

A Measure of Arbitrariness in Abductive Explanations

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Abstract

We study the framework of abductive logic programming extended with integrity constraints. For this framework, we introduce a new measure of the simplicity of an explanation based on its degree of *arbitrariness*: the more arbitrary the explanation, the less appealing it is, with explanations having no arbitrariness — they are called *constrained* — being the preferred ones. In the paper, we study basic properties of constrained explanations. For the case when programs in abductive theories are stratified we establish results providing a detailed picture of the complexity of the problem to decide whether constrained explanations exist.

1 Introduction

Abduction is an important form of nonmonotonic reasoning. The concept was introduced in the late 19th century by the American philosopher Charles Sander Peirce (1995) as an inference scheme aimed at deriving potential explanations of observations.¹ A general characteristic of abductive reasoning is the existence of multiple abductive explanations, which are typically not equally compelling. Therefore, identifying a subclass, possibly narrow, of “preferred explanations” is an important problem. Following the Occam’s principle, a typical approach is to identify as “preferred” those explanations that are, in some sense, *simple*. Several concepts of simplicity were considered in the literature, most notably those based on minimality with respect to inclusion and cardinality. In the context of logic programming, abduction was first studied by Eshghi and Kowalski (1989), and then by Kakas and Mancarella (1990b) under the *brave reasoning* variant of the stable-model semantics. That work established *abductive logic programming* as an important subarea of abduction, where the background theory is represented by a logic program, often with negation in the bodies and disjunction in the heads, under any of the standard logic programming semantics (Kakas et al. 1992; Denecker and Kakas 2002). This paper is concerned with the problem of limiting the space of explanations in the framework of abductive

¹ Peirce gave abduction the following informal interpretation: “The surprising fact, *C*, is observed; but if *A* were true, *C* would be a matter of course: hence, there is reason to suspect that *A* is true.”

logic programming extended by integrity constraints. We introduce a new measure of the quality of an explanation in terms of its *arbitrariness* and propose to consider as “preferred” only those explanations that minimize arbitrariness. Our approach can be applied with any of the standard semantics of logic programs.

Example 1

Let us consider the following scenario. It is Saturday and Bob is known to ski on Saturdays if it is not snowing. And when he is not skiing, he is on campus. We can represent this information by the logic program:

$$\{ \textit{saturday}. \quad \textit{skiing} \leftarrow \textit{saturday}, \textbf{not} \textit{snows}. \quad \textit{on_campus} \leftarrow \textbf{not} \textit{skiing}. \}$$

If it’s Saturday and we see Bob on campus, we can abduce it is snowing. Otherwise, Bob would be skiing. To put it differently, given our background knowledge, the fact *snows* is an abductive explanation (explanations will be formally defined in the next section) to our observation *on_campus*. $\square$

In this example, there is only one explanation of the observation (assuming all that is relevant to our reasoning has been mentioned). However, a general feature of abductive reasoning is the existence of *multiple* explanations. Typically, they are not all equally likely. Thus, narrowing down the range of possible explanations to a smaller set of “most likely” or “preferred” ones becomes an important problem. The key to it is a well-motivated notion of “preferred”. We will now present an example meant to develop intuitions behind the notion of “preferredness”.

Example 2

Let us consider the following scenario modeling security breaches in an information system. A security breach at a component of the system may only occur when a person with an account makes an unauthorized access. Regular staff personnel have accounts on the system if they complete training and have their security clearance current. Visitors may also be granted an account but only with an approval by the head of the IT department. This situation can be described by the following program:

$$\begin{aligned} \textit{account}(X) &\leftarrow \textit{staff}(X), \textit{trained}(X), \textit{current}(X). \\ \textit{account}(X) &\leftarrow \textit{visitor}(X), \textit{approved}(X). \\ \textit{security_breach}(W) &\leftarrow \textit{unauthorized_access}(W, X), \textit{account}(X). \end{aligned}$$

Let us also assume that *tom* and *mary* are regular staff members and *dan* is a visitor (there may also be other individuals in these groups and additional ones not mentioned as staff or visitors in the program), and that the system has information that *tom* completed training. That is, the program also contains facts

$$\textit{staff}(\textit{tom}). \quad \textit{staff}(\textit{mary}). \quad \textit{visitor}(\textit{dan}). \quad \textit{trained}(\textit{tom}).$$

If we observe *security_breach(warehouse)* (the security of *warehouse* was compromised), there are several possible explanations. Below we list some of them:

$$\begin{aligned} E_{\textit{tom}} &= \{ \textit{unauthorized_access}(\textit{warehouse}, \textit{tom}), \textit{current}(\textit{tom}) \} \\ E_{\textit{mary}} &= \{ \textit{unauthorized_access}(\textit{warehouse}, \textit{mary}), \textit{trained}(\textit{mary}), \textit{current}(\textit{mary}) \} \\ S_u &= \{ \textit{unauthorized_access}(\textit{warehouse}, U), \textit{staff}(U), \textit{trained}(U), \textit{current}(U) \}, \\ &\text{where } U \text{ is a name in the domain, possibly not mentioned in the program,} \end{aligned}$$

$$E_{dan} = \{unauthorized_access(warehouse, dan), approved(dan)\}$$

$$V_u = \{unauthorized_access(warehouse, U), visitor(U), approved(U)\},$$

where U is a name in the domain, possibly not mentioned in the program

$$E_{tom, dan} = \{unauthorized_access(warehouse, tom), current(tom), \\ unauthorized_access(warehouse, dan), approved(dan)\}.$$

