

On dual tableau-based decision procedures for relational fragments

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Abstract. We give a brief survey of the main results on dual tableau-based decision procedures for fragments of the logic of binary relations and we outline our current work in the design of relational decision procedures for logics for order-of-magnitude reasoning. Then we shortly hint at some further research plans on other relational decision procedures.

1 Introduction

In the last decades, the relational representation of many non-classical propositional logics has been studied and analyzed in a systematic way [20]. A homogeneous relational framework grounded on the logic of binary relations $RL(\mathbf{1})$, presented in [19] and called *relational dual tableau*, turned out to be an effective logical means to represent in a modular way the three fundamental parts of a formal proof system, namely its syntax, semantics, and deduction system. Proof systems based on the logic of relations have been proposed for many non-classical logics such as modal and intuitionistic logics, relevant and many-valued logics, etc. Relational proof systems have also been defined for reasoning in logics of information and data analysis, for reasoning about time and space, for fuzzy set- and rough set-based reasoning, for order-of-magnitude and qualitative reasoning, for dynamic reasoning about programs, etc.

Formulae of $RL(\mathbf{1})$ have the form xRy , where x, y are individual variables and R is a relational term of $RL(\mathbf{1})$. We recall that the set of relational terms of $RL(\mathbf{1})$ is the smallest set (with respect to inclusion) built from relational variables and constants (considering also the constant $\mathbf{1}$) with the operators ‘ $\cap$ ’, ‘ $\cup$ ’, ‘ $;$ ’ (binary) and ‘ $\neg$ ’, ‘ $^{-1}$ ’ (unary).

The possibility of representing non-classical logics in $RL(\mathbf{1})$ relies on the fact that once the Kripke-style semantics of the logic under consideration is known, formulae can be treated as relations. In particular, since in Kripke-style semantics formulae are interpreted as collections of objects, in their relational representation they are seen as right ideal relations. In the case of binary relations this means that $(R ; \mathbf{1}) = R$ is satisfied, where ‘ $;$ ’ is the composition operation on binary relations and ‘ $\mathbf{1}$ ’ is the universal relation.

One of the most useful features of the relational methodology is that, given a logic with a relational formalization, it is possible to construct its relational dual tableau in a systematic and modular way. On the other hand, using a uniform formalism to represent different logics could allow one to detect similarities among the logics which might be hidden by other formalisms.

Though the relational logic $\text{RL}(\mathbf{1})$ is undecidable, it contains several decidable fragments. In many cases, however, dual tableau proof systems are not by themselves decision procedures for decidable fragments of $\text{RL}(\mathbf{1})$. This is mainly due to the way decomposition and specific rules are defined and to the strategy of proof construction.

Over the years, great efforts have been spent in the construction of dual tableau proof systems for various logics known to be decidable; little care has been taken, however, in the design of dual tableau-based decision procedures for them. On the other hand, it is well known that when a proof system is designed and implemented, it is important to have decision procedures for decidable logics. In [11], for instance, an optimized relational dual tableau for $\text{RL}(\mathbf{1})$, based on Binary Decision Graphs, has been implemented; however, such an implementation turns out not to be effective for decidable fragments.

In this paper we give a short survey of the main results on dual tableau-based decision procedures for fragments of the logic of binary relations. In particular, we outline a deterministic dual tableau-based decision procedure RDT_{RL} , presented in [7], for a class of multimodal logics whose accessibility relations may satisfy the properties of reflexivity, transitivity, and heredity. Next, we hint at our current work of designing a dual tableau decision procedure in the flavour of RDT_{RL} for a logic for order-of-magnitude reasoning, called OMR_N . We recall that logics for order-of-magnitude reasoning are non-classical logics used for qualitative reasoning. In general they are multimodal logics over a strict linear order with some additional accessibility relations satisfying certain constraints. Finally, we briefly hint at some further research plans in the ambit of relational decision procedures and draw our conclusions.

2 Dual tableau-based decision procedures for fragments of the logic of binary relations

In [20], relational dual tableau-based decision procedures have been constructed for fragments of RL corresponding to well-known classes of first-order formulae in prenex normal form, namely the class of existential formulae ($\exists^*$), of universal formulae ($\forall^*$), and of formulae with universal quantifiers followed by existential quantifiers ($\forall^*\exists^*$). A dual tableau decision procedure for each of these fragments has been obtained from the general dual tableau system defined in [20] by simply restricting the applicability of the (;)-rule only in case (a) the variable used in the (;)-decomposition occurs on the current node of the dual tableau, (b) the application of the (;)-rule produces some new formulae not already occurring on the current node, and (c) no other rule is applicable.

