

Multiplicative Decomposition of Behaviours in Ludics*

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Abstract

Ludics is a theory that may be considered, at first glance, as a Brouwer-Heyting-Kolmogorov interpretation of Logic as a formula is denoted by the set of its proofs. More primitively, Ludics is a theory of interaction that models (a variant of) second-order multiplicative-additive Linear Logic. A formula is denoted by a set of objects called a behaviour, a proof by an object that satisfies some criteria. Our aim is to analyze the structure of behaviours in order to better understand and refine the usual notion of formula or type. More precisely, we study properties that guarantee a behaviour to be multiplicatively decomposable.

1 Introduction

While developments of Proof Theory related to computation mainly rest on the correspondence between Logic and Typed Lambda Calculus, Ludics, a logical theory developed by J.-Y. Girard around 2000 [7], may be considered as being closer to untyped lambda-calculus. By the way, formulas are usually in Logic defined at first, then proofs of formulas, and finally cuts between proofs may be considered. In Ludics, the ontological reversal is radical: the primitive notion is *interaction* that abstracts cut-elimination. Designs, as abstract formal proofs, are *supports of interaction*. For playing such a role, designs are made of potential interaction paths. Following this reading, designs may be understood as *strategies* of game semantics, except that, while strategies are defined according to a game already set, designs are a priori untyped.

Once the consubstantial notions of designs and interaction are set, usual logical concepts may be recovered. The denotation of a formula is a set of designs closed with respect to interaction, such a closed set is called a *behaviour*. Operations on behaviours rest on their set-theoretical expression. In particular, the linear connectives may be expressed by means of quite standard constructions. One of the most important results of the seminal paper [7] is a full completeness result for Ludics as a semantics of the second order multiplicative additive Linear Logic (MALL₂). Indeed it is possible to characterize proofs of a MALL₂-formula as particular designs belonging to the behaviour associated to this formula.

Subsequent works extended Ludics and corrected its initial limitations, in particular its strict linearity and sequentiality. Ludics nets were developed as a game model for concurrent interaction by F. Maurel and C. Faggian [4] and more thoroughly analyzed by P.-L. Curien and C. Faggian in [2]. K. Terui proposed in [9] a reformulation of Ludics by means of a notion of *c*-design, more suitable from a computational point of view. In particular *c*-designs may not be linear. M. Basaldella and C. Faggian [1] extended also Ludics to be able to deal with exponentials. However the study of Ludics must be continued as one of the main interests of

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Ludics is its wide expressivity. The structure of behaviours goes beyond what is needed for a model of Linear Logic hence may be of interest for designing program types, and some applications of Ludics use its *untyped* feature (e.g., [8]). Our long term aim is to define a grammar that may generate the set of behaviours, with connectives logically meaningful. This paper is a first step in that direction. We focus on the characterization of behaviours corresponding to linear formulas. We define a notion of *regular* behaviour and we show that such behaviours are exactly the ones decomposable by means of the linear multiplicative connectives.

This analysis is closely linked to the notion of *incarnation*. Incarnation is an important and even central notion, specific to Ludics. This is the core of the relation between the untyped feature of objects and closure operations. In particular, incarnation allows us to obtain internal completeness results for operations associated with linear connectives¹. This means that one knows how to decompose designs in the incarnation of $\mathbf{A} \otimes \mathbf{B}$ or $\mathbf{A} \oplus \mathbf{B}$ with respect to $\mathbf{A}$ and $\mathbf{B}$. In this paper we give a characterization of behaviours that may be *generated* from the linear connective $\otimes$. This result relies also on properties of incarnation.

In section 2, we focus on the notion of *path*: a path is a sequence of actions that may be followed during an interaction. We recall that a design in the incarnation of a behaviour may be defined by *visitable paths*, i.e., paths that are effectively visited by interaction with the dual of the behaviour. We introduce the operation of *shuffle* on paths: as in Game Semantics, interaction with a tensor of behaviours may be viewed as an interleaving of sequences of each behaviour. We make explicit in section 3 the relevance of visitable paths for describing linear connectives. In section 4, we define the notion of regular behaviours and we prove that regular behaviours correspond exactly to behaviours decomposable by multiplicative linear connectives.

2 Paths, Visitability, Incarnation

Main objects of Ludics² are *designs* together with a notion of *interaction* between them. As in Game Semantics, the basic steps of interaction are *actions* (moves). In Ludics, an action is either a special one called *daimon*, denoted $\star$ or a triple $(+/-, \xi, I)$ where the polarity $+$ or $-$ is relative to the point of view one adopts (one side or the other of the interaction); the *focus* ξ determines the address where this action may occur in an interaction. An action either terminates the interaction (if it is the daimon) or creates new addresses (denoted by means of the finite set I) on which the interaction may continue. The dynamics of the interaction consists in following two dual alternating sequences of actions, one in each design. Two kinds of sequences of actions, paths and chronicles, may equivalently be used to define designs. The latter is used by J.-Y. Girard to define a design as a clique of chronicles, the former is closer to the notion of *play* and is used in [1] as an alternate presentation of Ludics, making explicit the link between Ludics and Game Semantics. We use here the notion of paths that we defined in [6] to compute the incarnation of a behaviour generated by a set of designs without necessarily knowing the full behavior. In the following, we note a positive (resp. negative) action with a positive (resp. negative) subscript: κ^+ , κ'^+ (resp. κ^- , κ'^-), and without subscripts when we do not care of the polarity: κ , κ' .

2.1 Designs as Sets of Chronicles or Sets of Paths

The notions of paths and chronicles are closely linked. Both are alternating sequences of actions: the *view* of a path is a chronicle, a *shuffle* of chronicles is a path.

¹This latter result is one of the main ingredient of full completeness.

²The reader may find a thorough presentation of Ludics in [7].

Definition 1. • An **address** is a sequence of integers. A **base** β is a non-empty finite set of sequents: $\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n$ such that (i) each Δ_j is a finite set of addresses, (ii) at most one Γ_i may be empty and the other Γ_i contain each exactly one address, and (iii) each address appearing in β occurs only once.

- A sequence of actions s is **based on** β if each action of s is either justified³ by a previous action in s , or has its focus in β , or is a daimon which should be, in this case, the last action of s . An action κ of s is **initial** when the focus of κ is in β .
- Let s be a sequence of actions based on β , the **view** $\ulcorner s \urcorner$ is the subsequence of s such that:
 - $\ulcorner \epsilon \urcorner = \epsilon$ where ϵ is the empty sequence;
 - $\ulcorner w\kappa^+ \urcorner = \ulcorner w \urcorner \kappa^+$;
 - $\ulcorner w\kappa^- \urcorner = \ulcorner w_0 \urcorner \kappa^-$ where w_0 either is empty if κ^- is initial or is the prefix of w ending with the positive action which justifies κ^- .