The key question is whether there are principled ways to eliminate some of these explanations as less likely than others. $\square$

Most approaches to the problem of selecting preferred explanations follow the Occam's principle of parsimony that entities should not be multiplied unnecessarily and that among possible explanations the simplest one tends to be the right one. The first part of that principle is non-controversial. However, simplicity is a notoriously complex concept and different formalizations of it are possible! They range from the standard one based on the subset minimality, to its versions and refinements that require minimum cardinality, minimum weight, or minimality under prioritization of individual hypotheses (Eiter et al. 1997). In our example, the explanations E_{tom} , E_{mary} , S_U , $U \neq tom$, E_{dan} , and V_U , $U \neq dan$, are subset minimal and so, *preferred* under the subset minimality criterion. The explanations S_{tom} , V_{dan} , and $E_{tom, dan}$ are not. If we use a more restrictive criterion of minimum cardinality, the preferred explanations are E_{tom} and E_{dan} . Let us assume that there are reasons to view each of them as wrong (tom and dan can conclusively demonstrate they were not involved). Under the subset minimality criterion, we now prefer explanations E_{mary} , S_U , $U \neq tom$, and V_U , $U \neq dan$, while under the minimum cardinality criterion E_{mary} and V_U , $U \neq dan$, are preferred.

Let us look more carefully at these last two types of explanations. They are specific in that they speak about concrete individuals. However, the explanations of the latter type select an *arbitrary* individual in the domain (explicitly mentioned or not, as long as it is not dan), with no particular reason to choose one over another. In other words, investigators of the security breach would have no reason to start their investigation with any particular person and the person they would consider first would be arbitrarily selected. On the other hand, the explanation E_{dan} connects the structural information present in the program and the knowledge provided by the observation in a non-arbitrary (*constrained*) way — it “invents” no new entities and makes no arbitrary assumptions. Arguably, it would be the explanation pursued first by the investigators.

One might argue that the family of explanations V_U simply points to the *existence* of a person who managed to pass as a visitor with an approval for an account and made an unauthorized access to the IT system, and that this family of explanations can be represented as a single explanation involving existentially quantified formulas (along the lines of an early work by Pople (1973)). However, such “existential” explanations while no longer making arbitrary choices lack *specificity*.

In this paper we develop these intuitions into a formal measure of the quality of explanations based on how arbitrary they are. To this end, we introduce the notion of the *degree of arbitrariness* of an explanation: the smaller that value, the less arbitrary the explanation. We define *constrained* explanations as those with 0 degrees of arbitrariness, and propose them as *preferred*. Our most significant technical results establish the computational complexity of the problems to decide whether a given explanation is constrained, and whether a constrained explanation exists. We study the two problems for the case when programs modeling background knowledge in abductive theories are stratified. This is a common case in abductive logic programming. From the

technical standpoint it has two interesting aspects. First, the two main semantics of logic programs, the stable-model semantics (Gelfond and Lifschitz 1988) and the well-founded semantics (Van Gelder et al. 1991), as well as several others (but not the supported-model semantics (Marek and Subrahmanian 1992)) collapse in this case. Second, when the background knowledge is modeled by a stratified program under the stable-model semantics, all general semantics-independent schemas to define aductive explanations yield the same result. We show that for abductive theories with stratified logic programs, the two computational problems mentioned above are coNP-complete and Σ_2^P -complete, respectively. By considering special classes of stratified programs, we establish the main sources of the complexity of the problem.

The remainder of the paper is organized into five sections. First, we recall basic concepts of abductive logic programming and discuss four general schemata to define explanations. Next, we introduce the key concepts for the paper: the degree of arbitrariness in an explanation, and then arbitrary and constrained explanations. The following section presents our main complexity results. Next, we discuss related work where, in particular, we note that our results are also relevant in the area of view update repairs. We conclude by summarizing our contributions and pointing out problems for future work.

2 Abductive Logic Programs

We consider a fixed vocabulary σ consisting of relation and constant symbols (no function symbols). We write $\mathcal{D}$ for the set of constants in σ and assume that $\mathcal{D}$ is an infinite countable set (in some examples, for the sake of simplicity of presentation, we take $\mathcal{D}$ to be finite). We write $\mathcal{H}$ for the set of predicate symbols in σ . For $\mathcal{Q} \subseteq \mathcal{H}$, we define $\mathcal{Q}^\mathcal{D}$ to be the set of all ground atoms whose predicate symbols are in $\mathcal{Q}$. In particular, $\mathcal{H}^\mathcal{D}$ is the Herbrand base of σ .

