

Substructural Cut Elimination

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Abstract

In the paper “Structural Cut Elimination”, Pfenning gives a proof of the admissibility of cut for intuitionistic and classical logic. The proof is remarkable in that it does not rely on difficult termination metrics, but rather a nested lexicographical induction on the structure of formulas and derivations. A crucial requirement for this proof to go through is that contraction is not an inference rule of the system. Because of this, it is necessary to change the inference rules so that contraction becomes an admissible rule rather than an inference rule. This change also requires that weakening is admissible, hence it is not directly applicable to logics in which only contraction is admissible (e.g. relevance logic).

We present a sequent calculus which admits a unified structural cut elimination proof that encompasses Intuitionistic MALL and its affine, strict and intuitionistic extensions. A nice feature of the calculus is that, for instance, moving from linear to strict logic is as simple as allowing the presence of a rule corresponding to contraction.

1 Introduction

The most important theorem about any sequent calculus is that of *cut elimination*. This theorem usually ensures that the sequent calculus is consistent and internally sound. Additionally, the admissibility of the cut rule usually implies a host of useful inversion properties.

There are many different proofs of the admissibility of cut, but among these one of the most straightforward and elegant is the *structural cut admissibility* proof due to Pfenning [Pfe00]. For instance, given the following rules — which correspond to the implicational fragment of Intuitionistic MALL [Gir87],

$$\frac{}{a \Rightarrow a} \text{init} \quad \frac{\Delta \Rightarrow A \quad \Delta', B \Rightarrow C}{\Delta, \Delta', A \multimap B \Rightarrow C} \multimap \text{L} \quad \frac{\Delta, A \Rightarrow B}{\Delta \Rightarrow A \multimap B} \multimap \text{R}$$

the cut rule is

$$\frac{\Delta \Rightarrow A \quad \Delta', A \Rightarrow C}{\Delta, \Delta' \Rightarrow C} \text{cut}$$

and the statement of *cut admissibility* is

Theorem. *Given derivations $\mathcal{D} :: \Delta \Rightarrow A$ and $\mathcal{E} :: \Delta', A \Rightarrow C$ there exists a derivation $\mathcal{F} :: \Delta, \Delta' \Rightarrow C$.*

Pfenning’s proof of this theorem proceeds by a lexicographic induction on the structure of the cut formula A and the two input derivations $\mathcal{D}$ and $\mathcal{E}$. To see how this induction ordering naturally arises, we will consider two cases:

- First, we consider a case where the cut formula is not the principal formula of the last inference rule in $\mathcal{E}$:

$$\frac{\mathcal{D} \quad \frac{\Delta', A, B \Rightarrow C}{\Delta', A \Rightarrow B \multimap C} \multimap \text{R}}{\Delta, \Delta' \Rightarrow B \multimap C} \text{cut}$$

Here, the thick rule for cut is a reminder that this is the rule we are going to prove admissible. In particular, this means that both $\mathcal{D}$ and $\mathcal{E}$ are assumed to be cut-free derivations.

In this case, we argue as follows:

$$\begin{array}{l} \Delta, \Delta', B \Rightarrow C \\ \Delta, \Delta' \Rightarrow B \multimap C \end{array} \quad \begin{array}{l} \text{by cut on } A, \mathcal{D}, \mathcal{E}. \\ \text{by } \multimap \text{R}. \end{array}$$

Thus, in this case we appeal to the induction hypothesis on a subderivation, but keeping the cut formula the same. Note that although we are appealing to the induction hypothesis, we will write it as an appeal to the rule we are proving admissible.