Definition 2. A **path** p based on β is a finite alternating sequence of actions based on β such that

- A proper action is either justified by one of the previous actions in the sequence, or it is called **initial** with a focus in one of Γ_i (resp. Δ_i) if the action is negative (resp. positive).
- Let $w\kappa^+$ be a prefix of p , if κ^+ is not initial and is justified by κ^- then $\kappa^- \in \ulcorner w\kappa^+ \urcorner$. (There is no jump on positive action).
- Actions in p have distinct focuses.
- If present, a daimon ends the path. If it is the first action in p then one of Γ_i is empty.
- If there exists an empty Γ_i , then p is non-empty and begins either with $\boxtimes$ or with a positive action with a focus in Δ_i .

Chronicles are non-empty paths such that each non-initial negative action is justified by the immediately previous (positive) action. Then, by construction, views of paths are chronicles. More generally, if we consider the set p^* of all prefixes q of a path p , we obtain the set of chronicles $\ulcorner q \urcorner$ induced by p , this set is denoted $\ulcorner p^* \urcorner$. Conversely, it is possible to rebuild the path p from the set of chronicles $\ulcorner p^* \urcorner$. The relevant operation to build paths from chronicles is the operation of shuffle. The shuffle operation may more generally be defined on paths. The standard shuffle operation consists in interleaving sequences keeping each element and respecting order. We depart from this definition first by imposing that alternating polarities should also be satisfied, second by taking care of the daimon that should only appear at the end of a path. Being a path is preserved by the shuffle operation.

Definition 3. • Let p and q be two positive-ended negative paths on disjoint bases: β and γ , and such that at least one path does not end on a daimon. The **shuffle** of p and q , written $p \sqcup q$, is the set of sequences $p_1 q_1 \dots p_n q_n$, based on $\beta \cup \gamma$ such that:

- each sequence p_i and q_i is either empty or positive-ended,
- $p_1 \dots p_n = p$ and $q_1 \dots q_n = q$,
- if p_n ends with $\boxtimes$ then q_n is empty.
- The definition is extended to paths $p\kappa_1\boxtimes$ and $q\kappa_2\boxtimes$ where p and q are two positive-ended negative paths on disjoint bases: $p\kappa_1\boxtimes \sqcup q\kappa_2\boxtimes = (p\kappa_1\boxtimes \sqcup q) \cup (p \sqcup q\kappa_2\boxtimes)$
- The definition is extended to paths rp and rq where r is a positive-ended path and p and q are two positive-ended negative paths on disjoint bases: $rp \sqcup rq = r(p \sqcup q)$

³An action $\kappa = (+/-, \xi, i, J)$ is justified by an action $\kappa' = (-/+, \xi, I)$ when $i \in I$.

It is possible to build paths from a given set of chronicles, provided that these chronicles are pairwise coherent. Indeed, coherence ensures that, after a common positive-ended prefix, chronicles are made of negative paths either on disjoint bases or with first actions of same focus.

- Definition 4.** • Two chronicles $\mathbf{c}_1$ and $\mathbf{c}_2$ of same base are **coherent**, written $\mathbf{c}_1 \subset \mathbf{c}_2$, when the two following conditions are satisfied:
- Either one extends the other or they first differ on negative actions.
 - When they first differ on negative actions and these negative actions have distinct focuses then the focuses of following actions in $\mathbf{c}_1$ and $\mathbf{c}_2$ are pairwise distinct.
- Two paths p_1 and p_2 of same base are **coherent**, written $p_1 \subset p_2$, if $\ulcorner p_1^* \urcorner \cup \ulcorner p_2^* \urcorner$ is a clique with respect to $\subset$.
- A **design** of base β is a set of prefix-closed set of pairwise coherent chronicles of base β such that each chronicle without extension is positive-ended, and the set is non-empty if the base is positive.

We prove in [6] that a non-empty clique of non-empty paths may give rise to a design.⁴ Furthermore, when $\mathfrak{R}$ is a design, the closure by shuffle of $\mathfrak{R}$, denoted $\mathfrak{R}^\sqcup$ is a set of coherent paths (and we say that p is a positive-ended path of a design $\mathfrak{R}$ whenever p is in $\mathfrak{R}^\sqcup$). Hence that makes explicit the link between paths and chronicles of a design.

2.2 Duality and Reversible Paths, Interaction

- Definition 5.** Let p be a positive-ended alternating sequence.
- The **dual** of p (possibly empty) is the positive-ended alternating sequence of actions $\tilde{p}$ (possibly empty) such that⁵: If $p = w\mathfrak{x}$ then $\tilde{p} := \overline{w}$, otherwise $\tilde{p} := \overline{p}\mathfrak{x}$.
 - When p and $\tilde{p}$ are positive-ended paths, we say that p is **reversible**.

There exist paths such that their duals are not paths. Nevertheless, note that the dual of a chronicle is a path. A shuffle of two reversible paths may not be reversible, however we have the following property.

Lemma 1. Let p and q be two positive-ended paths and $r \in p \sqcup q$ be a path such that its dual $\tilde{r}$ is a path, then duals $\tilde{p}$ and $\tilde{q}$ are paths.

Proof. See the annex. □

Orthogonality is originally defined as an inductive process between two designs [7]. We give here an equivalent definition that emphasizes the role of path.

Definition 6. Two designs $\mathfrak{D}$ and $\mathfrak{E}$ are **orthogonal**, written $\mathfrak{D} \perp \mathfrak{E}$, if there exists a path p of $\mathfrak{D}$ such that $\tilde{p}$ is a path of $\mathfrak{E}$. Let $\mathbf{A}$ be a set of designs of same base, $\mathbf{A}^\perp = \{\mathfrak{E} ; \forall \mathfrak{D} \in \mathbf{A}, \mathfrak{D} \perp \mathfrak{E}\}$, $\mathbf{A}$ is a **behaviour** if $\mathbf{A} = \mathbf{A}^{\perp\perp}$.

The study of $\mathbf{A}$ may be restricted to the subset of its designs that are fully *visited* by interaction.