In [18], a relational dual tableau decision procedure has been provided for the relational logic corresponding to the modal logic $\mathbf{K}$. The system was constructed in the framework of the original methodology of relational proof systems, determined only by axioms and inference rules, without any external techniques. Its main feature is *uniqueness*, i.e., given a formula, the system generates in a deterministic way only one proof tree for it. Thus, the dual tableau presented in [18] is not only a base for an algorithm verifying validity of a formula, but is *itself* a deterministic decision procedure. Furthermore, due to a special dual clause representation of modal formulae, the system is very simple as it consists only of two rules. The system in [18] has been implemented in Prolog in [17].

Fragments of $\mathbf{RL(1)}$, characterized by some restrictions in terms of type $(R; S)$, have been presented in [8, 9] and, more recently, in [10]. Specifically, in [8], two fragments of $\mathbf{RL(1)}$, respectively called $(r; _)$ and $(\cup, \cap; _)$, have been introduced. The $(r; _)$ -fragment is the collection of the $\mathbf{RL(1)}$ -formulae xPy in which the composition operator ‘ $;$ ’ can occur only in the following restricted way. For each subterm of P of the form $R; S$, R must belong to a designated nonempty proper subset $\mathbb{RV}_1$ of $\mathbb{RV}$, whereas S can involve all the relational operators used to construct $\mathbf{RL(1)}$ -formulae, with the exception of the converse operator ‘ $^{-1}$ ’. In the relational interpretation of logics, the elements of $\mathbb{RV}_1$ are meant to denote accessibility relations (resp., roles) in modal (resp., description) logics. This fragment allows one to express the multimodal logic $\mathbf{K}$ and, therefore, also the description logic $\mathcal{ALC}$ [2, 1]. The $(\cup, \cap; _)$ -fragment is an extension of the $(r; _)$ -fragment in which the constraints on the composition operator ‘ $;$ ’ are more relaxed. In particular, the first argument in a term of type $R; S$ of the $(\cup, \cap; _)$ -fragment can be any term constructed from the relational variables of a proper nonempty subset of $\mathbb{RV}$, say again $\mathbb{RV}_1$, by applying only the operators ‘ $\cup$ ’ and ‘ $\cap$ ’. The restriction on the second argument is the same of the $(r; _)$ -fragment. The $(\cup, \cap; _)$ -fragment can express the description logic $\mathcal{ALC}(\cup, \cap)$ [2]. For each of these fragments, a dual tableau-based decision procedure is given differing from the general proof system described in [20] in the definition of the $(;)$ -rule and in some strictness conditions on the applicability of the rules. (These, in fact, are the two sources of non termination in dual tableaux for formulae of such fragments.) The versions of $(;)$ -rule introduced in [8] select the individual variables to decompose the $(;)$ -formula by analyzing the literals occurring on the current node of the dual tableau. No blocking condition is needed to guarantee termination.

In [9] a dual tableau-based decision procedure is defined for a fragment, called $(\cup, \cap, ^{-1}; _)$, of $\mathbf{RL(1)}$ which extends the $(\cup, \cap; _)$ -fragment by permitting a more liberal application of the composition operator ‘ $;$ ’. In fact, the first argument, R , in a term of type $R; S$ is allowed to be any term obtained from the relational variables of a proper nonempty subset of $\mathbb{RV}$, by applying the operators ‘ $^{-1}$ ’, ‘ $\cup$ ’, and ‘ $\cap$ ’. The $(\cup, \cap, ^{-1}; _)$ -fragment of $\mathbf{RL(1)}$ expresses the description logic $\mathcal{ALCI}(\cup, \cap)$ [2]. The dual tableau decision procedure for this fragment differs from the ones introduced in [8] in the definition of the $(;)$ -rule modified to handle more expressive $(;)$ -formulae containing the ‘ $^{-1}$ ’ operator.