By S we denote a semantics of logic programs. For now we do not commit to any particular semantics. As usual, we assume only that S is given in terms of Herbrand interpretations of σ , that is, subsets of $\mathcal{H}^\mathcal{D}$. For a logic program $\mathcal{P}$ (in the vocabulary σ), we denote by $sem_S(\mathcal{P})$ the set of Herbrand interpretations of σ that are models of $\mathcal{P}$ according to the semantics S . Since σ is fixed, we omit it from the notation. The most common choice for S is the stable-model semantics (Gelfond and Lifschitz 1988).

We now recall the concept of an abductive theory in the logic programming setting.

Definition 1 (ABDUCTIVE THEORY)

An *abductive theory* $\mathcal{T}$ over a vocabulary σ is a triple $\langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$, where

- $\mathcal{P}$ is a finite logic program over σ
- $\mathcal{A} \subseteq \sigma$ is a finite set of predicate names called the *abducible predicates*
- $\mathcal{C}$ is a finite set of first-order sentences over σ called *integrity constraints*,

and where every rule in $\mathcal{P}$ with an abducible predicate in the head is a ground fact. $\square$

Informally, the program $\mathcal{P}$ and the integrity constraints $\mathcal{C}$ provide a model (description) of the problem domain. The program $\mathcal{P}$ defines non-abducible predicates (those not in $\mathcal{A}$) in terms of abducible predicates (those in $\mathcal{A}$). The integrity constraints in $\mathcal{C}$ impose domain constraints on abducible and non-abducible predicates in the language. According to the definition, information about abducible predicates is given in terms of ground facts. They explicitly specify the extensions of abducible predicates. We refer to ground atoms based on abducible predicates as

abducibles. An *observation* is a set of ground facts based on *non-abducible* predicates. An observation may “agree” with the program $\mathcal{P}$ and the integrity constraints $\mathcal{C}$. But if it does not, we assume that this “disagreement” is caused by the incorrect information about the properties modeled by the *abducible* predicates. Abductive reasoning consists of inferring updates to the set of abducibles in the program (removal of some and inclusion of some new ones) so that the updated program, the integrity constraints and the observation “agree.” Each update that yields an agreement constitutes a possible *explanation* of the observation. Several notions of agreement have been proposed. They are defined in terms of entailment and satisfiability. A program $\mathcal{P}$ is *consistent in the semantics* S if $\text{sem}_S(\mathcal{P}) \neq \emptyset$. A consistent logic program $\mathcal{P}$ *skeptically* entails a sentence φ (in the same language as $\mathcal{P}$), written $\mathcal{P} \models_S \varphi$, if for every $M \in \text{sem}_S(\mathcal{P})$, $M \models \varphi$. A consistent logic program $\mathcal{P}$ *skeptically* entails a set Φ of sentences, written $\mathcal{P} \models_S \Phi$, if $\mathcal{P} \models_S \varphi$, for every $\varphi \in \Phi$.

We now present four concepts of “agreement”, three of which have received significant attention in the literature.

Definition 2 (AGREEMENT A (DENECKER AND KAKAS 2002))

Let $\mathcal{P}$ be a program and $\mathcal{C}$ a set of integrity constraints. An observation O *agrees* with $\mathcal{P}$ and $\mathcal{C}$ if (1) $\text{sem}_S(\mathcal{P}) \neq \emptyset$ ($\mathcal{P}$ is consistent); (2) $\mathcal{P} \models_S \mathcal{C}$; and (3) $\mathcal{P} \models_S O$. $\square$

To get this notion of agreement, it is necessary that the program be consistent and skeptically entail the observation. But, in addition, the program must also satisfy the integrity constraints in a very strong sense. Namely, all its models must satisfy them.

Definition 3 (AGREEMENT B (DENECKER AND KAKAS 2002))

Let $\mathcal{P}$ be a program and $\mathcal{C}$ a set of integrity constraints. An observation O *agrees* with $\mathcal{P}$ and $\mathcal{C}$ if (1) $\mathcal{P} \models_S O$; and (2) there is $M \in \text{sem}_S(\mathcal{P})$ such that $M \models \mathcal{C}$. $\square$

This definition is a relaxation of the previous one. To get this notion of agreement, at least one model of the program must satisfy the integrity constraints (not all, as before, and here is where the conditions are relaxed). This, in particular, means that the program is consistent. Moreover, exactly as before, the program must skeptically entail the observation.

Definition 4 (AGREEMENT C)

Let $\mathcal{P}$ be a program and $\mathcal{C}$ a set of integrity constraints. An observation O *agrees* with $\mathcal{P}$ and $\mathcal{C}$ if for every $M \in \text{sem}_S(\mathcal{P})$ such that $M \models \mathcal{C}$, $M \models O$. $\square$

Here to have an agreement, as in Definition 3, we require that at least one model of the program satisfies the integrity constraints and we weaken the other condition. Namely, we now only require that the observation holds in every model of the program satisfying the integrity constraints and not in every model of the programs.

