

Model Checking Existential Logic on Partially Ordered Sets

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Abstract

We study the problem of checking whether an existential sentence (that is, a first-order sentence in prefix form built using existential quantifiers and all Boolean connectives) is true in a finite partially ordered set (in short, a poset). A poset is a reflexive, antisymmetric, and transitive digraph. The problem encompasses the fundamental embedding problem of finding an isomorphic copy of a poset as an induced substructure of another poset.

Model checking existential logic is already NP-hard on a fixed poset; thus we investigate structural properties of posets yielding conditions for fixed-parameter tractability when the problem is parameterized by the sentence. We identify width as a central structural property (the width of a poset is the maximum size of a subset of pairwise incomparable elements); our main algorithmic result is that model checking existential logic on classes of finite posets of bounded width is fixed-parameter tractable. We observe a similar phenomenon in classical complexity, where we prove that the isomorphism problem is polynomial-time tractable on classes of posets of bounded width; this settles an open problem in order theory.

We surround our main algorithmic result with complexity results on less restricted, natural neighboring classes of finite posets, establishing its tightness in this sense. We also relate our work with (and demonstrate its independence of) fundamental fixed-parameter tractability results for model checking on digraphs of bounded degree and bounded clique-width.

Categories and Subject Descriptors F.4.1 [Theory of Computation]: Mathematical Logic and Formal Languages—Mathematical Logic; G.2.0 [Mathematics of Computing]: Discrete Mathematics—General

General Terms Theory

Keywords Partially Ordered Sets, Model Checking, Width, Parameterized Complexity.

1. Introduction

Motivation. The *model checking problem*, to decide whether a given logical sentence is true in a given structure, is a fundamental computational problem which appears in a variety of areas in computer science, including database theory, artificial intelligence,

constraint satisfaction, and computational complexity. The problem is computationally intractable in its general version, and hence it is natural to seek restrictions of the class of structures or the class of sentences yielding sufficient or necessary conditions for computational tractability.

Here, as usual in the complexity investigation of the model checking problem, computational tractability refers to *polynomial-time tractability* or, in cases where polynomial-time tractability is unlikely, a relaxation known as *fixed-parameter tractability with the sentence as a parameter*. The latter guarantees a decision algorithm running in $f(k) \cdot n^c$ time on inputs of size n and sentences of size k , where f is a computable function and c is a constant. For further discussion of the complexity setup adopted here, including its algorithmic motivations, we refer the reader to [9, 13].

The study of model checking first-order logic on restricted classes of finite *combinatorial structures* is an established line of research originating from the seminal work of Seese [20]. Results in this area have provided very general conditions for computational tractability, and even exact characterizations in many relevant cases [14]. As Grohe observes [13], though, it would be also interesting to investigate structural properties facilitating the model checking problem in the realm of finite *algebraic structures*, for instance groups or lattices.

In this paper, we investigate the class of finite *partially ordered sets*. A partially ordered set (in short, a *poset*) is the structure obtained by equipping a nonempty set with a reflexive, antisymmetric, and transitive binary relation. In other words, the class of posets coincides with the class of directed graphs satisfying a certain universal first-order sentence (axiom); namely, the sentence that enforces reflexivity, antisymmetry, and transitivity of the edge relation. In this sense, from a logical perspective, posets form an intermediate case between combinatorial and algebraic structures; they can be viewed as being stronger than purely combinatorial structures, as the non-logical vocabulary is presented by a first-order axiomatization; but weaker than genuinely algebraic structures, as the axiomatization is expressible in universal first-order logic (too weak of a fragment to define algebraic operations).

Posets are fundamental combinatorial objects [12, Chapter 8], with applications in many fields of computer science, ranging from software verification [16] to computational biology [18]. However, very little is known about the complexity of the model checking problem on classes of finite posets; to the best of our knowledge, even the complexity of natural syntactic fragments of first-order logic on basic classes of finite posets is open.

A prominent logic in first-order model-checking is *primitive positive* logic, that is, first-order sentences built using existential quantification ($\exists$) and conjunction ($\wedge$); the problem of model checking primitive positive logic is equivalent to the *constraint satisfaction problem* and the *homomorphism problem* [7]. However, restricted to posets, the problem of model checking primitive positive logic and even *existential positive* logic, obtained from primitive positive logic by including disjunction ($\vee$) in the logical

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CSL-LICS 2014, July 14–18, 2014, Vienna, Austria.

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<http://dx.doi.org/10.1145/2603088.2603110>

vocabulary, is trivial; because of reflexivity, every existential positive sentence is true on every poset!

As we observe (Proposition 2), the complexity scenario changes abruptly in *existential conjunctive* logic, that is, first-order sentences in prefix negation normal form built using $\exists$, $\wedge$, and negation ($\neg$). Here, the model checking problem is NP-hard even on a certain fixed finite poset; in the complexity jargon, the *expression* complexity of existential conjunctive logic is NP-hard on finite posets. In other words, as long as computational tractability is identified with polynomial-time tractability, any structural property of posets is algorithmically immaterial (in a sense that can be made precise). There is then a natural quest for relaxations of polynomial-time tractability yielding (i) a nontrivial complexity analysis of the problem, and (ii) a refined perspective on the structural properties of posets underlying tamer algorithmic behaviors; in this paper we achieve (i) and (ii) through the glasses of fixed-parameter tractability.

More precisely, as we discuss below, our contribution is a complete description of the parameterized complexity of model checking (all syntactic fragments of) existential first-order logic (first-order sentences in prefix normal form built using $\exists$, $\wedge$, $\vee$, and $\neg$), with respect to classes of finite posets in a hierarchy generated by fundamental poset invariants.¹

Model checking existential logic encompasses as a special case the fundamental *embedding problem*, to decide whether a given structure contains an isomorphic copy of another given structure as an *induced* substructure; in fact, the embedding problem reduces in polynomial-time to the problem of model checking certain existential (even conjunctive) sentences. The aforementioned fact that existential conjunctive logic is already NP-hard on a fixed finite poset leaves open the existence of a nontrivial classical complexity classification of the embedding problem. We provide such a classification by giving a complete description of the classical complexity of the embedding problem in the introduced hierarchy of poset invariants.

We hope that the investigation of the existential fragment prepares the ground (and possibly provides basic tools) for understanding the model checking problem for more expressive logics on posets.

Contribution. We now give an account of our contribution. We refer the reader to Figure 1 for an overview; the poset invariants and their relations are introduced in Section 3.

In contrast to the classical case, model checking existential logic on fixed structures is trivially fixed-parameter tractable; in fact, even the full first-order logic is trivially fixed-parameter tractable on any class of finite structures of bounded size. On the other hand, there exist classes of finite posets where existential logic is unlikely to be fixed-parameter tractable (in fact, there exist classes where even the embedding problem is W[1]-hard); but the reduction class given by the natural hardness proof is rather wild, in particular it has bounded depth but *unbounded width* (Proposition 4).

The *width* of a poset is the maximum size of a subset of pairwise incomparable elements (antichain); along with its *depth*, the maximum size of a subset of pairwise comparable elements (chain), these two invariants form the basic and fundamental structural properties of a poset, arguably its most prominent and natural features. Our main result establishes that width helps algorithmically (in contrast to depth); specifically, we prove that *model checking existential logic on classes of finite posets of bounded width is fixed-parameter tractable* (Theorem 5). This, together with Seese's

¹Note that existential *disjunctive* logic (first-order sentences in prefix negation normal form built using $\exists$, $\vee$, and $\neg$) is trivial on posets. In fact, every sentence in the fragment is either true on every poset, or false on every poset, and it is easy to check which of the two cases holds for any given sentence.

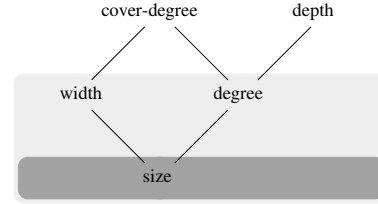


Figure 1. The (light or dark) gray region covers invariants such that, if a class of finite posets is bounded under the invariant, then model checking existential logic (or equivalently, by Proposition 1, model checking existential conjunctive logic, or deciding embedding) over the class is fixed-parameter tractable; the white region covers invariants such that there exists a class of finite posets bounded under the invariant where the problem is W[1]-hard. Similarly, the dark gray region covers invariants where the embedding problem is polynomial-time tractable, and the complement of the dark gray region (light gray or white) covers invariants where the problem is NP-hard. In classical complexity, as opposed to parameterized complexity, the tractability frontier of existential (conjunctive) logic and embedding are different (the former, since existential logic is already NP-hard on a fixed finite poset, is NP-hard everywhere).

algorithm (plus a routine reduction described in Proposition 6), allows us to complete the parameterized complexity classification of the investigated poset invariants, as depicted in Figure 1.

We believe that our tractability result essentially enlightens the fundamental feature of posets of bounded width that can be exploited algorithmically; namely, *bounded width posets admit a polynomial-time compilation to certain semilattice structures*, which are algorithmically tamer than the original posets, but equally expressive with respect to the problem at hand. The proof proceeds in two stages. We first prove that, on any class of finite relational structures, model checking existential logic is fixed-parameter tractable if and only if the embedding problem is fixed-parameter tractable (Proposition 1). Next, using the color coding technique of Alon, Yuster, and Zwick [1], we reduce an instance of the embedding problem on posets of bounded width to a suitable family of instances of the homomorphism problem of certain semilattice structures, which is polynomial-time tractable by classical results of Jeavons, Cohen, and Gyssens [15].

Our approach is reminiscent of the well established fact in order theory that finite posets correspond exactly (in a sense that can be made precise in category-theoretic terms) to finite distributive lattices. However, the algorithmic implications of this correspondence have been possibly overlooked. Indeed, using the correspondence and the known fact that the isomorphism problem is polynomial-time tractable on finite distributive lattices, we prove that *the isomorphism problem for posets of bounded width is polynomial-time tractable* (Theorem 8), which settles an open question in order theory [3, p. 284].

Motivated by the equivalence (in parameterized complexity) between embedding and model checking existential conjunctive logic (Proposition 1) on one hand, and the fact that existential conjunctive logic is already NP-hard on a fixed finite poset (Proposition 2) on the other hand, we also revisit the classical complexity of the embedding problem for finite posets and classify it with respect to the poset invariants studied in the parameterized complexity setting. The outcome is pictured in Figure 1; here, polynomial-time tractability of the embedding problem on posets of bounded size is optimal with respect to the studied poset invariants. We remark that the hardness results are technically involved (Theorem 6 and Theorem 7); in particular, bounded width is a known obstruction

for hardness proofs (for instance, the complexity of the dimension problem is unknown on bounded width posets).

We conclude mentioning that our work on posets relates with, but is independent of, general results by Seese [20] and Courcelle, Makowsky, and Rotics [4], respectively, on model checking first-order logic on classes of finite graphs of bounded degree and bounded clique-width. Namely, the order relation of a poset has bounded degree if and only if the poset has bounded depth and bounded *cover-degree* (that is, its cover relation has bounded degree); moreover, if a poset has bounded width, then it has bounded cover-degree (Proposition 3). However, *there exist classes of bounded width posets with unbounded degree* (for instance, chains), and *there exist classes of bounded width posets with unbounded clique-width* (Proposition 5), which excludes the direct application of the aforementioned results.