We recently proved that decidability holds also for the extended versions of the $(r; \cdot)$ -fragment and of the $(\cup, \cap; \cdot)$ -fragment where the distinction between variables in $\mathbb{RV}_1$ and in $\mathbb{RV} \setminus \mathbb{RV}_1$ is dropped and, therefore, the same relational variable can occur both on the left and on the right hand side of compositional terms. Then, in [10], a fragment of $\mathbf{RL}(\mathbf{1})$, called $(\{\mathbf{1}, \cup, \cap\}; \cdot)$, has been introduced admitting a restricted form of composition where the left subterm, R , of any term of type $(R; S)$ is allowed to be either the constant $\mathbf{1}$ or any term constructed from the relational variables by applying only the operators of relational intersection and union. Similarly, terms of type $(R; \mathbf{1})$ are admitted only if R is a Boolean term containing neither the complement operator nor the constant ‘ $\mathbf{1}$ ’. Decidability of the $(\{\mathbf{1}, \cup, \cap\}; \cdot)$ -fragment is proved by defining a dual tableau-based decision procedure where a suitable blocking mechanism has been introduced and rules for compositional and complemented compositional formulae have been appropriately modified to deal with the constant $\mathbf{1}$ while preserving termination. The fragment $(\{\mathbf{1}, \cup, \cap\}; \cdot)$ is a first result towards the use of entailment inside relational dual tableau-based decision procedures. We recall that relational entailment allows one to deal with properties of relational constants and of relational variables in dual tableau proofs without adding any specific rule to the basic set of decomposition rules. The $(\{\mathbf{1}, \cup, \cap\}; \cdot)$ -fragment allows one also to express some simple forms of inclusion between relations.

The paper [13] presents relational decision procedures in dual tableaux style for a class of relational logics admitting just one relational constant R with the properties of reflexivity, transitivity, and heredity (i.e., if xRy and xPz , then yPz , for all atomic relations P). These results are extended to multimodal logics in [7], where a class $\mathcal{RDL}^m$ of fragments of the logic $\mathbf{RL}$, allowing an unbounded number of relational constants which may enjoy the properties of reflexivity, transitivity, and heredity, is defined, and a dual tableau decision procedure for logics belonging to $\mathcal{RDL}^m$, called $\mathbf{RDT}_{\mathbf{RL}}$, is introduced.

The $\mathbf{RDT}_{\mathbf{RL}}$ system is constructed along the lines of the dual tableau methodology described in [20]. It consists of decomposition rules to analyze the structure of the formulae to be proved, of specific rules to deal with properties that can be enjoyed by the relational constants occurring in the formulae to be proved (such as reflexivity, transitivity, and heredity), and of axiomatic sets which specify the closure conditions. Each rule and axiom is applied in a deterministic way without resorting to external tools. Determinism of the decision procedure is guaranteed by enforcing an order of application of the rules obtained by associating with each rule a specific formula of the current node, called *pivot formula*, and by introducing a suitable order on the set of relational formulae. An important feature of the dual tableau procedure is a rule to handle the relational composition operator, that permits to decompose in a single step compositional formulae and negative compositional formulae sharing the same left object variable. In Table 1 we report the decomposition rule (**glob**;) to handle compositional and negative-compositional formulae sharing the same left object variable, and in Table 2 the specific rules (**ref**_{*i*}), (**tran**_{*i*}), and (**her**_{*i*}) to deal with relational constants R_i satisfying the properties of reflexivity, transitivity, and heredity, respectively. The

RDT_{RL} proof system is correct, complete, and terminating. Termination of the procedure is guaranteed by introducing some restrictions in rule applications to prevent infinite loops.

The class $\mathcal{RDL}^m$ can express several multimodal logics whose accessibility relations are reflexive, transitive, and satisfy the heredity property. Among them we recall the multimodal logic with reflexive and transitive frames and the description logic $\mathcal{ALC}_{R^+}$ [21].

Table 1. Decomposition rule for compositional formulae in RDT_{RL} .