- In the principal case:

$$\frac{\frac{\mathcal{D} \quad \Delta, A \Rightarrow B}{\Delta \Rightarrow A \multimap B} \multimap \text{R} \quad \frac{\mathcal{E}_1 \quad \Delta_1 \Rightarrow A \quad \mathcal{E}_2 \quad \Delta_2, B \Rightarrow C}{\Delta_1, \Delta_2, A \multimap B \Rightarrow C} \multimap \text{L}}{\Delta, \Delta_1, \Delta_2 \Rightarrow C} \text{cut}$$

we reason as follows:

$$\begin{array}{l} \mathcal{D}' :: \Delta, \Delta_1 \Rightarrow B \\ \Delta, \Delta_1, \Delta_2 \Rightarrow C \end{array} \quad \begin{array}{l} \text{by cut on } A, \mathcal{E}_1, \mathcal{D}. \\ \text{by cut on } B, \mathcal{D}', \mathcal{E}_2. \end{array}$$

Here, although the derivation $\mathcal{D}'$ may be much larger than the two input derivations, the second use of the cut rule is justified by the fact that the cut formula itself got strictly smaller.

If we add the contraction rule

$$\frac{\Delta, A, A \Rightarrow C}{\Delta, A \Rightarrow C} \text{contract}$$

to the above system, the above proof does not go through. When cutting against an contraction:

$$\frac{\mathcal{D} \quad \Delta \Rightarrow A \quad \frac{\mathcal{E} \quad \Delta', A, A \Rightarrow C}{\Delta, A \Rightarrow C} \text{contract}}{\Delta, \Delta' \Rightarrow C} \text{cut}$$

we would like to reason as follows:

$$\begin{array}{l} \mathcal{E}' :: \Delta, \Delta', A \Rightarrow C \\ \Delta, \Delta, \Delta' \Rightarrow C \\ \Delta, \Delta' \Rightarrow C \end{array} \quad \begin{array}{l} \text{by cut on } A, \mathcal{D}, \mathcal{E}. \\ \text{by cut on } A, \mathcal{D}, \mathcal{E}'. (*) \\ \text{by contract (repeated)}. \end{array}$$

But the line marked (*) is not a smaller cut according to our induction ordering.

For intuitionistic logic, Pfenning circumvents this problem by baking the rule of contraction into every single rule of inference in the system. Thus, the left implication rule would look as follows:

$$\frac{\Delta, A \supset B \Rightarrow A \quad \Delta, A \supset B, B \Rightarrow C}{\Delta, A \supset B \Rightarrow C} \supset \text{L}$$

With the above rule, contraction becomes an admissible rule of inference. Note, however, that the above rule requires *weakening* to be present in the system as well. Because of this, the above solution cannot directly be applied to logics that do not admit weakening.

2 Cut Elimination

From a proof search perspective, the rule of contraction is problematic, as it requires the search procedure to guess in advance whether a formula will be required several times or not. When weakening is allowed, this is not a problem as one can simply make extra copies whenever assumptions are consumed or the context is split. Any superfluous copies can then be discarded by means of weakening.

Rather than require a proof search procedure to be prescient, we do the obvious thing: whenever an assumption is used, it is not *forgotten*, but merely put aside so that it may potentially be used again further up in the proof. With this in mind, it makes sense to divide the assumptions into two contexts corresponding to whether they have already been used or not. Thus, the judgments of our system have two contexts on the left hand side of the sequent arrow. The difference between these contexts can be summed up as follows:

- The context Δ is a multiset of assumptions that *must* be used in the derivation of the sequent.
- The context Γ is a multiset of assumptions that *may* be used in the derivation of the sequent.

The sequent arrows themselves have an annotation to show which structural rules are present in the derivation of the sequent in question. These annotations are elements of the free semilattice over the set $\{\mathbf{w}, \mathbf{c}\}$, in other words they must be one of either $\cdot$, $\mathbf{w}$, $\mathbf{c}$, or $\mathbf{w} + \mathbf{c}$. We use $+$ for the join operation on this structure.

The syntax of formulas is standard:

$$A, B ::= a \mid A \& B \mid A \otimes B \mid A \multimap B \mid A \oplus B \mid \top \mid \mathbf{0} \mid \mathbf{1}$$

The inference rules can be seen in Figure 1.