Throughout the paper, we mark with $\star$ all statements whose proofs are omitted; we refer to [2] for a full version.

2. Preliminaries

For all integers $k \geq 1$, we let $[k]$ denote the set $\{1, \dots, k\}$.

Logic. In this paper, we focus on relational first-order logic. A *vocabulary* σ is a finite set of *relation symbols*, each of which is associated to a natural number called its *arity*; we let $\text{ar}(R)$ denote the arity of $R \in \sigma$. An *atom* α (over vocabulary σ) is an equality of variables ($x = y$) or is a predicate application $Rx_1 \dots x_{\text{ar}(R)}$, where $R \in \sigma$ and $x_1, \dots, x_{\text{ar}(R)}$ are variables. A *formula* (over vocabulary σ) is built from atoms (over σ), conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$), universal quantification ($\forall$), and existential quantification ($\exists$). A *sentence* is a formula having no free variables. We let $\mathcal{FO}$ denote the class of first-order sentences in *prefix negation normal form*, that is, for each $\phi \in \mathcal{FO}$, the quantifiers occur in front of the sentence and the negations occur in front of the atoms.

Let ρ be a subset of $\{\forall, \exists, \wedge, \vee, \neg\}$ containing at least one quantifier and at least one binary connective. We let $\mathcal{FO}(\rho) \subseteq \mathcal{FO}$ denote the *syntactic fragment* of $\mathcal{FO}$ -sentences built using only logical symbols in ρ . We call $\mathcal{FO}(\exists, \wedge, \vee, \neg)$ the *existential fragment*, $\mathcal{FO}(\exists, \wedge, \neg)$ the *existential conjunctive fragment*, and $\mathcal{FO}(\exists, \wedge)$ the *existential conjunctive positive* (or *primitive positive*) fragment.

Structures. Let σ be a relational vocabulary. A *structure* $\mathbf{A}$ (over σ) is specified by a nonempty set A , called the *universe* of the structure, and a relation $R^{\mathbf{A}} \subseteq A^{\text{ar}(R)}$ for each relation symbol $R \in \sigma$. A structure is *finite* if its universe is finite.

All structures considered in this paper are finite.

Given a structure $\mathbf{A}$ and $B \subseteq A$, we denote by $\mathbf{A}|_B$ the substructure of $\mathbf{A}$ induced by B , namely the universe of $\mathbf{A}|_B$ is B and $R^{\mathbf{A}|_B} = R^{\mathbf{A}} \cap B^{\text{ar}(R)}$ for all $R \in \sigma$.

Let $\mathbf{A}$ and $\mathbf{B}$ be σ -structures. A *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $h: A \rightarrow B$ such that $(a_1, \dots, a_{\text{ar}(R)}) \in R^{\mathbf{A}}$ implies $(h(a_1), \dots, h(a_{\text{ar}(R)})) \in R^{\mathbf{B}}$, for all $R \in \sigma$ and all $(a_1, \dots, a_{\text{ar}(R)}) \in A^{\text{ar}(R)}$; a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ is *strong* if $(a_1, \dots, a_{\text{ar}(R)}) \notin R^{\mathbf{A}}$ implies $(h(a_1), \dots, h(a_{\text{ar}(R)})) \notin R^{\mathbf{B}}$. An *embedding* from $\mathbf{A}$ to $\mathbf{B}$ is an injective strong homomorphism from $\mathbf{A}$ to $\mathbf{B}$. An *isomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a bijective embedding from $\mathbf{A}$ to $\mathbf{B}$.

In graph theory, an injective strong homomorphism is also called a “strong embedding”, and the term “embedding” is used in the weaker sense of injective homomorphism; here, we adopt the order-theoretic (and model-theoretic) terminology.

For a structure $\mathbf{A}$ and a sentence ϕ over the same vocabulary, we write $\mathbf{A} \models \phi$ if the sentence ϕ is *true* in the structure $\mathbf{A}$. When $\mathbf{A}$ is a structure, f is a mapping from variables to the universe of $\mathbf{A}$, and

$\psi(x_1, \dots, x_n)$ is a formula over the vocabulary of $\mathbf{A}$, we liberally write $\mathbf{A} \models \psi(f(x_1), \dots, f(x_n))$ to indicate that ψ is satisfied by $\mathbf{A}$ and f .

A structure $\mathbf{G} = (V, E^{\mathbf{G}})$ with $\text{ar}(E) = 2$ is called a *digraph*, and a *graph* if $E^{\mathbf{G}}$ is irreflexive and symmetric. We let $\mathcal{G}$ denote the class of all graphs. Let $\mathbf{G}$ be a digraph. The *degree* of $g \in G$, in symbols $\text{degree}(g)$, is equal to $|\{(g', g) \in E^{\mathbf{G}} \mid g' \in G\} \cup \{(g, g') \in E^{\mathbf{G}} \mid g' \in G\}|$, and the *degree* of $\mathbf{G}$, in symbols $\text{degree}(\mathbf{G})$, is the maximum degree attained by the elements of $\mathbf{G}$.

A digraph $\mathbf{P} = (P, \leq^{\mathbf{P}})$ is a *poset* if $\leq^{\mathbf{P}}$ is a *reflexive*, *antisymmetric*, and *transitive* relation over P , that is, respectively, $\mathbf{P} \models \forall x(x \leq x)$, $\mathbf{P} \models \forall x \forall y((x \leq y \wedge y \leq x) \rightarrow x = y)$, and $\mathbf{P} \models \forall x \forall y \forall z((x \leq y \wedge y \leq z) \rightarrow x \leq z)$.

A *chain* in $\mathbf{P}$ is a subset $C \subseteq P$ such that $p \leq^{\mathbf{P}} q$ or $q \leq^{\mathbf{P}} p$ for all $p, q \in C$ (in particular, if P is a chain in $\mathbf{P}$, we call $\mathbf{P}$ itself a chain). We say that p and q are *incomparable* in $\mathbf{P}$ (denoted $p \parallel^{\mathbf{P}} q$) if $\mathbf{P} \not\models p \leq q \vee q \leq p$. An *antichain* in $\mathbf{P}$ is a subset $A \subseteq P$ such that $p \parallel^{\mathbf{P}} q$ for all $p, q \in A$ (in particular, if P is an antichain in $\mathbf{P}$, we call $\mathbf{P}$ itself an antichain).

Let $\mathbf{P}$ be a poset and let $p, q \in P$. We say that q *covers* p in $\mathbf{P}$ (denoted $p \prec^{\mathbf{P}} q$) if $p <^{\mathbf{P}} q$ and, for all $r \in P$, $p \leq^{\mathbf{P}} r <^{\mathbf{P}} q$ implies $p = r$. The *cover graph* of $\mathbf{P}$ is the digraph $\text{cover}(\mathbf{P})$ with vertex set P and edge set $\{(p, q) \mid p \prec^{\mathbf{P}} q\}$. If $\mathcal{P}$ is a class of posets, we let $\text{cover}(\mathcal{P}) = \{\text{cover}(\mathbf{P}) \mid \mathbf{P} \in \mathcal{P}\}$. It is well known that computing the cover relation corresponding to a given order relation, and vice versa the order relation corresponding to a given cover relation, is feasible in polynomial time [19].

In the figures, posets are represented by their *Hasse diagrams*, that is a diagram of their cover relation where all edges are intended oriented upwards.

Let $\mathcal{P}$ be the class of all posets. A *poset invariant* is a mapping $\text{inv}: \mathcal{P} \rightarrow \mathbb{N}$ such that $\text{inv}(\mathbf{P}) = \text{inv}(\mathbf{Q})$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic. Let inv be any invariant over $\mathcal{P}$. Let $\mathcal{P}$ be any class of posets. We say that $\mathcal{P}$ is *bounded* with respect to inv if there exists $b \in \mathbb{N}$ such that $\text{inv}(\mathbf{P}) \leq b$ for all $\mathbf{P} \in \mathcal{P}$. Two poset invariants are *incomparable* if there exists a class of posets bounded under the first but unbounded under the second, and there exists a class of posets bounded under the second but unbounded under the first.

Problems. We refer the reader to [9] for the standard algorithmic setup of the model checking problem, including the underlying computational model, encoding conventions for input structures and sentences, and the notion of *size* of the (encoding of an) input structure or sentence. We also refer the reader to [9] for further background in parameterized complexity theory (including the notion of *fpt many-one reduction* and *fpt Turing reduction*).

Here, we mention that a *parameterized problem* (Q, κ) is a *problem* $Q \subseteq \Sigma^*$ together with a *parameterization* $\kappa: \Sigma^* \rightarrow \mathbb{N}$, where Σ is a finite alphabet. A parameterized problem (Q, κ) is *fixed-parameter tractable* (with respect to κ), in short *fpt*, if there exists a decision algorithm for Q , a computable function $f: \mathbb{N} \rightarrow \mathbb{N}$, and a polynomial function $p: \mathbb{N} \rightarrow \mathbb{N}$, such that for all $x \in \Sigma^*$, the running time of the algorithm on x is at most $f(\kappa(x)) \cdot p(|x|)$. We provide evidence that a parameterized problem is not fixed-parameter tractable by proving that the problem is $\text{W}[1]$ -hard under *fpt many-one reductions*; this holds unless the exponential time hypothesis fails [9].

The (parameterized) computational problems under consideration are the following. Let σ be a relational vocabulary, $\mathcal{C}$ be a class of σ -structures, and $\mathcal{L} \subseteq \mathcal{FO}$ be a class of σ -sentences. The *model checking problem* for $\mathcal{C}$ and $\mathcal{L}$, in symbols $\text{MC}(\mathcal{C}, \mathcal{L})$, is the problem of deciding, given $(\mathbf{A}, \phi) \in \mathcal{C} \times \mathcal{L}$, whether $\mathbf{A} \models \phi$. The parameterization, given an instance $(\mathbf{A}, \phi)$, returns the size of the encoding of ϕ . The *embedding problem* for $\mathcal{C}$, in symbols

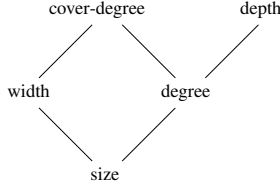


Figure 2. The order of poset invariants induced by Proposition 3.

$\text{EMB}(\mathcal{C})$, is the problem of deciding, given a pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A}$ is a σ -structure and $\mathbf{B}$ is a σ -structure in $\mathcal{C}$, whether $\mathbf{A}$ embeds into $\mathbf{B}$. The parameterization, given an instance $(\mathbf{A}, \mathbf{B})$, returns the size of the encoding of $\mathbf{A}$. The problems $\text{HOM}(\mathcal{C})$ and $\text{ISO}(\mathcal{C})$ are defined similarly in terms of homomorphisms and isomorphisms respectively.