- Definition 7.** Let $\mathbf{A}$ be a behaviour, $\mathfrak{D}$ be a design in $\mathbf{A}$.
- Let p be a path of $\mathfrak{D}$, p is **visitable** in $\mathbf{A}$ if there exists a design $\mathfrak{R}$ in $\mathbf{A}^\perp$ such that $\tilde{p}$ is a path of $\mathfrak{R}$. We note $V_{\mathbf{A}}$ the set of visitable paths of $\mathbf{A}$.
 - The design $\mathfrak{D}$ is **incarnated** in the behaviour $\mathbf{A}$ if $\mathfrak{D}$ is minimal in $\mathbf{A}$ with respect to inclusion. The **incarnation** $|\mathbf{A}|$ of a behaviour $\mathbf{A}$ is the set of its incarnated designs.

⁴More precisely, it gives rise to a net of designs.

⁵The notation $\bar{\kappa}$ is simply $(\pm, \xi, I) = (\mp, \xi, I)$ and may be extended on sequences by $\bar{\epsilon} = \epsilon$ and $\overline{w\kappa} = \overline{w}\bar{\kappa}$.

It is worth noticing that $V_{\mathbf{A}} = \widetilde{V_{\mathbf{A}^\perp}}$. Furthermore, we proved in [6] that the design $\mathfrak{D}$ is an incarnated design of the behaviour $\mathbf{A}$ iff there is a maximal clique C of visitable paths of $\mathbf{A}$, such that $\ulcorner C \urcorner = \mathfrak{D}$. In particular, when $\mathbf{A}$ is a behaviour and p is a visitable path of $\mathbf{A}$, the design $\ulcorner p \urcorner^c$ belongs to $\mathbf{A}$. The construction $\mathfrak{D}^c$, in the more general case of a design $\mathfrak{D}$, is the “completion with respect to incarnation” of $\mathfrak{D}$, that is the design obtained from $\mathfrak{D}$ by adding chronicles $\mathfrak{c}(-, \xi, I)\mathfrak{x}$ whenever $\mathfrak{c} \in \mathfrak{D}$, I is a finite set of integers, $\mathfrak{c}(-, \xi, I)$ is a chronicle of $|\mathbf{A}|$ and $\mathfrak{c}(-, \xi, I) \notin \mathfrak{D}$.

Proposition 1. *Let $\mathfrak{D}$ be a design in a behaviour $\mathbf{A}$, consider a design $\mathfrak{C} \subset \mathfrak{D}$ then $\mathfrak{C}^c \in \mathbf{A}$.*

Proof. See the Annex. $\square$

3 Visitable paths and Linear connectives

Before establishing properties concerning visitability (in particular proposition 2 that determines which paths of a tensor of behaviours are visitable), we recall below the definition of the tensor $\otimes$ operation and its main properties. In the following we note $\mathfrak{D}\mathfrak{a}\mathfrak{i}$ the design consisting only of the action daimon: $\mathfrak{D}\mathfrak{a}\mathfrak{i} = \{\mathfrak{x}\}$.

- Definition 8.** • *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two positive designs of same base $\vdash \xi$:*
- *If $\mathfrak{A}$ or $\mathfrak{B}$ is $\mathfrak{D}\mathfrak{a}\mathfrak{i}$, then $\mathfrak{A} \otimes \mathfrak{B} = \mathfrak{D}\mathfrak{a}\mathfrak{i}$.*
 - *Otherwise $\mathfrak{A} = (+, \xi, I)\mathfrak{A}'$ and $\mathfrak{B} = (+, \xi, J)\mathfrak{B}'$: if $I \cap J = \emptyset$ then $\mathfrak{A} \otimes \mathfrak{B} = (+, \xi, I \cup J)(\mathfrak{A}' \cup \mathfrak{B}')$ otherwise $\mathfrak{A} \otimes \mathfrak{B}$ is not defined.*
- *Two behaviours $\mathbf{G}$ and $\mathbf{H}$ are alien if $\mathfrak{A} \otimes \mathfrak{B}$ is defined for all $\mathfrak{A} \in \mathbf{G}$ and $\mathfrak{B} \in \mathbf{H}$.*
 - *Let $\mathbf{G}$ and $\mathbf{H}$ be two positive alien behaviours, $\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{A} \otimes \mathfrak{B} ; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}^{\perp\perp}$*

Theorem 1 (internal completeness [7]). *Let $\mathbf{G}$ and $\mathbf{H}$ be two alien positive behaviours, $\mathbf{G} \otimes \mathbf{H} = \{\mathfrak{A} \otimes \mathfrak{B} ; \mathfrak{A} \in \mathbf{G}, \mathfrak{B} \in \mathbf{H}\}$.*

The first step toward a characterization of multiplicative behaviours consists in calculating which paths of a tensor are visitable. As the connective $\otimes$ models a kind of concurrency, it is natural to consider that a visitable path of a tensor of behaviours should be a reversible shuffle of visitable paths of the behaviours. When $\mathbf{A}$ and $\mathbf{B}$ are two alien behaviours, we note $\mathbf{A}_{[\mathbf{B}]}$ the set of designs $\{\mathfrak{D}\mathfrak{a}\mathfrak{i}\} \cup \{(+, \xi, I \cup J)\mathfrak{A} ; (+, \xi, I)\mathfrak{A} \in \mathbf{A}, (+, \xi, J)\mathfrak{B} \in \mathbf{B}\}$. It is quite immediate to prove that $\mathbf{A}_{[\mathbf{B}]}$ is a behaviour.

Proposition 2. *Let $\mathbf{A}$ and $\mathbf{B}$ be alien positive behaviours, then $V_{\mathbf{A} \otimes \mathbf{B}} = \{p ; p \text{ is reversible and } p \in V_{\mathbf{A}_{[\mathbf{B}]}} \sqcup V_{\mathbf{B}_{[\mathbf{A}]}}\}$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be alien positive behaviours with base $\vdash \xi$.