To see how these inference rules arise, consider the behaviour of the $\multimap$ L rule in *strict* logic (i.e. with contraction but not weakening)¹. A first attempt at a suitable $\multimap$ L rule would be the following

$$\frac{\Gamma; \Delta_1 \xrightarrow{x} A \quad \Gamma; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap \text{L}$$

This rule is too strict, however, as we might want to use the assumption $A \multimap B$ again. To enable this, we copy this formula into both of the “may use” contexts:

$$\frac{\Gamma, A \multimap B; \Delta_1 \xrightarrow{x} A \quad \Gamma, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap \text{L}$$

This is better, but still too restrictive; if both subderivations want to use a hypothesis in Δ_1 , say, the proof can’t go through. To fix this, we additionally give each of the premises a copy of the “must use” context of the other premise. Thus, the final rule looks as follows:

$$\frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap \text{L}$$

Using the promote rule, we can then move assumptions back into the “must use” context. Thus, depending on the annotation of the sequent, we get the following four systems:

¹One could also refer to this logic as either *relevance* logic or *relevant* logic, but as these names refer to a host of related logics, we prefer to use the word *strict* for the logic arising from disallowing the structural rule of weakening.

$$\begin{array}{c}
\frac{}{\Gamma; a \xrightarrow{x} a} \text{init} \quad \frac{}{\Gamma; \Delta \xrightarrow{x} \top} \top R \quad \frac{}{\Gamma; \Delta, \mathbf{0} \xrightarrow{x} C} \mathbf{0}L \quad \frac{}{\Gamma; \cdot \xrightarrow{x} \mathbf{1}} \mathbf{1}R \\
\\
\frac{\Gamma, A_1 \& A_2; \Delta, A_i \xrightarrow{x} C}{\Gamma; \Delta, A_1 \& A_2 \xrightarrow{x} C} \&L_i \quad \frac{\Gamma; \Delta \xrightarrow{x} A \quad \Gamma; \Delta \xrightarrow{y} B}{\Gamma; \Delta \xrightarrow{x+y} A \& B} \&R \quad \frac{\Gamma, \mathbf{1}; \Delta \xrightarrow{x} C}{\Gamma; \Delta, \mathbf{1} \xrightarrow{x} C} \mathbf{1}L \\
\\
\frac{\Gamma, A \otimes B; \Delta, A, B \xrightarrow{x} C}{\Gamma; \Delta, A \otimes B \xrightarrow{x} C} \otimes L \quad \frac{\Gamma, \Delta_2; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} B}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} A \otimes B} \otimes R \\
\\
\frac{\Gamma, A \oplus B; \Delta, A \xrightarrow{x} C \quad \Gamma, A \oplus B; \Delta, B \xrightarrow{y} C}{\Gamma; \Delta, A \oplus B \xrightarrow{x+y} C} \oplus L \quad \frac{\Gamma; \Delta \xrightarrow{x} A_i}{\Gamma; \Delta \xrightarrow{x} A_1 \oplus A_2} \oplus R_i \\
\\
\frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap L \quad \frac{\Gamma; \Delta, A \xrightarrow{x} B}{\Gamma; \Delta \xrightarrow{x} A \multimap B} \multimap R \\
\\
\frac{\Gamma; \Delta, A \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x+\mathbf{c}} C} \text{promote} \quad \frac{\Gamma, A; \Delta \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathbf{w}} C} \text{demote}
\end{array}$$

Figure 1: Sequent calculus for IMALL with promote and demote rules.

- If the label is $\cdot$, then so is the label on all sequents in the derivation. Thus, the only way to get rid of assumptions is to decompose them using the corresponding inference rule, and there is no way to reintroduce assumptions from the “may use” context into the “must use” context. The resulting system is thus simply IMALL.
- If the label is $\mathbf{w}$, we can apply the demote rule freely, thus allowing us to discard assumptions at will. This corresponds to an affine extension of IMALL.
- If the label is $\mathbf{c}$, we can move assumptions back into the “must use” context, thus corresponding to using an assumption more than once. We are not able, however, to discard assumptions — every assumption must be used at least once, corresponding to a strict extension of IMALL.
- Finally, if the label is $\mathbf{w} + \mathbf{c}$, we can move assumptions freely between the two contexts. This essentially collapses the two contexts into one, and the resulting system is equivalent to LJ.