3. Basic Results

In this section, we set the stage for our parameterized and classical complexity results in Section 4 and Section 5 respectively. We start observing some basic reducibilities between the problems under consideration.

Proposition 1 $(\star)$. *Let $\mathcal{C}$ be a class of structures. The following are equivalent.*

- (i) $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ is fixed-parameter tractable.
- (ii) $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \neg))$ is fixed-parameter tractable.
- (iii) $\text{EMB}(\mathcal{C})$ is fixed-parameter tractable.

In particular, $\text{EMB}(\mathcal{C})$ polynomial-time (thus fpt) many-one reduces to $\text{MC}(\mathcal{C}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$.

The next observation is that model checking existential conjunctive logic (and thus the full existential logic) on posets is unlikely to be polynomial-time tractable, even if the poset is fixed. Let $\mathbf{B}$ be the bowtie poset defined by the universe $B = [4]$ and the covers $i \prec^{\mathbf{B}} j$ for all $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

Proposition 2 $(\star)$. $\text{MC}(\{\mathbf{B}\}, \mathcal{FO}(\exists, \wedge, \neg))$ is NP-hard.

In contrast, model checking existential logic on any fixed poset $\mathbf{P}$ is trivially fixed-parameter tractable (the instance is a structure of constant size, and a sentence taken as a parameter). However, there are classes of posets where the embedding problem, and hence, by Proposition 1, the problem of model checking existential logic, is unlikely to be fixed-parameter tractable, as we now show.

First, we introduce a family of poset invariants and relate them as in Figure 2. Let $\mathbf{P}$ be a poset.

- The *size* of $\mathbf{P}$ is the cardinality of its universe, $|\mathbf{P}|$.
- The *width* of $\mathbf{P}$, in symbols $\text{width}(\mathbf{P})$, is the maximum size attained by an antichain in $\mathbf{P}$.
- The *depth* of $\mathbf{P}$, in symbols $\text{depth}(\mathbf{P})$, is the maximum size attained by a chain in $\mathbf{P}$.
- The *degree* of $\mathbf{P}$, in symbols $\text{degree}(\mathbf{P})$, is the degree of the order relation of $\mathbf{P}$, that is, $\text{degree}(\leq^{\mathbf{P}})$.
- The *cover-degree* of $\mathbf{P}$, in symbols $\text{cover-degree}(\mathbf{P})$, is the degree of the cover relation of $\mathbf{P}$, that is, $\text{degree}(\text{cover}(\mathbf{P}))$.

Proposition 3 $(\star)$. *Let $\mathcal{P}$ be a class of posets.*

- (i) $\mathcal{P}$ has bounded degree if and only if $\mathcal{P}$ has bounded depth and bounded cover-degree.
- (ii) If $\mathcal{P}$ has bounded width, then $\mathcal{P}$ has bounded cover-degree.

(iii) $\mathcal{P}$ has bounded size if and only if $\mathcal{P}$ has bounded width and bounded degree.

The previous proposition, together with the observation that bounded width and bounded degree (bounded width and bounded depth, bounded cover-degree and bounded depth, respectively) are incomparable, justifies the order in Figure 2, whose interpretation is the following: invariant inv is below invariant inv' if and only if, for every class $\mathcal{P}$ of posets, if $\mathcal{P}$ is bounded under inv , then $\mathcal{P}$ is bounded under inv' .

The emerging hierarchy of poset invariants will provide a measure of tightness for our positive algorithmic results, once we will manage to surround them with complexity results on covering neighboring classes.

To this aim, we immediately observe that there exists a class of posets of bounded depth where the embedding problem, and hence model-checking existential first-order logic, is $\text{W}[1]$ -hard. Given any graph $\mathbf{G} \in \mathcal{G}$, construct a poset $r(\mathbf{G}) = \mathbf{P}$ by taking $|G|$ pairwise disjoint 3-element chains, and covering the bottom of the i th chain by the top of the j th chain if and only if i and j are adjacent in $\mathbf{G}$. Note that $\text{depth}(\mathbf{P}) \leq 3$. Hence, the class $\mathcal{P}_{\text{depth}} = \{r(\mathbf{G}) \mid \mathbf{G} \in \mathcal{G}\}$ has bounded depth.

Proposition 4. $\text{EMB}(\mathcal{P}_{\text{depth}})$ is $\text{W}[1]$ -hard.

Proof. CLIQUE fpt many-one reduces to $\text{EMB}(\mathcal{P}_{\text{depth}})$ by mapping $(\mathbf{G}, k)$ to $(r(\mathbf{K}_k), r(\mathbf{G}))$. $\square$

The goal of the technical part of the paper is to establish the facts leading from Figure 2 to Figure 1:

- For the parameterized complexity of model checking existential logic, we have tractability on bounded degree classes by Seese's algorithm [20], and hardness on (certain) bounded depth classes by Proposition 4. In Section 4, we establish tractability on bounded width classes by Theorem 5, and hardness on (certain) bounded cover-degree classes by Proposition 6.
- For the classical complexity of the embedding problem (Section 5), Proposition 7 establishes tractability on bounded size classes, Theorem 6 establishes hardness on (certain) bounded width classes, and Theorem 7 establishes hardness on (certain) bounded degree classes.

We conclude the section by relating our work on posets of bounded width with previous work on digraphs of bounded clique-width, and showing that our results are indeed independent.

Clique-width is a prominent invariant of undirected as well as directed graphs which generalizes treewidth [5]; in particular, it is known that monadic second-order logic (precisely, MSO_1) is fixed-parameter tractable on digraphs of bounded clique-width [4], thus:

Observation 1. $\text{MC}(\mathcal{P}, \mathcal{FO})$ is fixed-parameter tractable for any class $\mathcal{P}$ of posets such that the clique-width of $\mathcal{P}$ is bounded.

Since it is possible to compute the cover relation from the order relation (and vice versa) in polynomial time, one might wonder whether using the clique-width of the cover graph would allow us to efficiently model check wider classes of posets. This turns out not to be the case:

Observation 2 (follows from Examples 1.32, 1.33 and Corollary 1.53 of [6]). *For any class $\mathcal{P}$ of posets, the clique-width of $\mathcal{P}$ is bounded if and only if the clique-width of $\text{cover}(\mathcal{P})$ is bounded.*

A natural class of posets which is easily observed having clique-width bounded by 2 (despite having unbounded treewidth) is the class of *series parallel posets*. However, we show that there exist classes of posets of bounded width which do not have bounded clique-width (if not Theorem 5 would follow from Observation 1).

Proposition 5 (*). *There exists a class $\mathcal{P}$ of posets which has bounded width but does not have bounded clique-width.*

4. Parameterized Complexity

In this section, we study the parameterized complexity of the problems under consideration. The section is organized as follows.

- In Subsection 4.1, we develop a fixed-parameter tractable algorithm for the embedding problem on posets of bounded width (Theorem 4), which yields that model checking existential logic on such posets is fixed-parameter tractable (Theorem 5).
- In Subsection 4.2, we provide a reduction proving W[1]-hardness of model checking existential logic on posets of bounded cover-degree (Proposition 6).

4.1 Embedding is FPT on Bounded Width Posets

We first outline our proof strategy. The core of the proof lies in defining a suitable compilation of bounded width posets. We then proceed in two steps:

- proving that the homomorphism problem is polynomial-time tractable on such compilations, and
- reducing the embedding problem between two bounded width posets to fpt many instances of the homomorphism problem between compilations of these posets.

For (i), we prove that the compilation admits a semilattice polymorphism (Lemma 1), and use the classical result by Jeavons et al. that the homomorphism problem is polynomial-time tractable on semilattice structures (Theorem 1). For (ii), we use color coding and hash functions (Theorem 2) to link a homomorphism between two compilations to the existence of an embedding between the compiled posets (Lemma 2).

4.1.1 Known Facts

The proof uses known facts about semilattice structures and hash functions, collected below.

Semilattice Polymorphisms. Let σ be a finite relational vocabulary, and let $\mathbf{A}$ be a σ -structure. Let $f: A^m \rightarrow A$ be an m -ary function on A . We say that f is a *polymorphism* of $\mathbf{A}$ (or, $\mathbf{A}$ admits f) if f preserves all relations of $\mathbf{A}$, that is, for all $R \in \sigma$, where $\text{ar}(R) = r$, if

$$(a_{1,1}, a_{1,2}, \dots, a_{1,r}), \dots, (a_{m,1}, a_{m,2}, \dots, a_{m,r}) \in R^{\mathbf{A}},$$

then

$$(f(a_{1,1}, a_{2,1}, \dots, a_{m,1}), \dots, f(a_{1,r}, a_{2,r}, \dots, a_{m,r})) \in R^{\mathbf{A}}.$$

We say that a function $f: A^2 \rightarrow A$ is a *semilattice* function over A if f is idempotent, associative, and commutative on A , that is, $f(a, a) = a$, $f(a, f(a', a'')) = f(f(a, a'), a'')$, and $f(a, a') = f(a', a)$ for all $a, a', a'' \in A$.

Theorem 1 ([15]). *Let $\mathbf{A}$ be a σ -structure, and let f be a semilattice function over A . If f is a polymorphism of $\mathbf{A}$, then $\text{HOM}(\mathbf{A})$ is polynomial-time tractable.*

Hash Functions. Let M and N be sets, and let $k \in \mathbb{N}$. A *k-perfect family of hash functions* from M to N is a family Λ of functions from M to N such that for every subset $K \subseteq M$ of cardinality k there exists $\lambda \in \Lambda$ such that $\lambda|_K$ is injective.

Theorem 2. [Theorem 13.14, [9]] *Let C be a finite set. There exists an algorithm that, given C and $k \in \mathbb{N}$, computes a k-perfect family $\Lambda_{C,k}$ of hash functions from C to $[k]$ of cardinality $2^{O(k)} \cdot \log^2 |C|$ in time $2^{O(k)} \cdot |C| \cdot \log^2 |C|$.*

4.1.2 Semilattice Compilation

Let $\mathbf{P}$ be a poset. Let $(i_1, \dots, i_a) \in \mathbb{N}^a$ be a tuple of numbers. A *chain partition* of $\mathbf{P}$ is a tuple $(\mathbf{C}_1, \dots, \mathbf{C}_a)$ such that $\emptyset \neq C_{i_j} \subseteq P$ for all $j \in [a]$, $P = \bigcup_{j \in [a]} C_{i_j}$, $C_{i_j} \cap C_{i_{j'}} = \emptyset$ for all $1 \leq j < j' \leq a$, $\mathbf{C}_{i_j}$ is the substructure of $\mathbf{P}$ induced by C_{i_j} , and $\mathbf{C}_{i_j}$ is a chain.