- Let $p \in V_{\mathbf{A} \otimes \mathbf{B}}$. As p is a visitable path, p is reversible. Furthermore there exist designs $\mathfrak{D} \in \mathbf{A} \otimes \mathbf{B}$ and $\mathfrak{C} \in (\mathbf{A} \otimes \mathbf{B})^\perp$ such that p is a path of $\mathfrak{D}$ and $\tilde{p}$ is a path of $\mathfrak{C}$. Using the independence property ([7], Th. 20), there exist designs $\mathfrak{D}_1 \in \mathbf{A}$ and $\mathfrak{D}_2 \in \mathbf{B}$ such that $\mathfrak{D} = \mathfrak{D}_1 \otimes \mathfrak{D}_2$. If $p = \mathfrak{x}$, then either $\mathfrak{D}_1$ or $\mathfrak{D}_2$ is $\mathfrak{D}\mathfrak{a}\mathfrak{i}$, hence either $V_{\mathbf{A}_{[\mathbf{B}]}}$ or $V_{\mathbf{B}_{[\mathbf{A}]}}$ is the set $\{\mathfrak{x}\}$, thus $V_{\mathbf{A}_{[\mathbf{B}]}} \sqcup V_{\mathbf{B}_{[\mathbf{A}]}} = \{\mathfrak{x}\}$. So let us consider the other cases, *i.e.*, designs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are distinct from $\mathfrak{D}\mathfrak{a}\mathfrak{i}$.
Let $(+, \xi, I)$ (resp. $(+, \xi, J)$) be the first action of $\mathfrak{D}_1$ (resp. $\mathfrak{D}_2$). Behaviours being alien, we have that $I \cap J = \emptyset$. Note that each action of p is either $(+, \xi, I \cup J)$, or an action

of design $\mathfrak{D}_1$ or an action of design $\mathfrak{D}_2$, or $\star$ (at most once), the three first cases being exclusive of one another. Without loss of generality, if $\star$ is present, we suppose that this action is part of $\mathfrak{D}_1$ and not of $\mathfrak{D}_2$.

Let $p_1 = (+, \xi, I)q_1$ (resp. $p_2 = (+, \xi, J)q_2$) where q_1 (resp. q_2) is the subsequence of p made of actions of $\mathfrak{D}_1$ (resp. $\mathfrak{D}_2$). Let $p = w\kappa^-\kappa^+w'$ with κ^+ be a non-initial positive action, remark that there exists a sequence w_0 and $\ulcorner w\kappa^-\kappa^+\urcorner = w_0\kappa^-\kappa^+$, furthermore this view is a chronicle, hence if κ^+ is in p_1 , so is κ^- . Thus a subsequence $\kappa^-\kappa^+$ in p is either a subsequence of p_1 or a subsequence of p_2 . As $\mathfrak{D}_1 \otimes \mathfrak{D}_2 \perp \mathfrak{E}$, it follows from the adjunction theorem ([7], Th. 14) that there exists a design $(\mathfrak{E})\mathfrak{D}_2$ independent from $\mathfrak{D}_1$ such that $\mathfrak{D}_1 \perp (\mathfrak{E})\mathfrak{D}_2$, thus $(\mathfrak{E})\mathfrak{D}_2 \in \mathbf{A}^\perp$. Similarly, $(\mathfrak{E})\mathfrak{D}_1 \in \mathbf{B}^\perp$. Furthermore, interaction being deterministic, we have that p_1 (resp. p_2) is a path of $\mathfrak{D}_1$ and that $\tilde{p}_1$ (resp. $\tilde{p}_2$) is a path of $(\mathfrak{E})\mathfrak{D}_2$, thus $p_1 \in V_{\mathbf{A}}$ and $p_2 \in V_{\mathbf{B}}$. Finally remark that $p \in (+, \xi, I \cup J)q_1 \sqcup (+, \xi, I \cup J)q_2$ (there is no change of paths on a positive action). Hence $p \in V_{\mathbf{A}[\mathbf{B}]} \sqcup V_{\mathbf{B}[\mathbf{A}]}$.

- Let $p \in V_{\mathbf{A}}$, $p' \in V_{\mathbf{B}}$. If one of p or p' is the daimon, then their shuffle is the daimon, hence in $V_{\mathbf{A} \otimes \mathbf{B}}$. Otherwise, $p = (+, \xi, I)p_1$ and $p' = (+, \xi, J)p'_1$. As $p \in V_{\mathbf{A}}$, there exist designs $\mathfrak{D} \in \mathbf{A}$ and $\mathfrak{E} \in \mathbf{A}^\perp$ such that p is a path of $\mathfrak{D}$ and $\tilde{p}$ is a path of $\mathfrak{E}$. Similarly there exist designs $\mathfrak{D}' \in \mathbf{B}$ and $\mathfrak{E}' \in \mathbf{B}^\perp$ such that p' is a path of $\mathfrak{D}'$ and $\tilde{p}'$ is a path of $\mathfrak{E}'$. Let q be a reversible path in $(+, \xi, I \cup J)p_1 \sqcup (+, \xi, I \cup J)p'_1$. We prove that q which is a path of $\mathfrak{D} \otimes \mathfrak{D}'$ is visitable in $\mathbf{A} \otimes \mathbf{B}$. Let us consider the completion of $\tilde{q}$, i.e., the design $\mathfrak{G} = \ulcorner \tilde{q} \urcorner^c$. We prove by contradiction that $\mathfrak{G} \in (\mathbf{A} \otimes \mathbf{B})^\perp$. Let $\mathfrak{F} \in \mathbf{A}$ and $\mathfrak{F}' \in \mathbf{B}$. Suppose that $\mathfrak{F} \otimes \mathfrak{F}' \not\perp \mathfrak{G}$. Let r be the longest path in $\mathfrak{G}$ such that $\tilde{r}$ is a path in $\mathfrak{F} \otimes \mathfrak{F}'$. Note that r cannot be empty and does not end with a daimon (otherwise $\mathfrak{G} \perp \mathfrak{F} \otimes \mathfrak{F}'$).
 - Either r ends with a negative action κ^- . The view $\ulcorner r \urcorner$ is a chronicle of $\mathfrak{G}$, hence there exists a (unique) positive action κ^+ such that $\ulcorner r \urcorner \kappa^+$ is a chronicle of $\mathfrak{G}$. By construction of $\mathfrak{G}$, $r\kappa^+$ is a path of $\ulcorner \tilde{q} \urcorner$. The action κ^+ cannot be the daimon otherwise $\mathfrak{G} \perp \mathfrak{F} \otimes \mathfrak{F}'$. Note that the action κ^+ appears only once in $\ulcorner \tilde{q} \urcorner$, hence also in $\tilde{q}$ as Ludics is linear and $\tilde{q}$ is a path. Thus $\overline{\kappa^+}$ appears only once in q . Without loss of generality we can suppose that $\overline{\kappa^+}$ appears once in p . Let $\bar{r}_1$ be the subsequence of $\bar{r}$ with actions in $\mathfrak{F}$. Remark that $\bar{r}_1$ is a path in $\ulcorner \tilde{p} \urcorner$. As $r\kappa^+$ is a path, κ^+ is justified in r (the base contains a unique address), thus $\overline{\kappa^+}$ is a negative action justified in $\bar{r}_1$ (as the justifier of $\overline{\kappa^+}$ should be in $\mathfrak{F}$). Hence also $\bar{r}_1\kappa^+$ is a path in $\ulcorner \tilde{p} \urcorner$. The design $\ulcorner \tilde{p} \urcorner^c$ is included in $\mathfrak{E}$, hence by proposition 1, its completion $\ulcorner \tilde{p} \urcorner^c$ belongs to $\mathbf{A}^\perp$. So $\ulcorner \tilde{p} \urcorner^c \perp \mathfrak{F}$, hence a contradiction.
 - Or r ends with a positive action κ^+ . The action κ^+ is not the daimon. The view $\ulcorner \tilde{r} \urcorner$ is a chronicle of $\mathfrak{F} \otimes \mathfrak{F}'$ that ends with the negative action κ^- , hence there exists a (unique) positive action κ'^+ such that $\ulcorner \tilde{r} \urcorner \kappa'^+$ is a chronicle of $\mathfrak{F} \otimes \mathfrak{F}'$. Let κ'^- be the justifier of κ^+ , i.e., the last action of the view $\ulcorner \tilde{r} \urcorner$. The action κ'^- appears in $\ulcorner r \urcorner$ so there exists a unique chronicle $\mathfrak{c} \in \ulcorner r \urcorner$ that ends with κ'^- hence, by construction of the design $\mathfrak{G}$, the chronicle $\mathfrak{c}\kappa'^+$ is in $\ulcorner r \urcorner$. Thus $r\kappa'^+$ is a path of $\mathfrak{G}$, contradiction.
 Thus $\mathfrak{F} \otimes \mathfrak{F}' \perp \mathfrak{G}$. Finally, by construction, q is a path in $\mathfrak{D} \otimes \mathfrak{D}'$ and $\tilde{q}$ is a path in $\mathfrak{G}$. So $q \in V_{\mathbf{A} \otimes \mathbf{B}}$.