As an example of a derivation in this system, here is a proof of $a \multimap (a \otimes a)$ using contraction:

$$\frac{\frac{\frac{}{\cdot; a \xrightarrow{\cdot} a} \text{init}}{\cdot; a \xrightarrow{\cdot} a} \text{init} \quad \frac{\frac{}{\cdot; a \xrightarrow{\cdot} a} \text{init}}{a; \cdot \xrightarrow{\mathbf{c}} a} \text{promote}}{\cdot; a \xrightarrow{\mathbf{c}} a \otimes a} \otimes R \quad \frac{}{\cdot; \cdot \xrightarrow{\mathbf{c}} a \multimap (a \otimes a)} \multimap R$$

Note that although the context Γ above is not the usual persistent context seen in e.g. DILL [BP96], it admits weakening nonetheless:

Theorem (Weakening). *The following rule is admissible:*

$$\frac{\Gamma; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C} \text{weaken}$$

Furthermore, it is strongly admissible, in the sense that it does not change the shape of the resulting derivation.

Proof. By structural induction on the derivation of $\Gamma; \Delta \xrightarrow{x} C$. $\square$

Conversely, contraction is also admissible. Because of the dyadic context and the rules for moving across this context, proving this requires a mutual induction between three related forms of contraction:

Theorem (Contraction). *The following inference rules are admissible*

$$\frac{\Gamma, A, A; \Delta \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x} C} \text{ucontract} \quad \frac{\Gamma, A; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x} C} \text{contract} \quad \frac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{pcontract}$$

Proof. By structural induction on the given derivations. We will show a few of the more interesting cases:

- pcontract, principal case for $\multimap$ L:

$$\begin{array}{c} \mathcal{D} \qquad \qquad \mathcal{E} \\ \frac{\Gamma, \Delta_2, A \multimap B; \Delta_1, A \multimap B \xrightarrow{x} A \quad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B, A \multimap B \xrightarrow{x+y} C} \multimap \text{L} \\ \hline \Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y+\mathfrak{c}} C \text{ pcontract} \\ \hline \Gamma, \Delta_2; \Delta_1, A \multimap B \xrightarrow{x} A \qquad \text{by contract on } \mathcal{D}. \\ \mathcal{D}' :: \Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x+\mathfrak{c}} A \qquad \text{by promote.} \\ \Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y+\mathfrak{c}} C \qquad \text{by } \multimap \text{L on } \mathcal{D}', \mathcal{E}. \end{array}$$

- contract, principal case for promote:

$$\begin{array}{c} \mathcal{D} \\ \frac{\Gamma; \Delta, A, A \xrightarrow{x} C}{\Gamma, A; \Delta, A \xrightarrow{x+\mathfrak{c}} C} \text{promote} \\ \hline \Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C \text{ contract} \\ \hline \Gamma; \Delta, A \xrightarrow{x+\mathfrak{c}} C \qquad \text{by pcontract on } \mathcal{D}. \end{array}$$

$\square$

With contraction and weakening in place, we can now prove the central cut admissibility result. Again, because we have two contexts, we must have two cuts corresponding to which context the cut formula is in. The second cut rule, ucut, is a curious generalisation of the cut rule for the persistent context one sees in presentations of ILL.

Theorem (Cut admissibility). *The following inference rules are admissible*

$$\frac{\Gamma; \Delta_1 \xrightarrow{x} A \quad \Gamma; \Delta_2, A \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} C} \text{cut} \quad \frac{\Gamma; \Delta_1 \xrightarrow{x} A \quad \Gamma, A; \Delta_2 \xrightarrow{y} C}{\Gamma, \Delta_1; \Delta_2 \xrightarrow{x+y} C} \text{ucut}$$

Proof. By structural induction on the cut formula and the given derivations. Again, we show some of the more interesting cases:

- Principal cut for $\multimap$:

$$\frac{\frac{\mathcal{D} \quad \Gamma; \Delta, A \xrightarrow{x} B}{\Gamma; \Delta \xrightarrow{x} A \multimap B} \multimap R \quad \frac{\frac{\mathcal{E}_1 \quad \Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{y} A \quad \mathcal{E}_2 \quad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{z} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{y+z} C} \multimap L}{\Gamma; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C} \text{cut}$$