Definition 5 (AGREEMENT D (BARAL AND GELFOND 1994))

Let $\mathcal{P}$ be a program and $\mathcal{C}$ a set of integrity constraints. An observation O *agrees* with $\mathcal{P}$ and $\mathcal{C}$ if there is $M \in \text{sem}_S(\mathcal{P})$ such that $M \models \mathcal{C}$ and $M \models O$. $\square$

This is the final relaxations of the conditions for an agreement. There must be models of a program that satisfy the integrity constraints (just as in Definition 4). But it is enough that one of such models (not all, as in Definition 4) satisfies the observation. Each of these types of agreement yields the corresponding notion of an explanation.

Definition 6 (ABDUCTIVE EXPLANATION)

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$ be an abductive theory and O an observation. A pair $\Delta = (E, F)$, where E and F are disjoint finite sets of abducibles and $F \subseteq \mathcal{P}$, is an *explanation* (of type A, B, C , and D) if O is in agreement (of type A, B, C , and D , respectively) with $\mathcal{P}^\Delta = (\mathcal{P} \cup E) \setminus F$ and $\mathcal{C}$. We denote the sets of explanations of each type by $\Psi_A(O, \mathcal{T})$, $\Psi_B(O, \mathcal{T})$, $\Psi_C(O, \mathcal{T})$ and $\Psi_D(O, \mathcal{T})$. $\square$

The observations made above about each next concept of an agreement being less restrictive, imply the following relationships between the corresponding notions of explanations.

Proposition 1

For every abductive theory $\mathcal{T}$ and an observation O ,

$$\Psi_A(O, \mathcal{T}) \subseteq \Psi_B(O, \mathcal{T}) \subseteq \Psi_C(O, \mathcal{T}) \subseteq \Psi_D(O, \mathcal{T}).$$

$\square$

In general, the four concepts of explanations do not coincide. Which of them to choose may depend on a particular application domain. This issue is not of our concern here. Instead, we focus on the key challenge of abductive reasoning to find general principles that can narrow down a class of explanations and that are independent of what notion of an explanation is used. This is indeed an important problem as even in the case of explanations of type A multiple explanations are possible and not all of them are equally compelling.

The two most commonly used principles are subset minimality and cardinality minimality. We say that an explanation $\Delta = (E, F)$ is *subset minimal* if there is no *other* explanation $\Delta' = (E', F')$ such that $E' \subseteq E$ and $F' \subseteq F$. Similarly, $\Delta = (E, F)$ is *cardinality minimal* if there is no explanation $\Delta' = (E', F')$ such that $|E'| + |F'| < |E| + |F|$. We propose in the next section yet another general principle based on the degree of arbitrariness.

3 How arbitrary is a solution?

In this section we introduce the degree of *arbitrariness* in an explanation of an observation as a measure of how *arbitrary* the explanation is. Concepts introduced below apply to each of the four basic models of abductive reasoning, therefore we will use the generic term *explanation* without denoting its specific type. We start with examples to motivate our discussion.

Example 3

Let us consider the abduction problem presented in Example 2, and the explanations $\Delta_u = (S_u, \emptyset)$ and $\Delta_{dan} = (E_{dan}, \emptyset)$. In each explanation Δ_u , the constant u can be replaced with any other constant u' in the vocabulary and the result, $\Delta_{u'}$, is also an explanation. That is, the occurrences of u are not *constrained* by the program or, to put it differently, they are *arbitrary*. In contrast, replacing the constant dan in Δ_{dan} with any other constant does not yield an explanation (assuming there are no other visitors in the program but dan). Thus, there is no *arbitrariness* in Δ_{dan} . In other words, Δ_{dan} is *constrained* by the available information. $\square$

We will use the idea of “replaceability” to formalize the notions of arbitrariness and constrainedness, and their generalization, the degree of arbitrariness.

Definition 7 (OCCURRENCE)

Let $p(x)$ be a ground atom, where p has arity n and k is an integer such that $1 \leq k \leq n$. We denote by $p(x)[k]$ the constant in the position k in $p(x)$.

If E is a set of ground atoms, an *occurrence* of a constant c in E is an expression of the form $p(x)^k$, where $p(x)$ is an atom in E , and $p(x)[k] = c$. $\square$

Definition 8 (REPLACEMENT FUNCTION)

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$ be an abductive theory, E a set of abducibles (that is, $E \subseteq \mathcal{A}^{\mathcal{D}}$), and let c be a constant occurring in E . A *replacement function* for E and c determined by a *non-empty* set C of some (not necessarily all) occurrences of c in E , is a function $f_{E,C} : \mathcal{D} \rightarrow 2^{\mathcal{A}^{\mathcal{D}}}$ such that for each $x \in \mathcal{D}$, $f_{E,C}(x)$ is the set E' obtained by replacing with x each constant c in E referred to by an occurrence in C . $\square$

We observe that, given a set E and a constant c occurring in E in n places, the number of possible replacement functions is $2^n - 1$.