Example 1. *Let $\mathbf{Q}$ be the poset with universe $Q = D_1 \cup D_2$, where $D_1 = \{d_{11}, d_{12}, d_{13}, d_{14}\}$ and $D_2 = \{d_{21}, d_{22}, d_{23}, d_{24}\}$, and cover relation $d_{11} \prec^{\mathbf{Q}} d_{12} \prec^{\mathbf{Q}} d_{13} \prec^{\mathbf{Q}} d_{14}$, $d_{21} \prec^{\mathbf{Q}} d_{22} \prec^{\mathbf{Q}} d_{23} \prec^{\mathbf{Q}} d_{24}$, $d_{11} \prec^{\mathbf{Q}} d_{23}$, $d_{12} \prec^{\mathbf{Q}} d_{24}$, and $d_{22} \prec^{\mathbf{Q}} d_{13}$. Then, $(\mathbf{D}_1, \mathbf{D}_2)$ is a chain partition of $\mathbf{Q}$. See Figure 3 (left).*

Let $\mathbf{P}$ be the poset with universe $P = C_1 \cup C_2$, where $C_1 = \{c_{11}, \dots, c_{16}\}$ and $C_2 = \{c_{21}, \dots, c_{26}\}$, and cover relation $c_{11} \prec^{\mathbf{P}} \dots \prec^{\mathbf{P}} c_{16}$, $c_{21} \prec^{\mathbf{P}} \dots \prec^{\mathbf{P}} c_{26}$, $c_{11} \prec^{\mathbf{P}} c_{24}$, $c_{12} \prec^{\mathbf{P}} c_{25}$, $c_{13} \prec^{\mathbf{P}} c_{26}$, $c_{21} \prec^{\mathbf{P}} c_{14}$, $c_{22} \prec^{\mathbf{P}} c_{15}$, and $c_{23} \prec^{\mathbf{P}} c_{16}$. Then, $(\mathbf{C}_1, \mathbf{C}_2)$ is a chain partition of $\mathbf{P}$. See Figure 3 (right).

The mapping $e: Q \rightarrow P$ defined by $e(d_{11}) = c_{11}$, $e(d_{12}) = c_{12}$, $e(d_{13}) = c_{15}$, $e(d_{14}) = c_{16}$, $e(d_{21}) = c_{21}$, $e(d_{22}) = c_{22}$, $e(d_{23}) = c_{24}$, $e(d_{24}) = c_{25}$ embeds $\mathbf{Q}$ into $\mathbf{P}$.

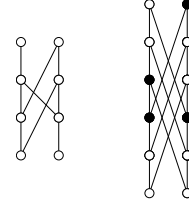


Figure 3. The posets $\mathbf{Q}$ (left) and $\mathbf{P}$ (right) in Example 1. The white points in P form the image of the embedding $e: Q \rightarrow P$ in Example 1.

Theorem 3. [Theorem 1, [8]] *Let $\mathbf{P}$ be a poset. Then, in time $O(\text{width}(\mathbf{P}) \cdot |P|^2)$, it is possible to compute both $\text{width}(\mathbf{P})$ and a chain partition of $\mathbf{P}$ of the form $(\mathbf{C}_1, \dots, \mathbf{C}_{\text{width}(\mathbf{P})})$.*

We are now ready to define the aforementioned compilations. Note that our compilations will depend not only on the poset itself, but also on a chain decomposition of the poset and a family of colorings (the significance of the latter will become clear in the proof of Lemma 2).

Let $\mathbf{P}$ be a poset such that $\text{width}(\mathbf{P}) \leq w$, and let $(\mathbf{C}_1, \dots, \mathbf{C}_w)$ be a chain partition of $\mathbf{P}$. Let $w' \leq w$ and let $(i_1, \dots, i_{w'})$ be a subtuple of $(1, \dots, w)$, that is, $(i_1, \dots, i_{w'})$ is obtained from $(1, \dots, w)$ by deleting $w - w'$ indices. For all $j \in [w']$, let $k_{i_j} \in \mathbb{N}$ be such that $k_{i_j} \leq |C_{i_j}|$. Λ_j be a family of functions from C_{i_j} to $[k_{i_j}]$, and $(\lambda_1, \dots, \lambda_{w'}) \in \Lambda_1 \times \dots \times \Lambda_{w'}$.

For a suitable relational vocabulary σ depending on w' and k_{i_j} for all $j \in [w']$, we define the σ -structure

$$\text{compil}(\mathbf{P}, \mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}, \lambda_1, \dots, \lambda_{w'}),$$

which we call the *compilation* of $\mathbf{P}$ with respect to the *coordinatization* $(\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}})$ and the *coloring* $(\lambda_1, \dots, \lambda_{w'})$, as follows (we use $\text{compil}(\mathbf{P})$ as a shorthand if the coordinatization and the coloring are contextually clear).

The relational vocabulary σ of $\text{compil}(\mathbf{P})$ consists of one binary relation symbol L , two unary relation symbols $I_{\{j, j'\}}$ and $O_{\{j, j'\}}$ for each 2-element subset $\{j, j'\}$ of $[w']$, and one binary relation symbol $R_{(j, k)}$ for each $j \in [w']$ and $k \in [k_{i_j}]$.

The universe of $\text{compil}(\mathbf{P})$ is

$$\text{compil}(\mathbf{P}) = C_{i_1} \times C_{i_2} \times \dots \times C_{i_{w'}}.$$

Let $\mathbf{c} = (c_1, \dots, c_{w'})$ and $\mathbf{c}' = (c'_1, \dots, c'_{w'})$ be elements of $\text{compil}(\mathbf{P})$, and let $K_{(j,k)} = \{c \in C_{i_j} \mid \lambda_j(c) = k\}$. The interpretation of the vocabulary σ in $\text{compil}(\mathbf{P})$ is the following:

- (i) The interpretation of L is the set of all pairs $(\mathbf{c}, \mathbf{c}')$ such that $c_1 \leq^{\mathbf{P}} c'_1, \dots, c_{w'} \leq^{\mathbf{P}} c'_{w'}$.
- (ii) For each 2-element subset $\{j, j'\}$ of $[w']$, $I_{\{j,j'\}}$ and $O_{(j,j')}$ are interpreted, respectively, over $I_{\{j,j'\}} = \{\mathbf{c} \mid c_j \parallel^{\mathbf{P}} c_{j'}\}$ and $O_{(j,j')} = \{\mathbf{c} \mid c_j <^{\mathbf{P}} c_{j'}\}$.
- (iii) For each $j \in [w']$ and $k \in [k_{i_j}]$, $R_{(j,k)}$ is interpreted over the subset of the interpretation of L defined by

$$\{(\mathbf{c}, \mathbf{c}') \in L^{\text{compil}(\mathbf{P})} \mid c_j \in K_{(j,k)}, c_j = c'_{j'}\}.$$

Example 2. Let $\mathbf{Q}$ and $(\mathbf{D}_1, \mathbf{D}_2)$ be as in Example 1. Let the subtuple of $(1, 2)$ be $(1, 2)$ itself. Let $k_1 = k_2 = 4 = |D_1| = |D_2|$. Let $\mu_1: D_1 \rightarrow [k_1]$ be defined by $\mu_1(d_{11}) = 1, \mu_1(d_{12}) = 2, \mu_1(d_{13}) = 3$, and $\mu_1(d_{14}) = 4$. Let $\mu_2: D_2 \rightarrow [k_2]$ be defined by $\mu_2(c_{21}) = 1, \mu_2(c_{22}) = 2, \mu_2(c_{23}) = 3$, and $\mu_2(c_{24}) = 4$. Then, $\text{compil}(\mathbf{Q}, \mathbf{D}_1, \mathbf{D}_2, \mu_1, \mu_2)$ is depicted in Figure 4.

Let $\mathbf{P}$ and $(\mathbf{C}_1, \mathbf{C}_2)$ be as in Example 1. Let the subtuple of $(1, 2)$ be $(1, 2)$ itself. Let $k_1 = k_2 = 4 \leq 6 = |C_1| = |C_2|$. Let $\lambda_1: C_1 \rightarrow [k_1]$ be defined by $\lambda_1(c_{11}) = 1, \lambda_1(c_{12}) = 2, \lambda_1(c_{13}) = 4, \lambda_1(c_{14}) = 1, \lambda_1(c_{15}) = 3$, and $\lambda_1(c_{16}) = 4$. Let $\lambda_2: C_2 \rightarrow [k_2]$ be defined by $\lambda_2(c_{21}) = 1, \lambda_2(c_{22}) = 2, \lambda_2(c_{23}) = 3, \lambda_2(c_{24}) = 3, \lambda_2(c_{25}) = 4$, and $\lambda_2(c_{26}) = 1$. Then, $\text{compil}(\mathbf{P}, \mathbf{C}_1, \mathbf{C}_2, \lambda_1, \lambda_2)$ is depicted in Figure 5.

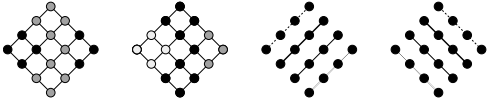


Figure 4. Describing the structure $\text{compil}(\mathbf{Q}, \mathbf{D}_1, \mathbf{D}_2, \mu_1, \mu_2)$ in Example 2. From left to right. The first picture displays the interpretation of L (thin solid edges) and $I_{\{1,2\}}$ (gray points) induced by (i) and (ii). The second picture displays the interpretation of L (thin solid edges), $O_{(2,1)}$ (light gray points), and $O_{(1,2)}$ (dark gray points) induced by (i) and (ii). The third picture displays the interpretation of $R_{(1,1)}$ (dotted edges), $R_{(1,2)}$ (medium solid edges), $R_{(1,3)}$ (thick solid edges), and $R_{(1,4)}$ (dashed edges), as induced by (iii) and λ_1 . Similarly, the fourth picture displays the interpretation of $R_{(2,1)}$, $R_{(2,2)}$, $R_{(2,3)}$, and $R_{(2,4)}$ induced by (iii) and λ_2 .

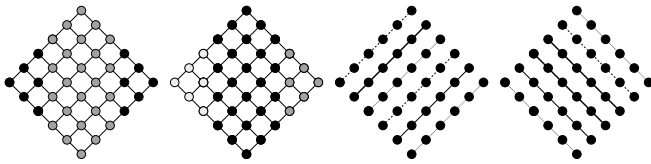


Figure 5. Describing the structure $\text{compil}(\mathbf{P}, \mathbf{C}_1, \mathbf{C}_2, \lambda_1, \lambda_2)$ in Example 2, along the lines of Figure 4.

The intuition underlying the compilation procedure is the following. The universe of $\text{compil}(\mathbf{P})$ is the Cartesian product of a family of chains $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}$ partitioning the universe of $\mathbf{P}$. The interpretation of L in $\text{compil}(\mathbf{P})$ is the natural lattice order inherited by $\text{compil}(\mathbf{P})$ from $\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}$. For $\{i, j\} \subseteq [w']$, the interpretations of $I_{\{j,j'\}}$ and $O_{(j,j')}$ in $\text{compil}(\mathbf{P})$ record, respectively, incomparabilities and comparabilities between the j th and j' th coordinate (corresponding to elements in the chains $\mathbf{C}_{i_j}$ and $\mathbf{C}_{i_{j'}}$, respectively) of the tuples in $\text{compil}(P)$. Finally, for each $j \in [w']$ and $k \in [k_{i_j}]$, the interpretation of $R_{(j,k)}$ in $\text{compil}(\mathbf{P})$ is the restriction of the lattice order of $\text{compil}(\mathbf{P})$ to those pairs of tuples

in $\text{compil}(P)$ such that their j th coordinate is colored k by λ_j ; the $R_{(j,k)}$'s responsibility is to implement the color coding technique (in our setting), as it will become clear in the proof of Claim 3.