Let $\mathcal{D}'$ be the entire first premise of this cut, i.e. the proof of $\Gamma; \Delta \xrightarrow{x} A \multimap B$. We now reason as follows (with implicit appeals to the weakening lemma):

$$\begin{array}{ll} \mathcal{E}'_1 :: \Gamma, \Delta_2, \Delta; \Delta_1 \xrightarrow{x+y} A & \text{by ucut on } A \multimap B, \mathcal{D}', \mathcal{E}_1. \\ \mathcal{E}'_2 :: \Gamma, \Delta_1, \Delta; \Delta_2, B \xrightarrow{x+z} C & \text{by ucut on } A \multimap B, \mathcal{D}', \mathcal{E}_2. \\ \mathcal{F} :: \Gamma, \Delta_2, \Delta; \Delta, \Delta_1 \xrightarrow{x+y} B & \text{by cut on } A, \mathcal{E}'_1, \mathcal{D}. \\ \Gamma, \Delta, \Delta_1, \Delta_2; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by cut on } B, \mathcal{F}, \mathcal{E}'_2. \\ \Gamma; \Delta, \Delta_1, \Delta_2 \xrightarrow{x+y+z} C & \text{by contract (repeated).} \end{array}$$

- Cutting against a promotion:

$$\begin{array}{ll} \mathcal{E} & \\ \frac{\frac{\mathcal{D} \quad \Gamma; \Delta', A \xrightarrow{y} C}{\Gamma; \Delta \xrightarrow{x} A \quad \Gamma, A; \Delta' \xrightarrow{y+c} C} \text{promote}}{\Gamma, \Delta; \Delta' \xrightarrow{x+y+c} C} \text{ucut} & \\ \Gamma; \Delta', \Delta \xrightarrow{x+y} C & \text{by cut on } A, \mathcal{D}, \mathcal{E}. \\ \Gamma, \Delta; \Delta' \xrightarrow{x+y+c} C & \text{by promote (repeated).} \end{array}$$

□

3 Soundness and Completeness

In this section, we will show that the sequent calculus introduced in the previous section is sound and complete with regard to the standard presentation of IMALL and the structural rules of contraction and weakening. Again, we annotate the sequent arrow to keep track of which structural rules are used. The inference rules can be seen in Figure 2.

Theorem (Completeness). *If $\Delta \xRightarrow{x} C$ then $\Gamma; \Delta \xrightarrow{x} C$ for any Γ .*

Proof. By induction over the structure of $\Delta \xRightarrow{x} C$. In the case of contract:

$$\begin{array}{l} \mathcal{D} \\ \frac{\Delta, A, A \xRightarrow{x} C}{\Delta, A \xrightarrow{x+c} C} \text{contract} \\ \frac{\Delta, A \xrightarrow{x+c} C}{\Gamma; \Delta, A \xrightarrow{x+c} C} \text{to show} \end{array}$$

we appeal to the contraction lemma:

$$\begin{array}{ll} \Gamma; \Delta, A, A \xrightarrow{x} C & \text{by the induction hypothesis on } \mathcal{D}. \\ \Gamma; \Delta, A \xrightarrow{x+c} C & \text{by pcontract.} \end{array}$$

