

Axiomatic and Tableau-Based Reasoning for $Kt(H, R)$

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The context of this work is automated reasoning for modal logics. Here, we explore a variety of different deduction approaches in the spectrum between the purely axiomatic approach and the explicitly semantic approach. Our investigation is focussed on a propositional modal logic with non-trivial interactions between the modalities. The logic, called $Kt(H, R)$, has forward and backward looking modal operators defined by two accessibility relations H and R . The frame conditions are reflexivity and transitivity of H , and *stability* of R with respect to H . The stability condition is defined as $H ; R ; H \subseteq R$, where $;$ denotes relational composition. This means in a Kripke frame, for any two states u and v , whenever there is an H -transition from u , followed by an R -transition and an H -transition, to v , then there is also an R -transition from u to v .

The logic $Kt(H, R)$ originates with recent work on a bi-intuitionistic tense logic, called BISK_T, which is studied with the motivation to develop a theory of relations on graphs and applications to spatial reasoning [11]. Using the standard embedding of intuitionistic logics into modal logic, BISK_T can be embedded into $Kt(H, R)$ and properties such as decidability, the finite model property and complexity of $Kt(H, R)$ carry over to BISK_T. Moreover, deduction methods for $Kt(H, R)$ and implementations can be used for BISK_T.

$Kt(H, R)$ is of independent interest because the modal axiom(s) corresponding to the stability condition can be used to ascribe levels of awareness to agents in a multi-agent setting. The standard model for formalising knowledge and actions performed by agents, or events happening in an agent environment, uses the *S5*-modality as knowledge operator and *K*-modalities as action operators. In $Kt(H, R)$, the $[H]$ -modality and the $[R]$ -modality can be seen as modelling knowledge and action operators. (The formalisation is slightly more general, because the negative introspection axiom is not assumed for the $[H]$ -modality but this is not critical because it can be easily added to the logic.) $[H]\phi$ is read to mean ‘the agent knows ϕ ’ and $[R]\phi$ is read to mean ‘always after executing action R , ϕ holds’. In this context, the axiom

$$S = [R]\phi \rightarrow [H][R][H]\phi$$

corresponding to the stability condition $H ; R ; H \subseteq R$, can be viewed as saying ‘the agent knows that, after performing an action R , it knows the effects of the action’. Thus, it states the agent has (strong) awareness of performing action R and its effects.

The logic $Kt(H, R)$ has an alternative axiomatisation in which the stability axiom S is equivalent to the two axioms

$$A = [R]\phi \rightarrow [H][R]\phi \quad \text{and} \quad P = [R]\phi \rightarrow [R][H]\phi.$$

From an agent perspective, Axiom A says ‘the agent knows, when action R is performed, then ϕ necessarily holds’; in other words, the agent is aware of action R . Axiom P says ‘after performing action R the agent knows ϕ holds’, i.e., it knows the post-condition has been realised. In some sense, Axioms A and P can be viewed as weak forms of no learning and perfect recall. No learning is typically formalised as $[R][H]\phi \rightarrow [H][R]\phi$, and perfect recall as $[H][R]\phi \rightarrow [R][H]\phi$.

A contribution of this work is a series of labelled semantic tableau calculi for the logic $Kt(H, R)$. Labelled semantic tableau calculi of the pure semantic kind explicitly and directly construct Kripke models during the inference process. They use *structural rules* which are direct reflections of the background theory given by a set of characterising frame conditions. For example, for axiom

$$4 = [H]\phi \rightarrow [H][H]\phi$$

the structural rule is $H(s, t), H(t, u) / H(s, u)$ and ensures H will be a transitive relation. Transitivity is the frame condition of Axiom 4. For logics with semantic characterisations, labelled tableau calculi using structural rules may be developed by systematic methods. A general method is described in [10, 12].

Alternatively, the background theory can be accommodated as *propagation rules* [3]. The propagation rule for Axiom 4 is $s : [H]\phi, H(s, t) / t : [H]\phi$. Propagation rules accommodate the background theory not by representations of the frame conditions but by representations of inferences with the Hilbert axioms [6]. Propagation rules can be seen to attempt to speed up the inference process by not returning complete concrete models but only skeleton models and performing just enough inferences to determine both satisfiability and unsatisfiability.