Example 4

Let us consider the set $E = \{p(1,2), s(2,3)\}$ and the constant 2. The possible replacement functions are: (1) f_{E,C_1} , where $C_1 = \{p(1,2)^2\}$; (2) f_{E,C_2} , where $C_2 = \{s(2,3)^1\}$; and (3) f_{E,C_3} , where $C_3 = \{p(1,2)^2, s(2,3)^1\}$. $\square$

Definition 9 (INDEPENDENCE OF REPLACEMENT FUNCTIONS)

Let f_{E,C_1} and f_{E,C_2} be replacement functions for a set $E \subseteq \mathcal{A}^{\mathcal{D}}$ and for constants c_1 and c_2 , respectively. We say that f_{E,C_1} and f_{E,C_2} are *independent* if $c_1 \neq c_2$ or if $C_1 \cap C_2 = \emptyset$. $\square$

Let us consider the replacement functions presented in Example 4. While the functions f_{E,C_1} and f_{E,C_2} are independent, the functions f_{E,C_3} and f_{E,C_1} (similarly, f_{E,C_3} and f_{E,C_2}) are not independent.

Definition 10 (DEGREE OF ARBITRARINESS)

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$ be an abductive theory, O an observation, $\Delta = (E, F)$ an explanation for O wrt $\mathcal{T}$, and let ξ be an arbitrary constant in $\mathcal{D}$ not occurring in $E \cup O \cup \mathcal{P}$. The *degree of arbitrariness* of Δ , denoted $\delta(\Delta)$, is the maximum number of *pairwise independent* replacement functions $f_{E,C}$ (not necessarily all for the same constant) such that $\Delta' = (f_{E,C}(\xi), F)$ is an explanation for O wrt $\mathcal{T}$. $\square$

Since the domain $\mathcal{D}$ is infinite, one can always find a constant ξ not occurring in $E \cup O \cup \mathcal{P}$. Moreover, since E is finite, there is an upper bound on the number of pairwise independent replacement functions one can have for E . Finally, the specific choice of the replacement constant ξ does not affect the maximum. Thus, the degree of arbitrariness is well defined. The examples below illustrate the concepts we have introduced above.

Example 5

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \emptyset \rangle$, where the program $\mathcal{P}$ consists of the facts $\{p(1), p(2), q(1), q(2), q(3)\}$ and of a single rule $t \leftarrow p(X), \text{not } q(X)$. Let us suppose that p and q are abducible predicates and that $O = \{t\}$. The following pairs of sets of abducibles form explanations for O wrt $\mathcal{T}$:

$\Delta_1 = (\emptyset, \{q(1)\})$. $\Delta_2 = (\emptyset, \{q(2)\})$. $\Delta_3 = (\{p(3)\}, \{q(3)\})$. $\Delta_x = (\{p(x)\}, \emptyset)$, where $x \notin \{1, 2, 3\}$.

It is evident that $\delta(\Delta_1) = \delta(\Delta_2) = 0$. Similarly, it is evident that $\delta(\Delta_3) = \delta(\Delta_x) = 1$ (the only constant in the “add” part of these explanations can be replaced with a fresh constant ξ and the result is an explanation). $\square$

In Example 5, the explanation Δ_3 is not satisfactory. Once we decide to remove $q(3)$, there is no reason why we have to add $p(3)$. Adding any atom $p(\xi)$, with $\xi \notin \{1, 2\}$ works equally well. Thus, the choice of the constant 3 in $p(3)$ is arbitrary and not grounded in the information available in the theory. Similarly, Δ_x , where $x \notin \{1, 2, 3\}$ is not satisfactory either. Here too, the choice of x is not grounded in the abductive theory and the observation. The explanations Δ_1 and Δ_2 do not show this arbitrariness.

Example 6

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \emptyset \rangle$, where $\mathcal{A} = \{q, r, t\}$ and $\mathcal{P}$ consists of the facts $q(a, b)$, $q(a, c)$, and $r(a, b, c)$, and two rules $p(X) \leftarrow q(X, Y), s(X, Y, Z)$, and $s(X, Y, Z) \leftarrow r(X, Y, Z), t(X, Z)$. Let us suppose $O = \{p(a)\}$. One can check that each of the following pairs of sets of abducibles is an explanation:

$$\begin{aligned} \Delta_{x_1, x_2} &= (\{q(a, x_1), r(a, x_1, x_2), t(a, x_2)\}, \emptyset), \text{ where } x_1 \neq c, x_1 \neq b \text{ and } x_2 \neq c \\ \Delta_{x_3} &= (\{r(a, b, x_3), t(a, x_3)\}, \emptyset), \text{ where } x_3 \neq c \\ \Delta &= (\{t(a, c)\}, \emptyset). \end{aligned}$$