□

$$\begin{array}{c}
\frac{}{a \xRightarrow{x} a} \text{init} \quad \frac{}{\Delta \xRightarrow{x} \top} \top R \quad \frac{}{\Delta, \mathbf{0} \xRightarrow{x} C} \mathbf{0} L \quad \frac{}{\cdot \xRightarrow{x} \mathbf{1}} \mathbf{1} R \\
\frac{\Delta, A_i \xRightarrow{x} C}{\Delta, A_1 \& A_2 \xRightarrow{x} C} \& L_i \quad \frac{\Delta \xRightarrow{x} A \quad \Delta \xRightarrow{y} B}{\Delta \xRightarrow{x+y} A \& B} \& R \quad \frac{\Delta \xRightarrow{x} C}{\Delta, \mathbf{1} \xRightarrow{x} C} \mathbf{1} L \\
\frac{\Delta, A, B \xRightarrow{x} C}{\Delta, A \otimes B \xRightarrow{x} C} \otimes L \quad \frac{\Delta_1 \xRightarrow{x} A \quad \Delta_2 \xRightarrow{y} B}{\Delta_1, \Delta_2 \xRightarrow{x+y} A \otimes B} \otimes R \\
\frac{\Delta, A \xRightarrow{x} C \quad \Delta, B \xRightarrow{y} C}{\Delta, A \oplus B \xRightarrow{x+y} C} \oplus L \quad \frac{\Delta \xRightarrow{x} A_i}{\Delta \xRightarrow{x} A_1 \oplus A_2} \oplus R_i \\
\frac{\Delta_1 \xRightarrow{x} A \quad \Delta_2, B \xRightarrow{y} C}{\Delta_1, \Delta_2, A \multimap B \xRightarrow{x+y} C} \multimap L \quad \frac{\Delta, A \xRightarrow{x} B}{\Delta \xRightarrow{x} A \multimap B} \multimap R \\
\frac{\Delta, A, A \xRightarrow{x} C}{\Delta, A \xRightarrow{x+\mathbf{c}} C} \text{contract} \quad \frac{\Delta \xRightarrow{x} C}{\Delta, A \xRightarrow{x+\mathbf{w}} C} \text{weaken}
\end{array}$$

Figure 2: Sequent calculus for IMALL with tracked contraction and weakening.

Stating the soundness theorem is, alas, not quite as straightforward. A problem in this regard is that we need to account for the fact that Γ can be weakened and thus can contain assumptions that are irrelevant to the given derivation. To account for this, we state the theorem as follows:

Theorem (Soundness). *If $\Gamma; \Delta \xrightarrow{x} C$ then $\Gamma', \Delta \xRightarrow{x} C$ for some $\Gamma' \subseteq \Gamma$. If $\mathbf{c} \not\leq x$ i.e. there are no occurrences of the promote rule in the given derivation, then Γ' may be chosen to be empty.*

Proof. By induction over the structure of $\Gamma; \Delta \xrightarrow{x} C$. We show here the case for $\multimap L$. The remaining cases are similar.

- Case $\multimap L$:

$$\begin{array}{c}
\mathcal{D} \qquad \mathcal{E} \\
\frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1, A \multimap B; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xRightarrow{x+y} C} \multimap L \\
\hline
\Gamma', \Delta_1, \Delta_2, A \multimap B \xRightarrow{x+y} C \text{ to show}
\end{array}$$

where $\Gamma' \subseteq \Gamma$. If $\mathbf{c} \not\leq x + y$ then $\mathbf{c} \not\leq x$ and $\mathbf{c} \not\leq y$, and we reason as follows:

$$\begin{array}{ll}
\mathcal{D}' :: \Delta_1 \xRightarrow{x} A & \text{by the induction hypothesis on } \mathcal{D}. \\
\mathcal{E}' :: \Delta_2, B \xRightarrow{y} C & \text{by the induction hypothesis on } \mathcal{E}. \\
\Delta_1, \Delta_2, A \multimap B \xRightarrow{x+y} C & \text{by } \multimap L \text{ on } \mathcal{D}', \mathcal{E}'.
\end{array}$$

If $\mathbf{c} \leq x + y$, we have $x + y = x + y + \mathbf{c}$, and reason as follows:

$$\begin{array}{ll}
\mathcal{D}' :: \Gamma_1, \Delta_1 \xRightarrow{x} A & \text{by the induction hypothesis on } \mathcal{D}. \\
\mathcal{E}' :: \Gamma_2, \Delta_2, B \xRightarrow{y} C & \text{by the induction hypothesis on } \mathcal{E}.
\end{array}$$

$$\begin{array}{ll}
\Gamma_1, \Gamma_2, \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C & \text{by } \multimap \text{L on } \mathcal{D}', \mathcal{E}' \\
\Gamma', \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y+c} C & \text{by contract (repeated).}
\end{array}$$

where $\Gamma' = (\Gamma_1 \uplus \Gamma_2) \setminus (\Delta_1 \uplus \Delta_2 \uplus \{A \multimap B\})$.