□

4 Regular Behaviours

We prove in this section that *regularity* as it is defined below is the main concept for characterizing behaviours that are tensors. The only other necessary constraint for also characterizing

additivity concerns in some sense finiteness of the structure. This last constraint is quite easy and is treated in [5]. In a few words, a regular behaviour is a behaviour such that all ‘reasonable’ paths are visitable in this behaviour. ‘Reasonable’ means not only reversible but also made of actions that are present in the incarnation of a behaviour: such a path is called *regular for* this behaviour. Obviously, a visitable path for a behaviour is a regular path for this behaviour. We present an equivalent definition of a regular behaviour: a behaviour is regular if the set of its visitable paths is the closure by shuffle and dual of chronicles in its incarnation.

Definition 9. *Let $\mathbf{A}$ be a behaviour, let $\mathbf{c}$ a chronicle occurring in $|\mathbf{A}|$ and let κ be a proper action of $\mathbf{c}$, the sequence of justifiers of κ in $\mathbf{c}$ until an initial action is called a **trivial chronicle for $\mathbf{A}$** .*

It is worth noticing that a trivial chronicle $\mathbf{c}$ is a chronicle such that each action is justified by the immediate previous one in $\mathbf{c}$. We may think of a trivial chronicle as (an abstraction of) a branch in the decomposition of a formula.

Definition 10. *A reversible path p is **regular for a behaviour $\mathbf{A}$** if p has the same base as $\mathbf{A}$ and if, for each proper action κ of p , the subsequence of justifiers of κ in p is a trivial chronicle for $\mathbf{A}$.*

*A behaviour $\mathbf{A}$ is **regular** when each regular path for $\mathbf{A}$ is visitable in $\mathbf{A}$.*

Remark that a trivial chronicle for a behaviour $\mathbf{A}$ is regular for $\mathbf{A}$, hence a trivial chronicle for a behaviour is visitable when this behaviour is regular. Remark also that a visitable path p of a behaviour is regular for this behaviour: as the path p is visitable, p is in a design of the incarnation, hence for each proper action κ such that $p = p_1 \kappa p_2$, we have that $\lceil p_1 \kappa \rceil$ is a chronicle in a design of the incarnation, hence the subsequence of justifiers of κ in p is a trivial chronicle. However it should be noticed that a regular path is not necessarily visitable, it may even not be a path in a design of the behaviour.

Regularity is stable with respect to linear logical operations [5]. We consider below the two most important cases $\cdot^\perp$ and $\otimes$.

Lemma 2. *Let $\mathbf{A}$ be a regular behaviour, then the behaviour $\mathbf{A}^\perp$ is regular.*

Proof. As $\widetilde{V_{\mathbf{A}}} = V_{\mathbf{A}^\perp}$, an action in $|\mathbf{A}^\perp|$ is either $\boxtimes$ or the dual of an action in $|\mathbf{A}|$. Let $\mathbf{c}$ be a trivial chronicle for $\mathbf{A}^\perp$ then there exists a path q visitable in $\mathbf{A}^\perp$ such that $\mathbf{c}$ is a subsequence of q . The path $\widetilde{q}$ is visitable in $\mathbf{A}$ and the sequence $\bar{\mathbf{c}}$ of opposite polarities is a subsequence of $\widetilde{q}$ and made of actions of immediate justification, hence $\bar{\mathbf{c}}$ is a trivial chronicle for $\mathbf{A}$. Let p be a regular path for $\mathbf{A}^\perp$, from the previous remark it follows that $\widetilde{p}$ is a reversible path made of actions in trivial chronicles of $\mathbf{A}$ or the daimon. Hence $\widetilde{p}$ is regular for $\mathbf{A}$. Thus $\widetilde{p} \in V_{\mathbf{A}}$ thus $p \in \widetilde{V_{\mathbf{A}}} = V_{\mathbf{A}^\perp}$. $\square$

Lemma 3. *Let $\mathbf{A}$ and $\mathbf{B}$ be regular alien positive behaviours, then the behaviour $\mathbf{A} \otimes \mathbf{B}$ is regular.*

Proof. Let p be a path regular for $\mathbf{A} \otimes \mathbf{B}$. Obviously a positive-ended prefix of p is also a path regular for $\mathbf{A} \otimes \mathbf{B}$. The path p is of the form $\kappa_0^+ \kappa_1^- \kappa_1^+ \dots \kappa_n^- \kappa_n^+$. We prove that $p \in V_{\mathbf{A} \otimes \mathbf{B}}$ by induction on n .

- Note that κ_0^+ is necessarily the first action in a design of the incarnation (as κ_0^+ is a regular path), hence we get the result when $n = 0$.