It is evident that if $x_1 \neq x_2$, then $\delta(\Delta_{x_1, x_2}) = 2$. Indeed changing all occurrences of x_1 or all occurrences of x_2 to a new constant ξ results in an explanation. In addition, the corresponding replacement functions for each constant and all its occurrences are obviously independent (they concern different constants). Finally, replacing either constant in only one position does not yield an explanation. More interestingly, if $x_1 = x_2 = x$, $\delta(\Delta_{x, x}) = 2$, too. Here, x has four occurrences in $\Delta_{x, x}$: $q(a, x)^2$, $r(a, x, x)^2$, $r(a, x, x)^3$ and $t(a, x)^2$. Let f_1 and f_2 be the replacement functions for x determined by the first two and the last two positions, respectively. Then f_1 and f_2 are independent, and both $(f_1(\xi), \emptyset)$ and $(f_2(\xi), \emptyset)$ are explanations. However, three mutually independent functions with this property cannot be found. Similarly, one can see that $\delta(\Delta_{x_3}) = 1$ (all occurrences of x_3 are free for a simultaneous change) and $\delta(\Delta) = 0$ (neither a nor c can be changed to a fresh constant). $\square$

In this example, all explanations are minimal and so, the subset-minimality criterion is not discriminating enough. However, Δ arguably is more compelling than the other ones. It uses no extraneous constants, and all constants occurring in it are constrained by the theory and an observation. The lowest degree of arbitrariness criterion correctly identifies Δ as the only preferred explanation. We will refer to it as the principle of *minimum arbitrariness*.

The most compelling are those explanations that have no arbitrariness at all. This suggests the notion of constrained explanations.

Definition 11 (CONSTRAINED AND ARBITRARY EXPLANATIONS)

Let $\mathcal{T}$ be an abductive theory $\langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$, O an observation, and Δ an explanation for O wrt $\mathcal{T}$. We say that Δ is *constrained* if $\delta(\Delta) = 0$. Otherwise, Δ is *arbitrary*. $\square$

In the remainder of this paper we discuss the principle of the lowest degree of arbitrariness, focusing our study primarily on the class of constrained explanations. We start with some general observations.

The principle of minimum arbitrariness can be used with all types of explanations we discussed in the previous sections. Moreover, it is “orthogonal” to other criteria one might consider when selecting preferred explanations such as the subset or cardinality minimality. Therefore, it can be combined with them. For instance, we might consider as preferred those subset-minimal explanations that have the smallest degree of arbitrariness (if we believe, that subset minimality is more important than minimum arbitrariness). For instance, coming back to Example 2, we see that the explanations $(E_{tom}, \emptyset)$, $(E_{mary}, \emptyset)$, $(S_u, \emptyset)$, $u \neq tom$, $(E_{dan}, \emptyset)$, and $(V_u, \emptyset)$, $u \neq dan$, are subset minimal. Selecting from among them only the constrained ones yields $(E_{tom}, \emptyset)$, $(E_{mary}, \emptyset)$, and $(E_{dan}, \emptyset)$. Alternatively, we could take as preferred those explanations with the minimum degree of arbitrariness that are subset minimal (if we believe that minimum arbitrariness should be the primary consideration). In Example 2, the explanation $(E_{tom, dan}, \emptyset)$ is constrained but not minimal. In general, the two concepts are different. Similar examples can be provided for the minimum cardinality criterion. We do not consider these and other possible combinations of

the principles in this paper but focus exclusively on the properties of the principle of minimum arbitrariness.

The degree of arbitrariness of an explanation (E, F) depends only on the “add” part E ; the “delete” component, F , has no effect on arbitrariness. Intuitively, the reason is that we can delete only those atoms that are in $\mathcal{P}$. Thus, if we replace a constant in an atom p in F with a fresh constant ξ , the effect simply is that p is no longer deleted. The same effect can be achieved by considering $F \setminus \{p\}$ in place of F . It is natural to impose on F some requirements, such as subset or cardinality minimality. However, as we noted above, we do not pursue this possibility here.

The next result shows that constrained explanations use only constants that are mentioned in an abductive theory and an observation. This property is consistent with the general principle of parsimony (Occam’s razor). It is important as it allows us to restrict the scope of searches for constrained explanations.²

Theorem 1

Let $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$ be an abductive theory and (E, F) a constrained explanation of an observation O . Then every constant symbol occurring in (E, F) occurs in $\mathcal{T}$ or in O .

Finally, we note that the minimum arbitrariness criterion does not impose any conditions on abducible predicates of arity 0 and some other criteria should be considered. Therefore, we are primarily interested in the case when every abducible predicate symbol has a positive arity.