□

4 Variations

Non-contracting Rules. It is well known that for some of the left rules of the sequent calculus it is unnecessary to copy the principal formula to all the premises of said rule. This is the case for e.g. the $\otimes$ L and $\multimap$ L rules, where the following more restricted rules have the same expressive power as their unrestricted counterparts in Figure 1:

$$\frac{\Gamma; \Delta, A, B \xrightarrow{x} C}{\Gamma; \Delta, A \otimes B \xrightarrow{x} C} \otimes \text{L} \quad \frac{\Gamma, \Delta_2, A \multimap B; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1; \Delta_2, B \xrightarrow{y} C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \xrightarrow{x+y} C} \multimap \text{L}$$

What is more surprising is that the left rules for $\&$ also admit this kind of non-contracting presentation:

$$\frac{\Gamma, A; \Delta, B \xrightarrow{x} C}{\Gamma; \Delta, A \& B \xrightarrow{x} C} \& \text{L}_1 \quad \frac{\Gamma, B; \Delta, A \xrightarrow{x} C}{\Gamma; \Delta, A \& B \xrightarrow{x} C} \& \text{L}_2$$

The benefit of this seems limited, however, as using these rules instead greatly complicates the proof of the admissibility of contraction.

Asymmetric splitting. One curious aspect of the system presented is the fact that although it is sound and complete at the level of provability, in general there will be many more derivations in the system with the promote rule than in the system with the contract rule. For example, there is essentially one proof of $a \xRightarrow{c} (a \otimes a)$:

$$\frac{\frac{\frac{}{a \xRightarrow{\cdot} a} \text{init} \quad \frac{}{a \xRightarrow{\cdot} a} \text{init}}{a, a \xRightarrow{\cdot} a \otimes a} \otimes \text{R}}{a \xRightarrow{c} a \otimes a} \text{contract}$$

In the present system, however, there are two derivations, corresponding to which branch of the $\otimes$ R rule gets the principal copy of a :

$$\frac{\frac{}{a \xRightarrow{\cdot} a} \text{init} \quad \frac{\frac{}{a \xRightarrow{\cdot} a} \text{init} \quad \frac{}{a \xRightarrow{\cdot} a} \text{promote}}{a; a \xRightarrow{\cdot} a} \otimes \text{R}}{a \xRightarrow{c} a \otimes a} \quad \frac{\frac{}{a \xRightarrow{\cdot} a} \text{init} \quad \frac{\frac{}{a \xRightarrow{\cdot} a} \text{promote} \quad \frac{}{a \xRightarrow{\cdot} a} \text{init}}{a; a \xRightarrow{\cdot} a} \otimes \text{R}}{a \xRightarrow{c} a \otimes a}$$

One way of fixing this is to make the $\otimes$ R rule, and indeed any rule that splits the context, asymmetric in how it handles the splitting.

First, observe that any instance of the promote rule can be permuted down until just above the rule that introduces the principal connective into the “may use” context. Assuming this has been done, we note that the following reduction is always permitted:

$$\frac{\frac{\frac{\vdots}{\Gamma, \Delta_2; \Delta_1, A \xrightarrow{x} C} \text{P} \quad \frac{\vdots}{\Gamma, \Delta_1; \Delta_2, A \xrightarrow{y} D}}{\Gamma, \Delta_2, A; \Delta_1 \xrightarrow{x+y+c} C} \otimes \text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+c} C \otimes D} \quad \sim \quad \frac{\frac{\frac{\vdots}{\Gamma, \Delta_2; \Delta_1, A \xrightarrow{x} C} \text{P} \quad \frac{\frac{\vdots}{\Gamma, \Delta_1; \Delta_2, A \xrightarrow{y} D} \text{P}}{\Gamma, \Delta_1, A; \Delta_2 \xrightarrow{y+c} D} \otimes \text{R}}{\Gamma; \Delta_1, \Delta_2, A \xrightarrow{x+y+c} C \otimes D} \otimes \text{R}$$

