

Backdoors into Two Occurrences

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Let F be a CNF-formula in the variables V . For a subset $X \subseteq V$, the formula $F \setminus X$ is the formula obtained by deleting from the clauses in F all occurrences of literals x and $\bar{x}$ for $x \in X$.

Let $\mathcal{C}$ be a class of CNF-formulas such that membership $F \in \mathcal{C}$, as well as the satisfiability problem for formulas $F \in \mathcal{C}$ can be decided in polynomial time. Well-known examples of such classes are e.g., the class Horn of Horn formulas, or the class 2-CNF of formulas in which every clause is of width at most 2.

- A strong $\mathcal{C}$ -backdoor set for F is a subset $X \subseteq V$ such that for every assignment $\alpha : X \rightarrow \{0, 1\}$, the formula $F[\alpha]$ is in $\mathcal{C}$.
- A weak $\mathcal{C}$ -backdoor set for F is a subset $X \subseteq V$ such that there is an assignment $\alpha : X \rightarrow \{0, 1\}$ such that the formula $F[\alpha]$ is in $\mathcal{C}$ and satisfiable.
- A deletion $\mathcal{C}$ -backdoor set for F is a subset $X \subseteq V$ such that the formula $F \setminus X$ is in $\mathcal{C}$.

The parameterized problem STRONG $\mathcal{C}$ -BACKDOOR SET is:

Instance: CNF-formula F , and $k \in \mathbb{N}$
Parameter: k
Question: Is there a strong $\mathcal{C}$ -backdoor set for F of size k ?

The problems WEAK $\mathcal{C}$ -BACKDOOR SET and DELETION $\mathcal{C}$ -BACKDOOR SET are defined analogously. The parameterized complexity of these problems for various tractable classes of formulas has been determined in the literature, a recent survey is [1].

A less well-known tractable class is the class CNF(2) of CNF-formulas in which every variable has at most two occurrences. The satisfiability problem for these formulas can be solved in linear time [3], and it has been shown to be complete for deterministic logarithmic space [2]. Here we study the parameterized complexity of the above problems about backdoor sets for this class.

For the base class CNF(2), it matters whether formulas are represented as multisets, with multiple occurrences of the same clause allowed, or as sets, since the number of occurrences of variables is counted differently in both cases.

If formulas are represented as multisets, then the smallest deletion CNF(2)-backdoor set is exactly the set of variables with more than two occurrences, hence we trivially have:

Proposition 1. *For formulas represented as multisets, the problem DELETION CNF(2)-BACKDOOR SET is in P.*

On the other hand, for formulas represented as sets, clauses can become equal - and hence identified - if literals are deleted or set to 0. To show that this actually makes a difference note the following theorem.

Theorem 1. *DELETION CNF(2)-BACKDOOR SET for formulas represented as sets is NP-hard, but fixed-parameter tractable.*

We will focus on the case of formulas represented as sets in the remainder of the paper.

Theorem 2. *WEAK CNF(2)-BACKDOOR SET is $W[2]$ -hard.*

In the survey [1] of Gaspers and Szeider, there is a generic construction to show the $W[2]$ -hardness of the problem WEAK $\mathcal{C}$ -BACKDOOR SET. Our reduction from HITTING SET is based on that construction, but is slightly simpler, and it has the advantage that the constructed formula $F(\mathcal{S})$ only depends on the input set system $\mathcal{S}$ and is independent from the parameter. The same simplification can also be made to other applications of the generic construction in [1], e.g. for $\mathcal{C} = \text{Horn}$ and $\mathcal{C} = \text{2-CNF}$.

Theorem 3. *WEAK CNF(2)-BACKDOOR SET for 3-CNF-formulas is fixed-parameter tractable.*

For the problems about strong backdoors, the situation is the same as for those about weak backdoors:

Theorem 4. *STRONG CNF(2)-BACKDOOR SET is $W[2]$ -hard.*

The reduction from HITTING SET is similar to that used in the proof of Theorem 2, and has the same property, that the constructed formula $F(\mathcal{S})$ only depends on the input set system $\mathcal{S}$ and is independent from the parameter.

Theorem 5. *STRONG CNF(2)-BACKDOOR SET for 3-CNF-formulas is fixed-parameter tractable.*

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References

1. S. Gaspers and S. Szeider. Backdoors to satisfaction. In H. L. Bodlaender, R. Downey, F. V. Fomin, and D. Marx, editors, *The Multivariate Algorithmic Revolution and Beyond*, volume 7370 of *Lecture Notes in Computer Science*, pages 287–317, 2012.
2. J. Johannsen. Satisfiability problems complete for deterministic logarithmic space. In V. Diekert and M. Habib, editors, *21st International Symposium on Theoretical Aspects of Computer Science (STACS 2004)*, Springer LNCS 2996, pages 317–325, 2004.
3. H. Kleine Büning and T. Lettmann. *Propositional Logic: Deduction and Algorithms*. Cambridge University Press, 1999.