We define a binary function

$$s: \text{compil}(P)^2 \rightarrow \text{compil}(P)$$

as follows. Let $\mathbf{c} = (c_1, \dots, c_{w'})$ and $\mathbf{c}' = (c'_1, \dots, c'_{w'})$ be elements in $\text{compil}(P)$. Let $j \in [w']$. Recalling that $\mathbf{C}_{i_j}$ is a chain, let $d_j = \min^{\mathbf{C}_{i_j}}(c_j, c'_j)$. Define

$$s(\mathbf{c}, \mathbf{c}') = (d_1, \dots, d_{w'}). \quad (1)$$

Clearly, s is idempotent, associative and commutative, and hence s is a semilattice function over $\text{compil}(P)$.

Lemma 1 (*). Let $\mathbf{P}$ be a poset, $(\mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}})$ be a coordinatization of $\mathbf{P}$, $(\lambda_1, \dots, \lambda_{w'})$ be a coloring of $\mathbf{P}$. Then, the function s in (1) is a polymorphism of $\text{compil}(\mathbf{P}, \mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}, \lambda_1, \dots, \lambda_{w'})$.

It follows from Lemma 1 and Theorem 1 that, for every poset $\mathbf{P}$ and every compilation $\mathbf{P}^*$ of $\mathbf{P}$, the problem $\text{HOM}(\mathbf{P}^*)$ is polynomial-time tractable; this settles the main result of this section.

4.1.3 Reduction

The following lemma reduces an instance of the poset embedding problem to a family of instances of the homomorphism problem for suitable compilations of the given posets. The lemma is illustrated in Example 3.

Lemma 2. Let $\mathbf{Q}$ and $\mathbf{P}$ be posets such that $\text{width}(\mathbf{Q}) \leq \text{width}(\mathbf{P}) = w$. Let $(\mathbf{C}_1, \dots, \mathbf{C}_w)$ be a chain partition of $\mathbf{P}$. The following are equivalent.

- (i) $\mathbf{Q}$ embeds into $\mathbf{P}$.
- (ii) There exist $w' \leq w$, a subtuple $(i_1, \dots, i_{w'})$ of $(1, \dots, w)$, a chain partition $(\mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_{w'}})$ of $\mathbf{Q}$ such that $|D_{i_j}| \leq |C_{i_j}|$ for all $j \in [w']$, and a tuple $(\mu_1, \dots, \mu_{w'})$ of bijections from D_{i_j} to $[|D_{i_j}|]$ for all $j \in [w']$, such that, for all tuples $(\Lambda_1, \dots, \Lambda_{w'})$, where Λ_j is a $|D_{i_j}|$ -perfect family of hash functions from C_{i_j} to $[|D_{i_j}|]$ for all $j \in [w']$, there exists a tuple $(\lambda_1, \dots, \lambda_{w'}) \in \Lambda_1 \times \dots \times \Lambda_{w'}$ such that

$$\mathbf{Q}^* \in \text{HOM}(\mathbf{P}^*),$$

where

$$\mathbf{Q}^* = \text{compil}(\mathbf{Q}, \mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_{w'}}, \mu_1, \dots, \mu_{w'}),$$

$$\mathbf{P}^* = \text{compil}(\mathbf{P}, \mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}, \lambda_1, \dots, \lambda_{w'}).$$

Proof of Lemma 2. (i) $\Rightarrow$ (ii): Let $e: \mathbf{Q} \rightarrow \mathbf{P}$ be an embedding of $\mathbf{Q}$ into $\mathbf{P}$.

Claim 1. There exist $w' \in \mathbb{N}$ such that $\text{width}(\mathbf{Q}) \leq w' \leq w$, a subtuple $(i_1, i_2, \dots, i_{w'})$ of $(1, 2, \dots, w)$, and a chain partition $(\mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_{w'}})$ of $\mathbf{Q}$ such that, for all $j \in [w']$, $e(D_{i_j}) = \{e(d) \mid d \in D_{i_j}\} \subseteq C_{i_j}$.

Proof. [Proof of Claim 1] To prove the claim, let $e(Q) = \{e(q) \mid q \in Q\}$. Let $(i_1, i_2, \dots, i_{w'})$ be the subtuple of $(1, 2, \dots, w)$ uniquely determined by deleting the index $i \in [w]$ if and only if $e(Q) \cap C_i = \emptyset$. For all $j \in [w']$, let $D_{i_j} = e^{-1}(C_{i_j})$, and let $\mathbf{D}_{i_j}$ be the substructure of $\mathbf{Q}$ induced by D_{i_j} . Then, $(\mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_{w'}})$ is a chain partition of $\mathbf{Q}$, and clearly $e(D_{i_j}) \subseteq C_{i_j}$ for all $j \in [w']$, which settles the claim. $\dashv$

We let $\mathbf{Q}^* = \text{compil}(\mathbf{Q}, \mathbf{D}_{i_1}, \dots, \mathbf{D}_{i_{w'}}, \mu_1, \dots, \mu_{w'})$ and $\mathbf{P}^* = \text{compil}(\mathbf{P}, \mathbf{C}_{i_1}, \dots, \mathbf{C}_{i_{w'}}, \lambda_1, \dots, \lambda_{w'})$ be the compilations of $\mathbf{Q}$ and $\mathbf{P}$ respectively, given by the colorings $(\mu_1, \dots, \mu_{w'})$ and $(\lambda_1, \dots, \lambda_{w'})$ defined as follows.

For each $j \in [w']$, let Λ_j be a $|D_{i_j}|$ -perfect family of hash functions from C_{i_j} to $[|D_{i_j}|]$. Let $j \in [w']$. Let $\lambda_j \in \Lambda_j$ be such that $\lambda_j|_{e(D_{i_j})}$ is injective; indeed such a λ_j exists, because $e(D_{i_j})$ is a subset of C_{i_j} of cardinality $|D_{i_j}|$ (as e is injective), and Λ_j is a $|D_{i_j}|$ -perfect family of hash functions from C_{i_j} to $[|D_{i_j}|]$. Let $e(D_{i_j}) = \{c_1, \dots, c_{|D_{i_j}|}\}$. Let $\lambda_j(c_1) = k_1, \dots, \lambda_j(c_{|D_{i_j}|}) = k_{|D_{i_j}|}$. We let μ_j be such that $\mu_j(e^{-1}(c_i)) = k_i$ for all $i \in [|D_{i_j}|]$. Clearly, μ_j is a bijection from D_{i_j} to $[|D_{i_j}|]$.

The following claim settles the forward direction.

Claim 2. *The function $h: Q^* \rightarrow P^*$ defined by*

$$h((d_1, \dots, d_{w'})) = (e(d_1), \dots, e(d_{w'}))$$

for all $(d_1, \dots, d_{w'}) \in Q^$ maps Q^* homomorphically to P^* .*

Proof. [Proof of Claim 2] Note that Q^* and P^* have the same vocabulary. To prove the claim, we check that h preserves all relations in the vocabulary. Below, $\mathbf{d} = (d_1, \dots, d_{w'})$ and $\mathbf{d}' = (d'_1, \dots, d'_{w'})$ are elements of Q^* .

L : If $(\mathbf{d}, \mathbf{d}') \in L^{Q^*}$, then $d_j \leq^{D_{i_j}} d'_j$ for all $j \in [w']$, then $d_j \leq^Q d'_j$ for all $j \in [w']$, then $e(d_j) \leq^P e(d'_j)$ for all $j \in [w']$, then $e(d_j) \leq^{C_{i_j}} e(d'_j)$ for all $j \in [w']$, then $((e(d_1), \dots, e(d_{w'})), (e(d'_1), \dots, e(d'_{w'}))) \in L^{P^*}$. Altogether this yields $(h(\mathbf{d}), h(\mathbf{d}')) \in L^{P^*}$.

$I_{\{j, j'\}}$: If $\mathbf{d} \in I_{\{j, j'\}}^{Q^*}$, then $d_j \parallel^Q d_{j'}$, then $e(d_j) \parallel^P e(d_{j'})$, then $(e(d_1), \dots, e(d_{w'})) \in I_{\{j, j'\}}^{P^*}$, that is, $h(\mathbf{d}) \in I_{\{j, j'\}}^{P^*}$.

$1 \leq j < j' \leq w'$, $O_{(j, j')}$: If $\mathbf{d} \in O_{(j, j')}^{Q^*}$, then $d_j <^Q d_{j'}$, then $e(d_j) <^P e(d_{j'})$, then $(e(d_1), \dots, e(d_{w'})) \in O_{(j, j')}^{P^*}$, that is, $h(\mathbf{d}) \in O_{(j, j')}^{P^*}$. The case $O_{(j', j)}$ is similar.

$j \in [w']$, $k \in [|D_{i_j}|]$, $R_{(j, k)}$: If $(\mathbf{d}, \mathbf{d}') \in R_{(j, k)}^{Q^*}$, then first observe that $(\mathbf{d}, \mathbf{d}') \in L^{Q^*}$, so that $(h(\mathbf{d}), h(\mathbf{d}')) \in L^{P^*}$ by the argument above. We have that $d_j = d'_j = d$ for some $d \in D_{i_j}$ such that $\mu_j(d) = k$. By construction, $\mu_j(d) = k$ if and only if there exists $c \in C_{i_j}$ such that $d = e^{-1}(c)$ and $\lambda_j(c) = k$. Therefore, $e(d_j) = e(d'_j) = e(d) = c$, so that $((e(d_1), \dots, e(d_{w'})), (e(d'_1), \dots, e(d'_{w'}))) \in R_{(j, k)}^{P^*}$, that is, $(h(\mathbf{d}), h(\mathbf{d}')) \in R_{(j, k)}^{P^*}$. $\dashv$

(ii) $\Rightarrow$ (i): Let Q^* and P^* be specified as in the statement of the lemma, and let $h: Q^* \rightarrow P^*$ be a homomorphism from Q^* to P^* . We define a function $e: Q \rightarrow P$ as follows. Below, $\mathbf{c} = (c_1, \dots, c_{w'})$, $\mathbf{c}' = (c'_1, \dots, c'_{w'})$, $\mathbf{c}'' = (c''_1, \dots, c''_{w'})$, $\mathbf{d} = (d_1, \dots, d_{w'})$, and $\mathbf{d}' = (d'_1, \dots, d'_{w'})$ are elements of Q^* .

Let $q \in Q$. Let $j \in [w']$ be such that $q \in D_{i_j}$.

Claim 3. *There exists a unique $p \in C_{i_j} \subseteq P$ such that:*

- if $h(\mathbf{d}) = \mathbf{c}$ and $d_j = q$, then $c_j = p$;
- $\mu_j(q) = \lambda_j(p)$.