- Suppose the property satisfied for n , and $p = p_1 \kappa_{n+1}^- \kappa_{n+1}^+$. By induction hypothesis $p_1 \in V_{\mathbf{A} \otimes \mathbf{B}}$. It follows from proposition 2 that there exist paths $a \in V_{\mathbf{A}}$ and $b \in V_{\mathbf{B}}$ such that $p_1 \in a \sqcup b$. Note that for each positive action κ^+ such that $p_1' \kappa^+$ is a prefix of p_1 , if κ^+ occurs in a then $\lceil p_1' \rceil \kappa^+$ is a subsequence of a except the first action whose ramification is an extension of the first action of a (it is a consequence of the definitions of shuffle and view operations). Furthermore κ_{n+1}^- and κ_{n+1}^+ are actions in the incarnation of $\mathbf{A} \otimes \mathbf{B}$ hence in visitable paths of $\mathbf{A}_{[\mathbf{B}]}$ or $\mathbf{B}_{[\mathbf{A}]}$. Without loss of generality, we can suppose that κ_{n+1}^- occurs in a visitable path of $\mathbf{A}_{[\mathbf{B}]}$. Let κ^+ be the justifier of κ_{n+1}^- in p and $p_1' \kappa^+$ be the prefix of p ending with κ^+ . We have that κ^+ is in a or is an extension of the first action of a as κ_{n+1}^- is in $\mathbf{A}_{[\mathbf{B}]}$. Note that $\lceil p \rceil = \lceil p_1' \rceil \kappa^+ \kappa_{n+1}^- \kappa_{n+1}^+$ is a chronicle. Hence the justifier of κ_{n+1}^+ is in a . Thus κ_{n+1}^+ occurs in $\mathbf{A}$ as the behaviours $\mathbf{A}$ and $\mathbf{B}$ are alien. It follows that κ_{n+1}^+ occurs in the incarnation of $\mathbf{A}$ (the incarnation of the tensor is the tensor of the incarnations).

From the previous elements we have also that $a \kappa_{n+1}^- \kappa_{n+1}^+$ is a path. Let us remark that $p \in a \kappa_{n+1}^- \kappa_{n+1}^+ \sqcup b$ and p is reversible. Hence by lemma 5 we have in particular that $a \kappa_{n+1}^- \kappa_{n+1}^+$ is a reversible path, hence is a regular path for $\mathbf{A}$. It follows that $a \kappa_{n+1}^- \kappa_{n+1}^+$ is visitable in $\mathbf{A}$ as $\mathbf{A}$ is a regular behaviour. Hence by proposition 2, $p \in V_{\mathbf{A} \otimes \mathbf{B}}$. □

We have the converse property except when one of the behaviours is the behaviour $\mathbf{0}_\xi$ for some base $\vdash \xi$. In fact, $\mathbf{0}_\xi$ is the behaviour reduced to the design $\mathfrak{D}\mathbf{ai}$. It is immediate to prove that $\mathbf{0}_\xi^{\perp\perp} = \mathbf{0}_\xi$ and that for all behaviour $\mathbf{A}$ of base disjoint from ξ , $\mathbf{A} \otimes \mathbf{0}_\xi = \mathbf{0}_\xi[\mathbf{A}]$. Furthermore $\mathbf{0}_\xi$ is regular hence $\mathbf{A} \otimes \mathbf{0}_\xi$ is regular even if $\mathbf{A}$ is not regular. In the following we omit the base so we simply note $\mathbf{0}$.

Lemma 4. *Let $\mathbf{A}$ and $\mathbf{B}$ be alien positive behaviours distinct from $\mathbf{0}$, if $\mathbf{A} \otimes \mathbf{B}$ is regular then $\mathbf{A}$ and $\mathbf{B}$ are regular.*

Proof. Let p be a regular path for $\mathbf{A}$, we can define a path p' extending the first action of p by the ramification of a first action of a design in $\mathbf{B}$. We have that p' is a regular path for $\mathbf{A}_{[\mathbf{B}]}$. Note that a regular path for $\mathbf{A}_{[\mathbf{B}]}$ is also a regular path for $\mathbf{A} \otimes \mathbf{B}$. Thus, as $\mathbf{A} \otimes \mathbf{B}$ is regular, $p \in V_{\mathbf{A} \otimes \mathbf{B}}$. Hence p is a shuffle of a visitable path in $\mathbf{A}$ and a visitable path in $\mathbf{B}$. As behaviours $\mathbf{A}$ and $\mathbf{B}$ are alien, they have no action in common except perhaps the daimon. Thus p' is a visitable path in $\mathbf{A}_{[\mathbf{B}]}$. It follows that $p \in V_{\mathbf{A}}$. So $\mathbf{A}$ is regular. Similarly, $\mathbf{B}$ is regular. □

The following proposition gives a more constructive presentation of regularity.

Definition 11. *Let $\mathbf{A}$ be a behaviour. $\mathbf{A}$ is called **shuffle-regular** if all positive-ended chronicles of designs of $|\mathbf{A}|$ and $|\mathbf{A}^\perp|$ are visitable and $V_{\mathbf{A}}$ and $V_{\mathbf{A}^\perp}$ are stable by reversible shuffle (that is, all reversible paths in designs of $|\mathbf{A}|$ and $|\mathbf{A}^\perp|$ are visitable).*

Proposition 3. *Let $\mathbf{A}$ be a behaviour of simple base, i.e., either $\vdash \xi$ or $\xi \vdash$. $\mathbf{A}$ is regular iff $\mathbf{A}$ is shuffle-regular.*

Proof. • Suppose that $\mathbf{A}$ is regular.

Let c be a positive-ended chronicle of a design $\mathfrak{D}$ belonging to $|\mathbf{A}|$. A chronicle is reversible and c is in the incarnation hence c is a regular path for $\mathbf{A}$ and then c is visitable.

Let p and q be two visitable paths of $\mathbf{A}$ and let r be a reversible path belonging to $p \sqcup q$.

Remark that one may extract from r sequences of immediate justification for each action

in r , and these sequences are either in p or q , thus are trivial chronicles. It follows that r is regular for $\mathbf{A}$ hence visitable in $\mathbf{A}$.

Finally it follows from lemma 2 and the same reasoning as before that all positive-ended chronicles of designs of $|\mathbf{A}^\perp|$ are visitable and $V_{\mathbf{A}^\perp}$ is stable by reversible shuffle.

