mid \Gamma'$ is an **R**-termination.

6 Conclusions

In this paper, validity of **R**-transitions and **R**-transition sequences is defined, the reducibility of **R**-transition sequences is defined and studied, the soundness

and completeness of **R**-calculus are redefined based on validity and reducibility of **R**-transition sequences, and **R**-calculus without cut is proved to be sound and complete in the new sense. By doing so, a model theory for **R**-calculus without cut is established.

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