4 Computational Complexity

Our primary technical contribution concerns reasoning about constrained explanations. We are interested in the following two problems: deciding whether a given explanation is constrained; and deciding whether a constrained explanation exists. We restrict attention to abductive theories with stratified programs and assume that these programs are interpreted by the stable-model semantics. This is an important class of abductive theories. First, stratified programs are regarded as semantically “non-controversial.” Indeed, the stable-model semantics and the well-founded semantics coincide on stratified programs and are generally accepted as providing them the correct meaning. Second, under the stable-model semantics, for abductive theories with stratified programs, the distinctions between the four types of explanations we introduced disappear. Formally, we have the following result.

Theorem 2

For every abductive theory $\mathcal{T} = \langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$, where $\mathcal{P}$ is stratified and interpreted under the stable-model semantics, and every observation O ,

$$\Psi_A(O, \mathcal{T}) = \Psi_B(O, \mathcal{T}) = \Psi_C(O, \mathcal{T}) = \Psi_D(O, \mathcal{T}). \quad \square$$

Because of Theorem 2, below we use the generic term *explanation* without specifying its type.

Let $\langle \mathcal{P}, \mathcal{A}, \mathcal{C} \rangle$ be an abductive logic theory. We represent $\mathcal{P}$ as the union of a set B of all abducibles in $\mathcal{P}$ and the set R of the remaining facts and rules. We will consider the complexity of the problems stated above under the assumption that R and $\mathcal{C}$ are fixed and input consists of B and an observation O . We start our study of the complexity by noting the following two upper bounds.

² The proofs of all results we give in the paper can be found in the online appendix (and also at <http://arxiv.org/abs/1405.2494>).

Theorem 3

Let $\mathcal{A}$ be a set of abducible predicates, R a (fixed) stratified program with no abducible predicates in the heads of its rules, and $\mathcal{C}$ a (fixed) set of integrity constraints.

1. The following problem is in coNP: given a set B of abducibles, an observation O , and a pair $\Delta = (E, F)$ of sets of abducibles, decide whether Δ is a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \mathcal{C} \rangle$.
2. The following problem is in Σ_2^P : given a set B of abducibles and an observation O , decide whether a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \mathcal{C} \rangle$ exists.

There are three sources of complexity: negation in programs, recursion in programs, and the presence of integrity constraints. The next three results show that each of these sources by itself pushes the complexity up to match the upper bounds of Theorem 3. The first of them concerns the case when the program in an abductive theory is non-recursive but with negation, and there are no integrity constraints.

Theorem 4

Let $\mathcal{A}$ be a set of abducible predicates and R a (fixed) non-recursive program with no abducible predicates in the heads of its rules.

1. The following problem is coNP-complete: given a set B of abducibles, an observation O , and a pair (E, F) of sets of abducibles, decide whether (E, F) is a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$.
2. The following problem is Σ_2^P -complete: given a set B of abducibles and an observation O , decide whether a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$ exists.

The next theorem shows that when there are no integrity constraints, disallowing negation, that is, restricting attention to Horn programs, also leads to the same complexity results as long as we allow recursion.

Theorem 5

Let $\mathcal{A}$ be a set of abducible predicates and R a (fixed) Horn program with no abducible predicates in the heads of its rules.

1. The following problem is coNP-complete: given a set B of abducibles, an observation O , and a pair (E, F) of sets of abducibles, decide whether (E, F) is a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$.
2. The following problem is Σ_2^P -complete: given a set B of abducibles and an observation O , decide whether a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$ exists.

The third result addresses the case of abductive theories with particularly simple programs, namely, non-recursive Horn, but with integrity constraints.

Theorem 6

Let $\mathcal{A}$ be a set of abducible predicates, R a (fixed) non-recursive Horn program with no abducible predicates in the heads of its rules, and $\mathcal{C}$ a (fixed) set of integrity constraints.

1. The following problem is coNP-complete: given a set B of abducibles, an observation O , and a pair (E, F) of sets of abducibles, decide whether (E, F) is a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \mathcal{C} \rangle$.

2. The following problem is Σ_2^P -complete: given a set B of abducibles and an observation O , decide whether a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \mathcal{C} \rangle$ exists.

On the other end of the spectrum, we have a particularly simple case when neither of the three sources of complexity is present: the case of abductive theories with non-recursive Horn programs and without integrity constraints. For this class of theories the two problems are tractable.

Theorem 7

Let $\mathcal{A}$ be a set of abducible predicates and R a (fixed) non-recursive Horn program with no abducible predicates in the heads of its rules. The following problems are in P.

1. Given a set B of abducibles, an observation O , and a pair (E, F) of sets of abducibles, decide whether (E, F) is a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$.
2. Given a set B of abducibles and an observation O , decide whether a constrained explanation for O wrt $\langle R \cup B, \mathcal{A}, \emptyset \rangle$ exists.