Proof. [Proof of Claim 3] Let $\mu_j(q) = k$. Since $\{\mathbf{d} \in Q^* \mid d_j = q\}$ is nonempty, there exists at least one element $p \in C_{i_j}$ such that $\mathbf{c}$ is in the image of h in P^* and $c_j = p$. Let $p, p' \in C_{i_j}$ be such that, for some $\mathbf{d}, \mathbf{d}' \in Q^*$ with $d_j = d'_j = q$, $h(\mathbf{d}) = \mathbf{c}$ and $c_j = p$, and $h(\mathbf{d}') = \mathbf{c}'$ and $c'_j = p'$. We prove that $p = p'$ and $\lambda_j(p) = k$. We distinguish two cases.

Case 1: $(\mathbf{d}, \mathbf{d}') \in L^{Q^*}$ or $(\mathbf{d}', \mathbf{d}) \in L^{Q^*}$. Assume $(\mathbf{d}, \mathbf{d}') \in L^{Q^*}$. Then, $(\mathbf{d}, \mathbf{d}') \in R_{(j, k)}^{Q^*}$. Then, $(\mathbf{c}, \mathbf{c}') \in R_{(j, k)}^{P^*}$, so that $c_j = c'_j$ by definition of $R_{(j, k)}^{P^*}$, that is $p = p'$, and $\lambda_j(p) = k$. The argument is similar if $(\mathbf{d}', \mathbf{d}) \in L^{Q^*}$.

Case 2: $(\mathbf{d}, \mathbf{d}') \notin L^{Q^*}$ and $(\mathbf{d}', \mathbf{d}) \notin L^{Q^*}$. Clearly it then holds that $\min^{Q_{i_j}}(d_j, d'_j) = q$ and $\min^{Q_{i_{j'}}}(d_{j'}, d'_{j'}) \leq^{Q_{i_{j'}}} d_{j'}, d'_{j'}$ for all $j' \in [w']$. Therefore,

$$((\min^{Q_{i_1}}(d_1, d'_1), \dots, \min^{Q_{i_{w'}}}(d_{w'}, d'_{w'})), \mathbf{d}) \in R_{j, k}^{Q^*},$$

and

$$((\min^{Q_{i_1}}(d_1, d'_1), \dots, \min^{Q_{i_{w'}}}(d_{w'}, d'_{w'})), \mathbf{d}') \in R_{j, k}^{Q^*}.$$

Let

$$h((\min^{Q_{i_1}}(d_1, d'_1), \dots, \min^{Q_{i_{w'}}}(d_{w'}, d'_{w'}))) = \mathbf{c}''.$$

Then, $(\mathbf{c}'', \mathbf{c}) \in R_{j, k}^{P^*}$ and $(\mathbf{c}'', \mathbf{c}') \in R_{j, k}^{P^*}$, so that $c''_j = c_j = c'_j$, that is $p = p'$, and $\lambda_j(p) = k$. $\dashv$

We define $e(q) = p$, where $p \in P$ is the unique element identified by Claim 3 relative to q . The following claim then settles the backwards direction.

Claim 4. *e embeds Q into P .*

Proof. [Proof of Claim 4] Let $q, q' \in Q$. It is sufficient to check that $q <^Q q'$ implies $e(q) <^P e(q')$, and $q \parallel^Q q'$ implies $e(q) \parallel^P e(q')$. Let $j, j' \in [w']$ be such that $q \in D_{i_j}$ and $q' \in D_{i_{j'}}$.

$q <^Q q'$ implies $e(q) <^P e(q')$: Assume $q <^Q q'$. Assume that $j \leq j'$ (the case $j' \leq j$ is similar). We distinguish two cases.

Case 1: If $j = j'$, then let $\mu_j(q) = k$ and $\mu_j(q') = k'$. Since $q, q' \in D_{i_j}$ and μ_j is a bijection from D_{i_j} to $[|D_{i_j}|]$, we have that $k \neq k'$. Hence, if $e(q) = p \in C_{i_j}$ and $e(q') = p' \in C_{i_j}$, then by the definition of e we have that $\lambda_j(p) = k \neq k' = \lambda_j(p')$, so that $p \neq p'$. We have that

$$((\text{bot}(\mathbf{D}_{i_1}), \dots, q, \dots, \text{bot}(\mathbf{D}_{i_{w'}})), (\text{bot}(\mathbf{D}_{i_1}), \dots, q', \dots, \text{bot}(\mathbf{D}_{i_{w'}}))) \in L^{Q^*},$$

where q and q' occur at the j th coordinate, and $\text{bot}(\mathbf{D}_{i_{j''}})$ is the bottom of chain $\mathbf{D}_{i_{j''}}$ for all $j'' \in [w'] \setminus \{j\}$. Let us set $h((\text{bot}(\mathbf{D}_{i_1}), \dots, q, \dots, \text{bot}(\mathbf{D}_{i_{w'}}))) = \mathbf{c} \in P^*$ and similarly $h((\text{bot}(\mathbf{D}_{i_1}), \dots, q', \dots, \text{bot}(\mathbf{D}_{i_{w'}}))) = \mathbf{c}' \in P^*$. Then

$$(\mathbf{c}, \mathbf{c}') \in L^{P^*},$$

so that, in particular, $c_j \leq^{C_{i_j}} c'_j$. We claim that $c_j = p$. Indeed, since h is a homomorphism, it is the case that $\mu_j(q) = \lambda_j(c_j) = k$, because there is a $R_{(j, k)}$ loop over the elements of $(\text{bot}(\mathbf{D}_{i_1}), \dots, q, \dots, \text{bot}(\mathbf{D}_{i_{w'}}))$ in Q^* . By Claim 3, there exists a unique element in C_{i_j} having the same color of q and occurring at the j th coordinate of any $h((\dots, q, \dots)) \in P^*$, and this element is $e(q) = p$ by definition. Similarly, $c'_j = p'$. Thus, since we observed that $p \neq p'$, we have that $p <^{C_{i_j}} p'$, and therefore, $e(q) = p <^P p' = e(q')$.

Case 2: If $j < j'$, then $e(q) = p \in C_{i_j}$ and $e(q') = p' \in C_{i_{j'}}$, so that $p \neq p'$ because $C_{i_j} \cap C_{i_{j'}} = \emptyset$. We have that

$$(\dots, q, \dots, q', \dots) \in O_{(j, j')}^{Q^*},$$

where q occurs at the j th coordinate and q' occurs at the j' th coordinate, so that, if $h((\dots, q, \dots, q', \dots)) = \mathbf{c} \in P^*$, then

$$\mathbf{c} \in O_{(j, j')}^{P^*},$$

that is, $c_j \leq^P c_{j'}$. We claim that $c_j = p$ and $c_{j'} = p'$, which implies $e(q) = p <^P p' = e(q')$. Indeed, since h is a homomorphism, it is the case that $\mu_j(q) = \lambda_j(c_j) = k$ and $\mu_{j'}(q') = \lambda_{j'}(c_{j'}) = k'$, because there is both a $R_{(j, k)}$ loop and a $R_{(j', k')}$ loop over $(\dots, q, \dots, q', \dots)$ in Q^* . Then, by Claim 3 and the definition of e , it is the case that $c_j = e(q) = p$ and $c_{j'} = e(q') = p'$.

$q \parallel^Q q'$ implies $e(q) \parallel^P e(q')$: Let $\mu_j(q) = k$ and $\mu_{j'}(q') = k'$. We have that

$$(\dots, q, \dots, q', \dots) \in I_{\{j, j'\}}^{Q^*},$$

where q occurs at the j th coordinate and q' occurs at the j' th coordinate, so that, if $h((\dots, q, \dots, q', \dots)) = c \in P^*$, then

$$c \in I_{\{j, j'\}}^{P^*},$$

that is, $c_j \parallel^P c_{j'}$. We claim that $c_j = e(q)$ and $c_{j'} = e(q')$, which implies $e(q) \parallel^P e(q')$. Indeed, since h is a homomorphism, it is the case that $\lambda_j(c_j) = k$ and $\lambda_{j'}(c_{j'}) = k'$, because there is both a $R_{(j, k)}$ loop and a $R_{(j', k')}$ loop over $(\dots, q, \dots, q', \dots)$ in Q^* . Then, by Claim 3 and the definition of e , it is the case that $c_j = e(q)$ and $c_{j'} = e(q')$. $\dashv$

The statement is proved. $\square$

Example 3. Let Q and P be the posets in Example 1, so that Q embeds into P via the map $e: Q \rightarrow P$ defined in the example (see Figure 3). Let $Q^* = \text{compil}(Q, D_1, D_2, \mu_1, \mu_2)$ and $P^* = \text{compil}(P, C_1, C_2, \lambda_1, \lambda_2)$ be the structures in Example 2, respectively compiling Q and P . The homomorphism $h: Q^* \rightarrow P^*$, corresponding to the embedding $e: Q \rightarrow P$ as by (the forward direction of) Lemma 2, is depicted in Figure 6.

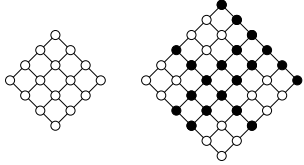


Figure 6. The structures Q^* (left) and P^* (right) in Example 3. The white points in P^* form the image of the homomorphism $h: Q^* \rightarrow P^*$ in Example 3. It is possible to check that h is a homomorphism by direct inspection of Figure 4 and Figure 5.

4.1.4 Algorithm

We are now ready to list the pseudocode of our main algorithm. The input is a pair (Q, P) of posets.

```

ALGORITHM( $Q, P$ )
1  if  $(|P| < |Q|$  or  $\text{width}(P) < \text{width}(Q))$  then reject
2   $w \leftarrow \text{width}(P)$ 
3  compute a chain partition  $(C_1, \dots, C_w)$  of  $P$ 
4  foreach  $1 \leq w' \leq w$ ,
    subtuple  $(i_1, \dots, i_{w'})$  of  $(1, \dots, w)$ ,
    chain partition  $(D_{i_1}, \dots, D_{i_{w'}})$  of  $Q$ ,
    coloring  $(\mu_1, \dots, \mu_{w'}) \in M_1 \times \dots \times M_{w'}$  do
5    if exists  $j \in [w']$  such that  $|C_{i_j}| < |D_{i_j}|$  then reject
6     $Q^* \leftarrow \text{compil}(Q, D_{i_1}, \dots, D_{i_{w'}}, \mu_1, \dots, \mu_{w'})$ 
7    foreach  $j \in [w']$  do
8       $\Lambda_j \leftarrow |D_{i_j}|$ -perfect family of hashing functions
        from  $C_{i_j}$  to  $[|D_{i_j}|]$ 
9    foreach  $(\lambda_1, \dots, \lambda_{w'}) \in \Lambda_1 \times \dots \times \Lambda_{w'}$  do
10      $P^* \leftarrow \text{compil}(P, C_{i_1}, \dots, C_{i_{w'}}, \lambda_1, \dots, \lambda_{w'})$ 
11     if  $Q^* \in \text{HOM}(P^*)$  then accept
12 reject

```

In Line 4, M_j is the set of all bijections from D_{i_j} to $[|D_{i_j}|]$, for all $j \in [w']$. We conclude proving that the algorithm above has the desired properties, from which the main result of the section follows.