$(\text{glob } ;) \frac{X \cup Y \cup \bigcup_{i \in I_\Phi} \{z_n(-(R_i ; S_{q_i}))z_0, z_n(R_i ; T_{j_i})z_0 : q_i \in Q_i, j_i \in J_i\}}{X \cup \bigcup_{i \in I_\Phi} \{z_{n_{q_i}}(-S_{q_i})z_0, z_{n_{q_i}}T_{j_i}z_0 : q_i \in Q_i, j_i \in J_i\}}$

where:

- $n \geq 1$ and Y is the set of literals with left variable z_n occurring in the current node;
- I_Φ is the set of indices of constants of $\mathbb{RC}$ occurring in the current node Φ ;
- for all $T \in \mathbb{RT}_{\mathcal{RDL}^m}$, $z_n T z_0 \notin X$ (the only formulae in the premise that are neither compositional nor negative-compositional and have z_n as left variable are in Y);
- $Q = \bigcup_{i \in I_\Phi} Q_i$ and $J = \bigcup_{i \in I_\Phi} J_i$ are sets of indices such that $Q_i \neq \emptyset$, for some $i \in I_\Phi$ (by this condition, if in the current node there is a formula $z_n(R_i ; T_{j_i})z_0$, for some $i \in I_\Phi$, and no formula of type $z_n(-(R_i ; S_{q_i}))z_0$ occurs in the current node, then $z_n(R_i ; T_{j_i})z_0$ cannot be decomposed anymore and therefore it is not repeated in the successive nodes of the dual tableau);
- $S_{q_i}, T_{j_i} \in \mathbb{RT}_{\mathcal{RDL}^m}$, for all $q_i \in Q_i, j_i \in J_i$, with $i \in I_\Phi$;
- the set $N = \{n_{q_i} : q_i \in Q_i, i \in I_\Phi\}$ satisfies the following conditions:
 - the elements of N are consecutive natural numbers,
 - $\min(N) = k+1$, where k is the largest number such that z_k occurs in the premise,
 - for all $n_{q_i}, n_{q'_{i'}} \in N$, we have $n_{q_i} < n_{q'_{i'}}$ if and only if $\langle R_i, S_{q_i} \rangle < \langle R_{i'}, S_{q'_{i'}} \rangle$;
- the pivot of $(\text{glob } ;)$ is the formula $z_n(-(R_i ; S_{q_i}))z_0$ with the minimal pair $\langle R_i, S_{q_i} \rangle$.

3 Current and future developments in qualitative reasoning and related settings

Qualitative reasoning (QR) is an area of Artificial Intelligence especially useful for handling situations in which purely numeric methods are too complex or the available knowledge is vague or incomplete. QR methods are alternatives to quantitative ones. Although quantitative approaches usually provide precise answers to precise questions and assumptions, their applicability is very limited. Indeed, most problems of the physical world cannot be solved by purely quantitative methods. Moreover, even if numeric methods are used in solving a problem, often we have to neglect some factors as unimportant in the given context. Qualitative representation make only as many distinctions (qualitative classes) as necessary to identify objects and to set essential relationships between them. Thus, the main aim of QR is to design methods for dealing with

Table 2. Reflexivity, transitivity, and heredity rules for RDT_{RL} .

Reflexivity rule:

$$(\text{ref}_i) \frac{X \cup \{z_n(R_i^s; T)z_0\}}{X \cup \{z_n(R_i^s; T)z_0\} \cup \{z_n(R_i^j; T)z_0 : j \in \{0, \dots, s-1\}\}},$$

where $n, s \geq 1$ and $T \in \mathbb{RT}_{\mathcal{RDL}^m}$ is either a non-compositional term or a compositional term $(R_j; T')$, with $j \neq i$, and $z_n(R_i^t; T) \notin X$, for all $t > s$.

Transitivity rule:

$$(\text{tran}_i) \frac{X \cup \{z_n(R_i; T)z_0\}}{X \cup \{z_n(R_i; T)z_0\} \cup \{z_n(R_i^2; T)z_0\}},$$

where $n \geq 1$ and $T \in \mathbb{RT}_{\mathcal{RDL}^m}$ is either a non-compositional term or a compositional term $(R_j; T')$, with $j \neq i$.

Heredity rule:

$$(\text{her}_i) \frac{X \cup \{z_n(-(R_i; T))z_0\} \cup \{z_n(-\mathbf{p}_j)z_0 : j \in J_\Phi\}}{X \cup \{z_n(-(R_i^s; T))z_0\} \cup \{z_n(-\mathbf{p}_j)z_0 : j \in J_\Phi\} \cup \{z_n(R_i; (-\mathbf{p}_j))z_0 : j \in J_\Phi\}}$$

where: $n \geq 1$, $T \in \mathbb{RT}_{\mathcal{RDL}^m}$, and $z_n(-\mathbf{p}_j)z_0 \notin X$, for any $\mathbf{p}_j \in \mathbb{RV}$.