5 Related Work

Abduction was introduced to artificial intelligence in early 1970s by Harry Pople Jr. (1973), where it is now commonly understood as the *inference to the best explanation* (Josephson and Josephson 1996). Over the years several criteria have been proposed to identify the preferred (best) explanations, all rooted in the Occam's razor parsimony principle. The most commonly considered one is subset-minimality (Bylander et al. 1991; Konolige 1992; Selman and Levesque 1990). A more restrictive condition of minimum cardinality has also been broadly studied (Peng and Reggia 1990). The abduction reasoning formalism we study in the paper uses logic programs to represent background knowledge in abductive theories. It is referred to as *abductive logic programming* (Eshghi and Kowalski 1989; Kakas et al. 1992; Dung 1991). Abductive explanations which allow the removal of hypotheses are first introduced by Inoue and Sakama (1995). The importance of abductive logic programming to knowledge representation was argued by Denecker and Schreye (1995). It was applied in diagnosis (Console et al. 1996), planning (Eshghi 1988; do Lago Pereira and de Barros 2004), natural language understanding (Balsa et al. 1995), and case-based reasoning (Satoh 1996). Denecker and Kakas (2002) provide a comprehensive survey of the area.

The complexity of abductive reasoning was first studied by Bylander et al. (1991). Eiter et al. (1997) studied the complexity of reasoning tasks in the abductive logic programming setting.

The profusion of abductive explanations was explicitly noted by Maher (2005) in his work on constraint abduction. Maher considers a different setting and handles the problem by different techniques. The key similarity is that in Maher's setting, as in ours, symbols from the vocabulary not present in the theory can give rise to alternative explanations.

None of the earlier works on abduction considered the concepts of constrainedness or arbitrariness. These concepts were proposed by us for the setting of view updates in deductive databases (Caroprese et al. 2012). View updating consists of modifying base relations to impose properties on view relations, that is, relations defined on the database by queries. The subject has received much attention in database research (cf. the survey papers by Fraternali and Paraboschi (1993), and Mayol and Teniente (1999)). There is a natural connection between view updating and abduction (Kakas and Mancarella 1990a; Console et al. 1995). The view plays the role of the background theory, all base relation symbols are abducible predicates, and requests for a view

update are observations. The role of integrity constraints is the same in both areas. However, there is an important distinction here. In view updating, integrity constraints concern only base relation, a restriction not present in abductive settings. Our present work adapts the notions of constrainedness and arbitrariness to the more general setting of abduction. Importantly, we introduce the new concept of the degree of arbitrariness, which allows us to compare explanations even when no constrained explanations exist.

6 Concluding Remarks

We proposed a new approach to limiting the space of explanations in abductive logic programming extended by integrity constraints. Specifically, we introduced the degree of arbitrariness as a measure of the quality of an explanation. It imposes a hierarchy on the space of explanations (possibly already narrowed down by means of other principles): the lower the degree of arbitrariness, the more compelling an explanation. Explanations with the degree of arbitrariness equal to 0 are particularly important. We presented a detailed account of the complexity of reasoning with constrained explanations when programs in abductive theories are stratified. In our discussion in Sections 2 and 3 we were implicitly assuming that the set $sem_S(P)$ consisted of two-valued interpretations. However, our definitions can also be extended to the three-valued well-founded semantics. Moreover, since for stratified programs the well-founded and the stable-model semantics coincide, all complexity results we obtained hold for that case, too. Finally, our approach applies to each of the four basic models of abductive reasoning, three of which have been studied before, with the remaining one (Definition 4) to the best of our knowledge being new. Our discussion and the results suggest that the notions of the degree of arbitrariness and constrainedness are important additions to the space of fundamental principles of abductive reasoning. We note, however, that as with other principles there are situations where the minimum degree of arbitrariness may not be the appropriate principle to use for abduction. For example, let us consider the transitive closure program containing two rules: $tc(x, y) \leftarrow r(x, y)$ and $tc(x, y) \leftarrow r(x, z), tc(z, y)$ with an abducible predicate r . If we observe $tc(a, b)$, the explanation $E_1 = (\{r(a, b)\}, \emptyset)$ is constrained while $E_2 = (\{r(a, z), r(z, b)\}, \emptyset)$ is arbitrary (z can be replaced by any constant in the vocabulary of the language). Thus, according to the minimum degree of arbitrariness criterion, E_1 is preferred to E_2 . However, if the relation tc is the ancestor relation, r is the parent relation and $ancestor(a, b)$ is observed, preferring the constrained explanation $\{parent(a, b)\}$ to $\{parent(a, c), parent(c, b)\}$, which is arbitrary, may be a point of dispute. Establishing conditions to help decide which criteria to use when is a grand challenge of the area of abduction. For the example we just gave, we note that the minimum cardinality principle would give the same result as the minimum arbitrariness one and would be open to the same question. Our work opens several avenues for future research. First, we do not have a clear picture of the complexity of the case of abductive theories with non-recursive Horn programs and with integrity constraints restricted only to abducible predicates. This case is of interest due to its view updating applications. Next, there is a challenging problem of resolving the complexity of reasoning with constrained explanations in the case of abductive theories with arbitrary programs.

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