Theorem 4 ($\star$). Let $\mathcal{P}$ be a class of posets of bounded width. There exists an algorithm deciding any instance (Q, P) of $\text{EMB}(\mathcal{P})$ in $2^{O(k \log k)} \cdot n^{O(1)}$ time, where $n = |P|$ and $k = |Q|$.

Theorem 5. Let $\mathcal{P}$ be a class of posets of bounded width. Then, $\text{MC}(\mathcal{P}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ is fixed-parameter tractable (with single exponential parameter dependence).

Proof. Directly from Proposition 1 and Theorem 4. $\square$

4.2 Embedding is W[1]-hard on Bounded Cover-Degree Posets

We construct a class $\mathcal{P}_{\text{cover-degree}}$ of bounded cover-degree posets such that $\text{EMB}(\mathcal{P}_{\text{cover-degree}})$ is W[1]-hard. By Proposition 1, it follows that $\text{MC}(\mathcal{P}_{\text{cover-degree}}, \mathcal{FO}(\exists, \wedge, \vee, \neg))$ is W[1]-hard.

Let $G = (V, E^G)$ be a graph and let $V = [n]$. Then $r(G) = P$ is the poset defined as follows. The universe of P is $P = \bigcup_{i \in [n]} P_i$ where, for all $i \in [n]$,

$$P_i = \{\perp_i, a_i, b_i, c_i, d_i, \top_i\} \\ \cup \{l_{(i,j)}, u_{(i,j)} \mid j \in [n], (i,j) \in E^G\}.$$

The order is defined by the following. For each $i, j \in [n]$:

- $a_i \prec^P b_i, a_i \prec^P c_i, b_i \prec^P d_i, c_i \prec^P d_i$, and $b_i \parallel^P c_i$;
- $\perp_i \prec^P l_{(i,1)} \prec^P \dots \prec^P l_{(i,n)} \prec^P a_i$;
- $d_i \prec^P u_{(i,1)} \prec^P \dots \prec^P u_{(i,n)} \prec^P \top_i$;
- $l_{(i,j)} \prec^P u_{(j,i)}$ if and only if $(i,j) \in E^G$.

The construction satisfies the following properties. Let $G \in \mathcal{G}$:

- (i) since $\text{cover-degree}(r(G)) \leq 3$, the class $\mathcal{P}_{\text{cover-degree}} = \{r(G) \mid G \in \mathcal{G}\}$ has bounded cover-degree;
- (ii) $r(G)$ can be constructed in polynomial time;
- (iii) for any $j, j' \in [n], j \neq j'$, we have $\perp_j \prec^P \top_{j'}$ if and only if $(j, j') \in E^G$.

Proposition 6 ($\star$). $\text{EMB}(\mathcal{P}_{\text{cover-degree}})$ is W[1]-hard.

5. Classical Complexity

In this section, we study the classical complexity of the embedding problem on the targeted classes of posets, and we prove a tractability result of independent interest on bounded width posets. We first observe the following fact.

Proposition 7 ($\star$). Let $\mathcal{P}$ be a class of posets of bounded size. Then, $\text{EMB}(\mathcal{P})$ is polynomial-time tractable.

Note that the above together with Proposition 2 rules out a polynomial-time tractability analogue of Proposition 1. The section is organized into three subsections, as follows.

- In Subsections 5.1 and 5.2, we prove that the embedding problem is NP-hard on bounded width and bounded degree posets respectively. This implies that Proposition 7 is tight with respect to the studied invariants.
- In Subsection 5.3, we show how the ideas developed in Section 4 may be used to obtain a polynomial-time tractable algorithm for the isomorphism of bounded width posets, an open problem in order theory [3, p. 284].

5.1 Embedding is NP-hard on Bounded Width Posets

In this subsection, we construct a class $\mathcal{P}$ of posets of bounded width such that $\text{EMB}(\mathcal{P})$ is NP-hard, which immediately implies NP-hardness of $\text{MC}(\mathcal{P}, \mathcal{FO}(\exists, \wedge, \neg))$.

The reduction, from the Boolean satisfiability problem (SAT), is technically involved. Intuitively, given a SAT instance ϕ , we construct two bounded width posets $\mathbf{Q}_\phi$ and $\mathbf{P}_\phi$. The two posets are such that, if ϕ is satisfiable, then $\mathbf{Q}_\phi$ embeds into $\mathbf{P}_\phi$ “nicely”, in the sense that certain chains of $\mathbf{Q}_\phi$ embed into certain families of chains in $\mathbf{P}_\phi$; conversely, every embedding of $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$ must be nice in the above sense, and any nice embedding of $\mathbf{Q}_\phi$ into $\mathbf{P}_\phi$ yields a satisfying assignment to ϕ .

Let $\mathcal{S}$ be the class of propositional formulas in conjunctive form, containing at least 3 clauses, where each clause contains at most 3 literals; also, no clause contains a pair of complementary literals, and each variable occurs in at least two clauses. Let $\phi(x_1, \dots, x_n) = \delta_1 \wedge \dots \wedge \delta_m$ be in $\mathcal{S}$. For $i \in [n]$ and $j \in [m]$, we write $x_i \in \delta_j$ if a literal on variable x_i occurs in clause δ_j , and we let $\text{var}(\delta_j) = \{x_i \mid i \in [n], x_i \in \delta_j\}$.

We proceed in two stages. First, we define a poset $\mathbf{Q}_\phi$ as follows. The universe Q_ϕ contains $Q_\phi^a = \{(\delta_i, j) \mid i \in [m], j \in [n]\}$, $Q_\phi^c = \{(\delta'_i, j) \mid i \in [m], j \in [n-1]\}$,

$$Q_\phi^v = \left\{ (x_i, (j, j')) \mid \begin{array}{l} i \in [n], x_i \in \delta_j, x_i \in \delta_{j'}, j < j', \\ \text{and } x_i \notin \delta_{j''} \text{ for all } j < j'' < j' \end{array} \right\},$$

and a set Q_ϕ^l of auxiliary elements introduced below.

For $q, q' \in Q_\phi$, we let $\ll^{\mathbf{Q}_\phi}$ denote the fact that, in the order of $\mathbf{Q}_\phi$, there is a chain of $|Q_\phi^a|$ fresh auxiliary elements, contained in Q_ϕ^l , between q and q' . The order relation of $\mathbf{Q}_\phi$ is defined by the following cover relations:

- (Q1) for all $(\delta_i, j), (\delta_{i+1}, j) \in Q_\phi^a$: if $i+1 \leq i'$ where i' is the minimum in $[m]$ such that $x_j \in \delta_{i'}$, then $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$; if $i' \leq i$ where i' is the maximum in $[m]$ such that $x_j \in \delta_{i'}$, then $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$; otherwise, $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta_{i+1}, j)$;
- (Q2) $(\delta_m, j) \ll^{\mathbf{Q}_\phi} (\delta_1, j+1)$, for $(\delta_m, j), (\delta_1, j+1) \in Q_\phi^a$;
- (Q3) $(\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_{i+1}, j)$, for $(\delta'_i, j), (\delta'_{i+1}, j) \in Q_\phi^c$;
- (Q4) $(\delta'_m, j) \prec^{\mathbf{Q}_\phi} (\delta'_1, j+1)$, for $(\delta'_m, j), (\delta'_1, j+1) \in Q_\phi^c$;
- (Q5) $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_i, (j', j''))$, for $(x_i, (j, j')), (x_i, (j', j'')) \in Q_\phi^v$;
- (Q6) $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_{i+1}, (k, k'))$, for $(x_i, (j, j')), (x_{i+1}, (k, k')) \in Q_\phi^v$ where j' is maximum in $[m]$ such that $x_i \in \delta_{j'}$ and k is minimum in $[m]$ such that $x_{i+1} \in \delta_k$.
- (Q7) $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta_i, j+1)$, for all $(\delta_i, j), (\delta_i, j+1) \in Q_\phi^a$ and $(\delta'_i, j) \in Q_\phi^c$;
- (Q8) $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (x_j, (i, i')) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j)$, for all $(\delta_i, j), (\delta_{i'}, j) \in Q_\phi^a$ and $(x_j, (i, i')) \in Q_\phi^v$.

Second, we define the poset $\mathbf{P}_\phi = r(\phi)$, using $\mathbf{Q}_\phi$ as a basis, as follows. The universe P_ϕ is the union of

$$\begin{aligned} P_\phi^a &= \bigcup_{(\delta_i, j) \in Q_\phi^a} \{(f, (\delta_i, j)) \mid f \in \{0, 1\}^{\text{var}(\delta_i)} \text{ satisfies } \delta_i\}, \\ P_\phi^c &= \bigcup_{(\delta'_i, j) \in Q_\phi^c} \{(f, (\delta'_i, j)) \mid f \in \{0, 1\}^{\text{var}(\delta_i)} \text{ satisfies } \delta_i\}, \\ P_\phi^v &= \bigcup_{(x_i, (j, j')) \in Q_\phi^v} \{(x_i, (j, j')), (\neg x_i, (j, j'))\}, \end{aligned}$$

and a set P_ϕ^l of auxiliary elements introduced below.