We also explore the extreme case of basing the tableau rules of the background theory on direct representations of Hilbert axioms, e.g., using the rule $s : [H]\phi / s : [H][H]\phi$ for Axiom 4. This is an example of what we call an *axiomatic rule*. Calculi with such rules are seldom seen in the literature, and some authors have suggested completeness and termination cannot be guaranteed with such rules. We show however complete and terminating tableau calculi based on such rules can be obtained.

In total we present fifteen different tableau systems for $Kt(H, R)$ which we prove sound and complete. These emerge in systematic ways from the two axiomatisations and corresponding semantic characterisations of the logic. Only in one case a cut rule is needed when using propagation rules. For the Axioms A and P , no cut is needed.

Each of the calculi is terminating when endowed with the unrestricted blocking mechanism. The unrestricted blocking mechanism is based on the use of the following (ub) rule.

$$(\text{ub}) \frac{}{s \approx t \mid s \not\approx t}$$

Adding the unrestricted blocking mechanism to a sound and constructively complete labelled tableau calculus forces termination, when the logic has the (effective) finite model property [8, 9].

Each of the tableau calculi can serve as a basis for an effective translation of satisfiability in $Kt(H, R)$ into first-order logic. The soundness and completeness of these translations is a consequence of the soundness and completeness of the calculi, and the fact that derivations are defined over a bounded number of modal formulae. The encoding of one of the calculi has the property that it maps satisfiability of a formula in $Kt(H, R)$ into the guarded fragment [1, 5]. This enables us to show: (i) $Kt(H, R)$ is decidable and has the effective finite model property, and (ii) satisfiability in $Kt(H, R)$ is PSPACE-complete. Important is the effective finite model property; using the results in [9] guarantees termination for all our tableau calculi, when unrestricted blocking is used.

The guarded fragment can be decided by ordered resolution [4]. Therefore, both ordered resolution and ordered resolution with selection as defined in [4] decide the axiomatic translation of satisfiability problems in $Kt(H, R)$.

To get insight into the relative performances of different approaches and the properties of different techniques, we implemented the tableau calculi by encoding them into first-order logic and using the SPASS-YARRALUMLA system. SPASS-YARRALUMLA is a bottom-up model generator based on the SPASS theorem prover [13]. SPASS-YARRALUMLA emulates the behaviour of semantic labelled tableau provers [2]. We used four forms of blocking available in SPASS-YARRALUMLA: (i) sound ancestor blocking; (ii) unrestricted blocking; (iii) sound ancestor blocking on non-disjoint worlds; and (iv) sound anywhere blocking on non-disjoint worlds. The encodings of the tableau calculi are implemented as an extension of the `m12dfg` tool used for the empirical evaluation of the axiomatic translation principle in [6].

The results of the experiments with implementations of the tableau calculi with SPASS-YARRALUMLA showed, on the whole, the encoding based on only propagation rules fared best for all forms of blocking tested. Similarly good results were obtained for the encodings involving a mixture of propagation and axiomatic rules based on the rules for Axioms A and P . For satisfiable problems the encodings based on only structural rules fared well, too, in two cases giving best results. In terms of blocking, for all encodings unrestricted blocking was most expensive on unsatisfiable problems. Sound ancestor blocking produced best results for all encodings on both satisfiable and unsatisfiable problems.

We also tested SPASS in auto mode, which uses a form of ordered resolution with dynamic selection, on the encodings. As expected many timeouts were observed for encodings based on correspondence properties, as these do not provide decision procedures. For all other encodings the performances were very close for unsatisfiable problems. It is interesting how much faster these performances were than the best performances for SPASS-YARRALUMLA. For unsatisfiable problems, tableau-like approaches need to construct a complete derivation tree in which every formula is grounded, and this is generally larger than the non-ground clause set derived with ordered resolution.

Though the focus has been on $Kt(H, R)$, the techniques and ideas used in this research are of general nature and provide a useful methodology for developing practical decision procedures for modal logics. Some aspects are completely rou-

tine. The main aspect for which creativity is required and is specific to $Kt(H, R)$ is the development of the tableau calculi based on propagation or axiomatic rules and the axiomatic translation to the guarded fragment. Here the contribution of the paper has been to extend the ideas of the axiomatic translation principle from [6]. Key is finding effective refinements and showing completeness and termination, which is in general non-trivial and will not always be possible. All in all, because of the ubiquity of modal logics, we believe this kind of systematic research of decidability, proof theory, refinements and relative efficiency is widely applicable and useful, and should be extended to more logics, more types of tableau approaches, other deduction approaches, different provers and more problem sets.

Full details can be found in [7].

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