- Suppose that $\mathbf{A}$ is shuffle-regular. Let p be a path, in the following we set n_p the number of positive actions of p justified each by a negative action that justifies at least two positive actions in p . In other words, n_p “measures” the distance to a trivial chronicle.
 - We first show that if a positive-ended chronicle $\mathfrak{c}$ is a trivial chronicle for $\mathbf{A}$ then $\mathfrak{c}$ is visitable. By definition of a trivial chronicle, there is a design $\mathfrak{D} \in |\mathbf{A}|$ and a chronicle $\mathfrak{d} \in \mathfrak{D}$ such that $\mathfrak{c}$ is a subsequence of $\mathfrak{d}$: all actions of $\mathfrak{c}$ occur in $\mathfrak{d}$ in the same order and $\mathfrak{c}$ and $\mathfrak{d}$ have the same positive last action. Let us notice that, in a trivial chronicle, each negative action justifies exactly one positive action. We show that $\mathfrak{c}$ is visitable by induction on $n_{\mathfrak{d}}$.
 - * If $n_{\mathfrak{d}} = 0$ then $\mathfrak{c} = \mathfrak{d}$ as the base has a unique address. Hence $\mathfrak{c}$ is visitable as $\mathbf{A}$ is shuffle-regular.
 - * Otherwise, the chronicle $\mathfrak{d}$ may be written $w_0\kappa^-w_1\kappa^+w_2$ such that κ^- justifies κ^+ and κ^+w_2 is the longest suffix of $\mathfrak{d}$ made of actions such that each action justifies the following in the sequence (or the following is a daimon), hence κ^- justifies at least two positive actions in $\mathfrak{d}$. As $\mathfrak{d} \in V_{\mathbf{A}}$, we also have $\tilde{\mathfrak{d}} \in V_{\mathbf{A}^\perp}$. Note that $w_0\kappa^-\kappa^+w_2$ is a chronicle hence a path. Note also that $\ulcorner \tilde{\mathfrak{d}} \urcorner = \ulcorner \overline{w_0\kappa^-} \urcorner \ulcorner \kappa^+w_2 \urcorner$ hence as $\overline{w_0\kappa^-}$ is a path and κ^+ is justified by κ^- we have that $\overline{w_0\kappa^-} \ulcorner \kappa^+w_2 \urcorner$ is a path. Let us note $\mathfrak{d}' = w_0\kappa^-\kappa^+w_2$. The completed design $\ulcorner \tilde{\mathfrak{d}} \urcorner^{\mathfrak{c}}$ is in the behaviour $\mathbf{A}$ (proposition 1). Hence there exists a subdesign $\mathfrak{E} \subset \ulcorner \tilde{\mathfrak{d}} \urcorner^{\mathfrak{c}}$ such that $\mathfrak{E} \in |\mathbf{A}|$. As $\tilde{\mathfrak{d}} \in V_{\mathbf{A}}$, we have necessarily that $\ulcorner \tilde{\mathfrak{d}} \urcorner \subset \mathfrak{E}$, hence also that the reversible path $\tilde{\mathfrak{d}}'$ is a path of $\mathfrak{E}$. As the behaviour $\mathbf{A}$ is shuffle-regular, $\tilde{\mathfrak{d}}'$ is visitable in $\mathbf{A}^\perp$, hence $\mathfrak{d}'$ is visitable in $\mathbf{A}$. Furthermore, the trivial chronicle $\mathfrak{c}$ is a subsequence of $\mathfrak{d}'$ and $n_{\mathfrak{d}'} \leq n_{\mathfrak{d}} - 1$. So by induction hypothesis, the chronicle $\mathfrak{c}$ is visitable in $\mathbf{A}$.
 - We show now that if a chronicle $\mathfrak{c}$ is regular for $\mathbf{A}$ then $\mathfrak{c}$ is visitable in $\mathbf{A}$. The proof is done by induction on $n_{\mathfrak{c}}$.
 - * If $n_{\mathfrak{c}} = 0$ then $\mathfrak{c}$ is a trivial chronicle for $\mathbf{A}$ thus it follows from the previous item that $\mathfrak{c}$ is visitable in $\mathbf{A}$.
 - * Otherwise, the chronicle $\mathfrak{c}$ may be written $w_0\kappa^-w_1\kappa^+w_2$ such that κ^- justifies κ^+ and κ^+w_2 is the longest suffix of $\mathfrak{c}$ made of actions such that each justifies the following in the sequence (or the following is a daimon), hence κ^- justifies at least two positive actions in $\mathfrak{c}$. Let us note $\mathfrak{c}_1 = w_0\kappa^-\kappa^+w_2$ and $\mathfrak{c}_2 = w_0\kappa^-w_1\mathfrak{x}$. Note that the sequences $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are chronicles hence reversible, furthermore regular as subsequences of a regular chronicle. Moreover both $n_{\mathfrak{c}_1}$ and $n_{\mathfrak{c}_2}$ are strictly less than $n_{\mathfrak{c}}$. Thus by induction hypothesis, these two chronicles are visitable in $\mathbf{A}$. Finally we note that the path $\tilde{\mathfrak{c}}$ belongs to $\tilde{\mathfrak{c}}_1 \sqcup \tilde{\mathfrak{c}}_2$. So, as $\mathbf{A}$ is a shuffle-regular behaviour, $\tilde{\mathfrak{c}}$ is visitable in $\mathbf{A}^\perp$, thus $\mathfrak{c}$ is visitable in $\mathbf{A}$.
 - Let p be a regular path for $\mathbf{A}$. We prove by induction on the length of p that p belongs to the shuffle of some of its views:
 - If $p = \epsilon$ or $p = \kappa^+$ or $p = \kappa^-\kappa^+$, the result is obvious as $\ulcorner p \urcorner = \{p\}$.
 - Otherwise p is of the form $w\kappa^-\kappa^+$. As p is a path, $\ulcorner w\kappa^-\kappa^+ \urcorner = w'\kappa^-\kappa^+$ is a

(positive-ended) chronicle. By induction hypothesis, w is a shuffle of a set S of some of its views. Note that elements of S are views of p . Notice that p is a shuffle of elements of $S \cup \{w'\kappa^-\kappa^+\}$.

Remark that each view is a chronicle regular for $\mathbf{A}$, hence visitable in $\mathbf{A}$ by the previous item. Thus, as the behaviour $\mathbf{A}$ is shuffle-regular, p is visitable in $\mathbf{A}$. $\square$

5 Conclusion

The additive decomposition of behaviours was already set in the seminal paper [7]. We complete here the linear decomposition of behaviours by means of the notion of regularity that characterizes tensors of behaviors. The full characterization of formulas finitely built from the linear constants and additive and multiplicative connectives follows quite easily [5]. Such behaviours should be “finite in height”, *i.e.*, the heights of its incarnated designs should be bounded. Our result goes beyond Faggian’s works [3] in that we give an incremental condition of decomposition.