Again, for $p, p' \in P_\phi$, we let $\ll^{\mathbf{P}_\phi}$ denote the fact that, in the order of $\mathbf{P}_\phi$, there is a chain of $|Q_\phi^a|$ fresh auxiliary elements, contained in P_ϕ^l , between p and p' . The order relation of $\mathbf{P}_\phi$ is defined by the following cover relation:

- (P1) for all $(f, (\delta_i, j)), (f', (\delta_{i'}, j')) \in P_\phi^a$, $(f, (\delta_i, j)) \prec^{\mathbf{P}_\phi} (f', (\delta_{i+1}, j))$ if and only if $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta_{i'}, j')$, and $(f, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (f', (\delta_{i+1}, j))$ if and only if $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j')$;
- (P2) for all $(f, (\delta'_i, j)), (f', (\delta'_{i'}, j')) \in P_\phi^c$, $(f, (\delta'_i, j)) \prec^{\mathbf{P}_\phi} (f', (\delta'_{i+1}, j))$ if and only if $(\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_{i'}, j')$;
- (P3) for all $(x_i, (j, j')), (\neg x_i, (j, j')), (x_i, (j', j'')), (\neg x_i, (j', j''))$ in P_ϕ^v , $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_i, (j', j''))$ and $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_i, (j', j''))$ if and only if $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_i, (j', j''))$;
- (P4) for all $(x_i, (j, j')), (\neg x_i, (j, j')), (x_{i+1}, (k, k')),$ and $(\neg x_{i+1}, (k, k'))$ in P_ϕ^v , $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_{i+1}, (k, k'))$, $(x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_{i+1}, (k, k'))$, $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (x_{i+1}, (k, k'))$, and $(\neg x_i, (j, j')) \prec^{\mathbf{P}_\phi} (\neg x_{i+1}, (k, k'))$, if and only if $(x_i, (j, j')) \prec^{\mathbf{Q}_\phi} (x_{i+1}, (k, k'))$.
- (P5) for all $(f, (\delta_i, j)), (f, (\delta_i, j+1)) \in P_\phi^a$ and $(f, (\delta'_i, j)) \in P_\phi^c$, $(f, (\delta_i, j)) \prec^{\mathbf{P}_\phi} (f, (\delta'_i, j)) \prec^{\mathbf{P}_\phi} (f, (\delta_i, j+1))$ if and only if $(\delta_i, j) \prec^{\mathbf{Q}_\phi} (\delta'_i, j) \prec^{\mathbf{Q}_\phi} (\delta_i, j+1)$;
- (P6) for all $(f, (\delta_i, j)), (f', (\delta_{i'}, j)), (g, (\delta_i, j)), (g', (\delta_{i'}, j)) \in P_\phi^a$ and $(x_j, (i, i')), (\neg x_j, (i, i')) \in P_\phi^v$, it holds that $(f, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (x_j, (i, i')) \ll^{\mathbf{P}_\phi} (f', (\delta_{i'}, j))$ and $(g, (\delta_i, j)) \ll^{\mathbf{P}_\phi} (\neg x_j, (i, i')) \ll^{\mathbf{P}_\phi} (g', (\delta_{i'}, j))$ if and only if $(\delta_i, j) \ll^{\mathbf{Q}_\phi} (x_j, (i, i')) \ll^{\mathbf{Q}_\phi} (\delta_{i'}, j)$, $f(x_j) = f'(x_j) = 1$, and $g(x_j) = g'(x_j) = 0$.

Note that $\text{width}(\mathbf{Q}_\phi) \leq 4$ and $\text{width}(\mathbf{P}_\phi) \leq 2^2 + 7^2 + 7^2 = 102$ for all $\phi \in \mathcal{S}$ (we remark that this width bound may be improved at the cost of a more complicated construction). Hence $\mathcal{P}_{\text{width}} = \{r(\phi) \mid \phi \in \mathcal{S}\}$ has bounded width.

Theorem 6 (*). $\text{EMB}(\mathcal{P}_{\text{width}})$ is NP-hard.

5.2 Embedding is NP-hard on Bounded Degree Posets

We reduce from the satisfiability problem. Let $\mathcal{S}$ be the class of propositional formulas in conjunctive form, where each clause contains exactly 3 pairwise non-complementary literals.

The idea of the reduction is the following. We encode a formula in $\mathcal{S}$ by a poset $\mathbf{P}$, whose universe partitions into three blocks, P_0 , P_1 and P_2 . The set P_1 contains several groups of 7 elements, where each element corresponds to one possible satisfying assignment of a clause, and the embedding encodes an assignment for the whole formula by forcing us to choose one element out of each group. The set P_2 ensures that each assignment chosen by the embedding is consistent for each pair of clauses. To preserve bounded degree while ensuring the consistency of each pair of clauses, it is necessary to use many groups in P_1 for each clause. Finally, P_0 ensures that each choice made by the embedding for a given clause is consistent across all groups corresponding to that clause.

We now formalize the ideas outlined above. Let $\phi(x_1, \dots, x_n) = \delta_1 \wedge \dots \wedge \delta_m$ be in $\mathcal{S}$. For $j \in [n]$ and $i \in [m]$, we write $x_j \in \delta_i$ if a literal on variable x_j occurs in clause δ_i , and we let $\text{var}(\delta_i) = \{x_j \mid j \in [n], x_j \in \delta_i\}$. For all $i \in [m]$, let $(g_{i,1}, \dots, g_{i,7})$ be a fixed ordering of the assignments in $\{0, 1\}^{\text{var}(\delta_i)}$ satisfying δ_i , and let $(i_1, i_2, \dots, i_{m-1}) = (1, \dots, i-1, i+1, \dots, m)$. We define our two posets $\mathbf{Q}_\phi$ and $\mathbf{P}_\phi$ below.

The poset $\mathbf{Q}_\phi$ has universe $Q_\phi = Q_0 \cup Q_1 \cup Q_2$, where

$$\begin{aligned} Q_0 &= \{c_{(i,j)}, c_{(i,m)}, c_{(m,j)} \mid i, j \in [m-1], i \neq j\}, \\ Q_1 &= \{f_{(i,j)} \mid i, j \in [m], i \neq j\}, \\ Q_2 &= \{d_{(i,j)} \mid 1 \leq i < j \leq m\}, \end{aligned}$$

and its cover relation is defined by the following:

- (E1) $f_{(i,j)}, f_{(j,i)} \prec^{\mathbf{Q}_\phi} d_{(i,j)}$ for all $1 \leq i < j \leq m$.

(E2) For all $i \in [m]$,

$$f_{(i,i_1)} \succ^{\mathbf{Q}_\phi} c_{(i,i_1)} \prec^{\mathbf{Q}_\phi} f_{(i,i_2)} \succ^{\mathbf{Q}_\phi} \dots \\ \dots \succ^{\mathbf{Q}_\phi} c_{(i,i_{m-1})} \prec^{\mathbf{Q}_\phi} f_{(i,i_{m-1})}.$$

The poset $\mathbf{P}_\phi$ has universe $P_\phi = P_0 \cup P_1 \cup P_2$ where,

$$P_0 = \{c_{(i,j),a}, c_{(i,m),a}, c_{(m,j),a} \mid i, j \in [m-1], i \neq j, a \in [7]\},$$

$$P_1 = \{f_{(i,j),a} \mid i, j \in [m], i \neq j, a \in [7]\},$$

$$P_2 = \{d_{(i,j),(a,a')} \mid 1 \leq i < j \leq m, (a, a') \in [7]^2\},$$

and its cover relation is defined by the following:

(D1) For all $1 \leq i < j \leq m$, it holds that $f_{(i,j),a}, f_{(j,i),a'} \prec^{\mathbf{P}_\phi} d_{(i,j),(a,a')}$ if and only if $g_{i,a}(x) = g_{j,a'}(x)$ for all $x \in \text{var}(\delta_i) \cap \text{var}(\delta_j)$.

(D2) For all $i \in [m]$ and $a \in [7]$,

$$f_{(i,i_1),a} \succ^{\mathbf{P}_\phi} c_{(i,i_1),a} \prec^{\mathbf{P}_\phi} f_{(i,i_2),a} \succ^{\mathbf{P}_\phi} \dots \\ \dots \succ^{\mathbf{P}_\phi} c_{(i,i_{m-1}),a} \prec^{\mathbf{P}_\phi} f_{(i,i_{m-1}),a}.$$

Since $\text{cover-degree}(\mathbf{P}_\phi) \leq 1 + 7 = 8$ and $\text{depth}(\mathbf{P}_\phi) \leq 3$, $\mathcal{P}_{\text{degree}} = \{\mathbf{P}_\phi \mid \phi \in \mathcal{S}\}$ has bounded degree by Proposition 3.

Theorem 7 (*). $\text{EMB}(\mathcal{P}_{\text{degree}})$ is NP-hard.

5.3 Isomorphism in Polytime on Bounded Width Posets

The insight on bounded width used to prove tractability of the embedding problem essentially scales to the isomorphism problem.

Theorem 8. Let $\mathcal{P}$ be a class of posets of bounded width. Then, $\text{ISO}(\mathcal{P})$ is polynomial-time tractable.

Proof. The proof utilizes three known facts from the literature.

Let $\mathbf{R}$ be any poset. For all $S \subseteq R$, let $[S]$ be *downset* generated by S in $\mathbf{R}$, i.e., $[S] = \{r \in R \mid \exists s \in S \text{ such that } r \leq_{\mathbf{R}} s\}$. Let $l(\mathbf{R})$ be the order defined by equipping the universe of all antichains in $\mathbf{R}$ by the relation $A \leq^{l(\mathbf{R})} A'$ if and only if $[A] \subseteq [A']$. Note that, if $\text{width}(\mathbf{R})$ is considered as a constant, the construction of $l(\mathbf{R})$ is polynomial-time computable from $\mathbf{R}$.

The three needed facts are the following. First, for any (finite) poset $\mathbf{R}$, the structure $l(\mathbf{R})$ is a (finite) distributive lattice [19, Proposition 5.5.5]. Second, the substructure of $l(\mathbf{R})$ generated by join irreducible elements is isomorphic to $\mathbf{R}$ [19, Theorem 5.5.6]; recall that, if $\mathbf{L} = (L, \leq)$ is a lattice, then $j \in L$ is *join irreducible* if, for all $l, l' \in L$, if j is the least upper bound of l and l' , then $j = l$ or $j = l'$. Third, the isomorphism problem restricted to finite distributive lattices is polynomial-time tractable [11].

Using the previous facts, we design the following algorithm. Let $w \in \mathbb{N}$ be the upper bound on the width of posets in $\mathcal{P}$. Let $(\mathbf{Q}, \mathbf{P})$ be an instance of $\text{ISO}(\mathcal{P})$. Let $|P| = n$. If $|Q| \neq n$, or $\text{width}(\mathbf{Q}) > w$, or $\text{width}(\mathbf{Q}) \neq \text{width}(\mathbf{P})$, then reject; the condition is checkable in time $O(w \cdot n^2)$ by Theorem 3. Otherwise, in polynomial time, compute $l(\mathbf{Q})$ and $l(\mathbf{P})$ and accept if and only if $l(\mathbf{Q})$ and $l(\mathbf{P})$ are isomorphic.

The algorithm clearly runs in polynomial time. For correctness, notice that $\mathbf{Q}$ and $\mathbf{P}$ are isomorphic if and only if $l(\mathbf{Q})$ and $l(\mathbf{P})$ are isomorphic. For the nontrivial direction (backwards), if f is an isomorphism from $l(\mathbf{Q})$ to $l(\mathbf{P})$, then let f' be the restriction of f to the join irreducible elements of $l(\mathbf{Q})$. It is easy to check that f' is bijective into the join irreducible elements of $l(\mathbf{P})$, hence, using the second fact mentioned above, f' is an isomorphism between $\mathbf{Q}$ and $\mathbf{P}$. $\square$

6. Conclusion

We embarked on the study of the model checking problem on posets; compared to graphs, the problem is largely unexplored, and we made

a first contribution by studying basic syntactic fragments (existential logic) and fundamental poset invariants (including width, depth, and degree). Our complexity classification for existential logic also carries over to the *jump number* (between size and width in Figure 1); a future direction is to extend our study to *dimension* (above width [3] and degree [10] in Figure 1).

Our main algorithmic result, fixed-parameter tractability of existential logic on bounded width posets, raises the natural question of whether model checking the full first-order logic is fixed-parameter tractable on classes of posets of bounded width. We propose this as a topic for future research.

Acknowledgments

This research was supported by ERC Starting Grant (239962) and FWF Austrian Science Fund (P26200).

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