The pivot of the rule (her_i) is $z_n(-(R_i; T))z_0$.

commonsense knowledge without using numerical computation. Over the years, QR has developed various techniques of qualitative representation and proved to have a wide variety of potential applications.

An approach in QR concerned with the analysis of physical systems in terms of relative magnitudes is order-of-magnitude reasoning (OMR). Note that in purely qualitative reasoning the lack of quantitative information may often lead to ambiguity. One of the aims of OMR is to avoid this problem. Thus, in a sense, order-of-magnitude methods are situated midway between numerical and qualitative formalisms. OMR approaches are usually based on a family of binary order-of-magnitude relations, among which are comparability, negligibility, and closeness relations. Such a relative order-of-magnitude paradigm is rich enough to capture many types of commonsense reasoning. For instance, as presented in [5], using OMR we may represent and reason in a qualitative way about the behaviour of a device to control automatically the temperature in a building. In this representation, temperature is assumed to be either VERY HOT, HOT, OK, COLD, or VERY COLD. Next, some general and specific axioms about the behaviour of the system are introduced. The axioms are expressed in terms of order-of-magnitude relations, for example: ‘If the temperature is OK and it is incremented in some value with respect to which the actual value is negligible, then humidification or extra humidification system must operate’. A detailed example of order-of-magnitude representation is discussed in [5].

The simplest logic for OMR, OMR_N for short, has been introduced in [5]. It is a multimodal logic with five propositional constants and four necessity operators determined by a strict linear order $<$, the negligibility relation N , and

their converses. The intended meaning of the negligibility relation defined on the set of real numbers is as follows: 0 is negligible with respect to any real number and each number from the qualitative class of sufficiently small numbers is negligible with respect to any number from the qualitative class of sufficiently large numbers. Models of OMR_N are Kripke structures with a strict linear order, the negligibility relation, and five landmarks which represent propositional constants and divide the universe of a model into seven qualitative classes. Sound and complete relational proof systems for OMR_N have been presented in [16]. However, the system presented in [16] is not a decision procedure for OMR_N , even though the logic is decidable, as proved in [12].

Recently, it has been shown that the logic $\text{OMR}_N(C)$, which is the extension of OMR_N with the comparability relation (see [4]), is also decidable, as shown in [3]. Its relational proof system, which, however, is not a decision procedure, can be found in [20, Chapter 15].

Thus, a natural question is how to modify the relational dual tableau systems for the logics OMR_N and $\text{OMR}_N(C)$ into their relational decision procedures. Our first step is the construction of a decision procedure along the lines of the one presented in [7] for a multimodal logic over a strict linear order. Since there are relational decision procedures for logics with transitive relations, to design a relational decision procedure for logics over a strict linear order we need, in particular, a rule for the property of connectedness (for all x and y , either xRy or yRx or $x = y$) and irreflexivity (for all x , not xRx). Recall that these properties are not definable by a standard modal formula, while they are definable by a relational formula. This fact can be very useful for the construction of a desired decision procedure.

In [6], a more complex logic for order-of-magnitude reasoning has been introduced. It includes modal operators determined by the relations of negligibility, non-closeness, and distance. In particular, a simplified version of it, called OMR_D , has been considered. This is the logic determined by a class of Kripke frames with a strict linear order and a distance relation. As it turns out, the relations of non-closeness and negligibility are definable in OMR_D . It has been shown that the logic OMR_D is sound and complete over Kripke models with a strict total order and distance relation.

A sound and complete dual tableau system for the relational logic associated with OMR_D has been constructed in [15]. Such a system can be used to verify whether a formula is OMR_D -provable. However, it does not provide any decision procedure for OMR_D , as it may generate infinite trees. In fact, the decision problem for the logic OMR_D is still an open question, which we plan to study. In case of a positive solution, we also intend to construct a dual tableau-based decision procedure for the relational logic OMR_D .

In addition, we intend to construct relational decision procedures for fragments of multimodal logics whose accessibility relations satisfy properties such as seriality, symmetry, euclideanity, partial functionality, and weak density. We have already designed specific rules for each of these properties to be integrated in the procedure presented in [7]. However, in order to preserve termination of

the procedure, we still have to define suitable conditions and restrictions on their application.

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