This work is a first step toward a full algebraic study of behaviours. Let us remark that properties of Ludics that serve for proving that Ludics is a fully abstract model of MALL_2 ,⁶ are satisfied for the entire Ludics and not only for behaviours interpreting MALL_2 formulas: interaction between objects, *i.e.*, cut-elimination, is at the heart of Ludics, thus it allows to consider the Ludics framework as a semantics for computation beyond what is given with MALL_2 : a behaviour denotes a type and (open) interaction between behaviours corresponds to composition of types. For future work, we plan to extend our analysis to the whole set of behaviours, defining a grammar for it in such a way that connectives of the grammar may be computationally (and logically) interpreted.

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⁶More precisely of a slightly modified polarized version of the second-order multiplicative-additive Linear Logic.

Annex

Lemma 5. *Let p and q be two positive-ended paths and r be a reversible path in $p \sqcup q$, then duals $\tilde{p}$ and $\tilde{q}$ are paths.*

Proof. We prove the result by contradiction: let us suppose that p and q are two paths, $r \in p \sqcup q$ is a path such that its dual $\tilde{r}$ is a path, and at least one of the duals $\tilde{p}$ and $\tilde{q}$ is not a path. Without loss of generality, we can suppose that $\tilde{p}$ is not a path. Remark first the following point. Let p, q, r defined as above then there exist p', q', r' satisfying the same requirements as for p, q, r and such that $r' = w \bowtie$ where w is a prefix of r . Indeed, as $\tilde{p}$ is not a path, there exists an action κ^- justified by κ^+ such that $p = w_1 \kappa^+ w_2 \kappa^- w_3$ and the (Restrictive Negative Jump) constraint is not satisfied on κ^- . Hence $p' := w_1 \kappa^+ w_2 \kappa^- \bowtie$ is a path such that $\tilde{p'}$ is not a path. Let $r = x_1 \kappa^- x_2$ and note that $w_1 \kappa^+ w_2$ is a subsequence of x_1 . Let q' be the subsequence of $x_1 \kappa^-$ with actions in q , then q' is a positive-ended prefix of q , hence a path. Finally let $r' := x_1 \kappa^- \bowtie$ then r' and $\tilde{r'}$ are paths such that $r' \in p' \sqcup q'$ (by following the same construction steps as for r). Hence for proving the lemma, it suffices to consider triples (p', q', r') satisfying the following: $r' \in p' \sqcup q'$ is such that $\tilde{r'}$ is a path (r' satisfies the restrictive negative jump), $r' = w \kappa^- \bowtie$ and the restrictive negative jump is not satisfied on action κ^- present in p' . Remark also that if lengths of p' and q' are less or equal to 2 then $\tilde{p'}$ and $\tilde{q'}$ are paths.

– Let (p_0, q_0, r_0) be such a triple with length of r_0 minimal with respect to all such triples (p', q', r') . Notice that κ^- is not initial, otherwise (Restrictive Negative Jump) would be satisfied for p_0 . As r_0 satisfies the (Restrictive Negative Jump), there exists a sequence $(\alpha_n^+ = \kappa^+, \alpha_n^-, \dots, \alpha_0^+, \alpha_0^- = \kappa^-)$ where κ^+ justifies κ^- and the sequence is defined as in the definition of (Restrictive Negative Jump). Let us suppose $r_0 = w' \alpha_1^- w'' \alpha_0^+ \kappa^- \bowtie$.

- Suppose α_0^+ is an action of p_0 , then it is also the case for its justifier α_1^- . Define $r_1 = w' \kappa^- \bowtie$. Remark that r_1 is a path and its dual is also a path. Furthermore, we can define q_1 (resp. p_1) as the subsequence of q_0 (resp. p_0) present in r_1 . Remark that $r_1 \in p_1 \sqcup q_1$ and $\tilde{p_1}$ is not a path. This contradicts the fact that r_0 is minimal.
- Otherwise α_0^+ is an action of q_0 , then it is also the case for its justifier α_1^- . If actions in w'' are actions of q_0 , we define r_1, q_1, p_1 as before and this yields a contradiction. Else let β^+ be the last action of p_0 in w'' . There is also an action γ^- of p_0 which immediately precedes β^+ in w'' . One can delete from r_0 the actions γ^- and β^+ . Then we get a shorter sequence r_1 together with paths p_1 and q_1 such that p_1 does not satisfy the (Restrictive Negative Jump). Hence a contradiction.

□

Definition 12 (Completion of designs). *Let $\mathcal{D}$ be a design, the completion of $\mathcal{D}$, written $\mathcal{D}^c$, is the design obtained from $\mathcal{D}$ by adding chronicles $\mathfrak{c}(-, \xi, I) \bowtie$ whenever $\mathfrak{c} \in \mathcal{D}$, I is a finite set of integers, $\mathfrak{c}(-, \xi, I) \bowtie$ is a chronicle and $\mathfrak{c}(-, \xi, I) \notin \mathcal{D}$.*

Proposition 4. *Let $\mathcal{D}$ be a design in a behaviour $\mathbf{A}$, consider a design $\mathfrak{H} \subset \mathcal{D}$ then $\mathfrak{H}^c \in \mathbf{A}$.*

Proof. Let $\mathfrak{E} \in \mathbf{A}^\perp$. Hence $\mathfrak{E} \perp \mathcal{D}$: there exists a path q in $\mathcal{D}$ and $\tilde{q}$ is in $\mathfrak{E}$. Let p be the longest positive-ended path in the design $\mathfrak{H}$ that is a prefix of q . Either $p = q$, hence $\mathfrak{E} \perp \mathfrak{H}$, and also $\mathfrak{E} \perp \mathfrak{H}^c$. Or there exist actions κ^-, κ^+ and a sequence w such that $q = p \kappa^- \kappa^+ w$. Consider the chronicle $\mathfrak{c}$ such that $\lceil p \kappa^- \rceil = \mathfrak{c} \kappa^-$. By construction, $\mathfrak{c} \in \mathfrak{H}$. Either $\mathfrak{c} \kappa^- \in \mathfrak{H}$ hence also $\mathfrak{c} \kappa^- \kappa^+ \in \mathfrak{H}$ as $\mathfrak{H} \subset \mathcal{D}$ and there is a unique positive action after a negative action. Contradiction as p is then not maximal. Or $\mathfrak{c} \kappa^- \bowtie \in \mathfrak{H}^c$ hence $\mathfrak{E} \perp \mathfrak{H}^c$. □