where p is used as shorthand for the promote rule. Assuming the above reduction is applied maximally, anything left over in Δ_2 in the first premise can simply be strengthened away, as it is not promoted back into the linear context. In other words, the following *asymmetric* version of the $\otimes R$ rule is sufficient:

$$\frac{\Gamma; \Delta_1 \xrightarrow{x} A \quad \Gamma, \Delta_1; \Delta_2 \xrightarrow{y} B}{\Gamma; \Delta_1, \Delta_2 \xrightarrow{x+y} A \otimes B} \otimes R$$

Adopting this change does not complicate the proof of the admissibility of cut.

5 Future Work

Focusing To highlight the similarities between the present system and Pfenning’s presentation, we have chosen to present it as an *unfocused* sequent calculus, even though Linear Logic yields itself well to a focused presentation [And92]. A consequence of this choice is that every left rule moves a copy of the principal formula into the “may use” context in all premises. This creates not only an extra copy of each formula that is decomposed on the left, but also a copy of many of the subformulas of this formula. In a focused system, these many copies are avoided by making just a single copy when initiating the focusing phase. Thus, the rule for promotion would be changed into the following form, where the brackets indicate the formula A is under focus:

$$\frac{\Gamma, A; \Delta, [A] \xrightarrow{x} C}{\Gamma, A; \Delta \xrightarrow{x+\epsilon} C} \text{promote}$$

It seems likely that the present system allows for a focused presentation, and that the cut admissibility argument can be extended to this focused presentation in the style of Simmons [Sim14].

Exponentials In this paper, we chose to use IMALL as the base system. An obvious next step would be to add exponentials to the system, to see whether the cut elimination result extends to full ILL. We conjecture that the $!$ modality can be implemented using the following inference rules:

$$\frac{\Gamma; \cdot \xrightarrow{x} A}{\Gamma; \cdot \xrightarrow{x} !A} !R \quad \frac{\Gamma, !A; \Delta, A \xrightarrow{x} C}{\Gamma, !A; \Delta \xrightarrow{x} C} !L_1 \quad \frac{\Gamma, !A; \Delta \xrightarrow{x} C}{\Gamma; \Delta, !A \xrightarrow{x} C} !L_2$$

Alternatively, one could add a third context representing the truly persistent assumptions.

Structural Modalities The way our system tracks the structural rules as part of the sequent suggests that it might be possible to internalise this as a modality. Thus, one might consider rules of the following form:

$$\frac{\Gamma; \Delta \xrightarrow{x} A}{\Gamma; \Delta \xrightarrow{x+s} [s]A} [s]R \quad \frac{\Gamma; \Delta \xrightarrow{x-s} A}{\Gamma; \Delta \xrightarrow{x} [\neg s]A} [\neg s]R$$

Here, $[s]A$ would indicate that A is provable in the presence of the structural rules in s , whereas $[\neg s]A$ conversely indicates that A is provable in the *absence* of these rules. It is not clear, however, whether there are sensible left rules to accompany the above rules in such a way that cut elimination still holds.

6 Conclusion

We have presented a sequent calculus for Intuitionistic MALL and the affine, strict, and intuitionistic extensions arising from the addition of contraction and weakening to this system. The structural cut admissibility argument we present works uniformly for all combinations of the structural rules. Moreover, proving the soundness and completeness of this system with regard to the usual presentation can be done by a straightforward induction.

The main novelty of this paper is the presentation of a structural cut admissibility argument for strict logic. In contrast, the admissibility of cut for linear, affine and intuitionistic logic is comparatively well known. This is not to say that cut elimination for logics with contraction but not weakening (e.g. relevance logic) has not been attempted before. In [DR02], Dunn and Restall present a cut elimination argument that proceeds along the lines of Gentzen’s original proof. Because of this, the termination metric is much more elaborate.

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