

Secure Equilibria in Weighted Games^{*}

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Abstract

We consider two-player non zero-sum infinite duration games played on weighted graphs. We extend the notion of secure equilibrium introduced by Chatterjee et al., from the Boolean setting to this quantitative setting. As for the Boolean setting, our notion of secure equilibrium refines the classical notion of Nash equilibrium. We prove that secure equilibria always exist in a large class of weighted games which includes common measures like sup, inf, lim sup, lim inf, mean-payoff, and discounted sum. Moreover we show that one can synthesize finite-memory strategy profiles with few memory. We also prove that the constrained existence problem for secure equilibria is decidable for sup, inf, lim sup, lim inf and mean-payoff measures. Our solutions rely on new results for zero-sum quantitative games with lexicographic objectives that are interesting on their own right.

Categories and Subject Descriptors B.6.3 [design aids]: automatic synthesis; F.1.2 [Modes of computation]: interactive and reactive computation

General Terms Theory

Keywords Games on weighted graphs, quantitative objectives, secure equilibria, mean-payoff and discounted sum.

1. Introduction

Two-player zero-sum infinite duration games played on graphs is a useful framework to formalize many important problems in computer science. System synthesis, and especially synthesis of reactive systems, is one of those important problems, see for example [20, 22]. In this application, the vertices and the edges of the graph represent the states and the transitions of the system; one player represents the

system to synthesize, whereas the other represents the environment with which the system interacts. In the classical setting, the environment is considered as *antagonist* and the objectives of the two players are *complementary*, leading to a so-called *zero-sum* game. There are many known results about those zero-sum games [13, 14, 16, 18].

Modeling the environment as completely adversarial is usually a bold abstraction of reality. However it is often used as it is simple and sound: a winning strategy against a completely adversarial model of the environment is winning against any environment which pursues its own objective. But this approach may fail to find a winning strategy whereas there exists a solution when the objective of the environment is taken into account. In this case, we need to consider the more general framework of *non zero-sum* games. The classical notion of rational behavior in this context is commonly formalized as Nash equilibria [19]. Nash equilibria capture rational behaviors when the players only care about their own payoff (internal criteria), and they are indifferent to the payoff of the other player (external criteria). In the setting of synthesis, the more appropriate notion is the *adversarial external criteria*, where the players are as harmful as possible to the other players without sabotaging their own objectives. This has inspired the study of refinements of Nash equilibria, like the notion of *secure equilibria* that captures the adversarial external criteria and is at the basis of *compositional synthesis algorithms* [9]. In secure equilibria, lexicographic objectives are considered: each player first tries to maximize his own payoff, and then tries to minimize the opponent's payoff. It is shown in [9] that secure equilibria are those Nash equilibria which represent enforceable contracts between the two players.

In this paper, we extend the notion of secure equilibria from the Boolean setting with ω -regular objectives of [9] to a quantitative setting where objectives are non necessarily ω -regular. More precisely, we consider two-player non zero-sum turn-based games played on weighted graphs, called *weighted games* with objectives defined by classical measures considered in the literature for infinite plays: *sup*, *inf*, *lim sup*, *lim inf*, *mean-payoff*, and *discounted sum* [10]. In our setting, the edges of the weighted graph are labelled with pairs of rational values that are used to assign two values to each infinite play: one value models the reward of player 1, and the other the reward of player 2.

Contributions Our contributions are threefold. (1) We show that all weighted games with the classical measures have *secure equilibria*. We also establish that there exist simple profiles of strategies that witness such equilibria: finite-memory strategies with a linear memory size are sufficient

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for most of the measures (polynomial size for inf and sup measures). (2) We provide polynomial or pseudo-polynomial (depending on the measures) algorithms for the automatic synthesis of such strategy profiles. We thus provide the necessary algorithms to extend the compositional synthesis framework of [8] to a quantitative setting. (3) We prove that one can decide the existence of a secure equilibrium whose outcome satisfies some constraints, for all measures except the discounted sum. In the latter case, we show that this problem is connected to a challenging open problem [3, 11]. Our solutions rely on the analysis of two-player zero-sum games with lexicographic objectives (for all the measures) for which we provide worst-case optimal algorithms. These so-called *lexicographic games* are interesting on their own right and, to the best of our knowledge, were not studied in the literature.

Related work Secure equilibria were first introduced in [9] in a Boolean setting, and for ω -regular objectives. We extend this work to a quantitative setting in which objectives are not necessarily ω -regular. Our contributions (1) and (2) are very much inspired by a recent work by Brihaye et al. [6] that provides general results for the existence of Nash equilibria in a large class of multi-player weighted graphs. We show that their main theorem is extendable to secure equilibria in the case of two-player games. However, to adapt their theorem we need non trivial new results on lexicographic games, while for Nash equilibria they can rely on well-known results for (non lexicographic) zero-sum two-player games. Previously to [6], existence results about different kinds of equilibria (Nash, secure, perfect, and subgame perfect) have been established for *quantitative reachability* objectives in multi-player weighted games [4, 5, 17]. In [23], among other results, the authors study the decision problem of both the existence and the constraint existence of Nash equilibria in multi-player weighted games for the mean-payoff measure. A part of our contribution (3) is inspired by some of their techniques. The *rational synthesis* problem is studied in [15], when a system interacts with agents that all have their own objectives. This problem asks to construct a strategy profile that enforces the objective of the system, and which is an equilibrium for the agents. The objectives are ω -regular or defined by deterministic latticed Büchi word automata (they do not include mean-payoff and discounted sum objectives), and secure equilibria are not considered explicitly. Lexicographic games were first considered in [2], for a mean-payoff measure that differs from the one studied in this paper. The proof technique that we develop is similar to their approach but requires non trivial adaptations. To the best of our knowledge, lexicographic games for all the other objectives (sup, inf, lim sup, lim inf, and discounted sum) have not been studied previously.

Structure of the paper In Section 2, we first recall classical notions on games and equilibria, we then introduce the three problems studied in this paper and our solutions for the sup, inf, lim sup, lim inf, mean-payoff, and discounted sum measures (Problems 1-3 and Theorems 1-3). In Section 3, for each of the three problems, we provide a general framework of weighted games in which the problem can be solved (Propositions 1-3). Those frameworks impose in particular the determinacy of two (one for each player) lexicographic games associated with the initial game. Lexicographic games are introduced in Section 3 and proved to be determined in Section 4, where their complexity is also established. With these results, in Section 3, we are

able to prove Theorems 1-2, because games with the considered measures all fall in the frameworks proposed for solving Problems 1-2. The framework proposed for the last problem requires an additional hypothesis that we study in Section 5. Theorem 3 can then be derived for all the measures, except the discounted sum. We also show that Problem 3 for this measure is linked to a challenging open problem [3, 11]. In Section 6, we give a conclusion. Due to lack of space, some proofs are just sketched or omitted (see [7] for a full version of the paper).

2. Weighted Games and Studied Problems

In this section, we recall the notions of weighted game, Nash equilibrium, secure equilibrium, and we state the problems that we want to solve.

2.1 Weighted Games

We consider two-player turn-based non zero-sum weighted games with weights seen as rewards. The two players want to maximize their payoff (this payoff is computed from the weights, for example as in Definition 2).

Definition 1. A two-player non zero-sum weighted game is a tuple $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$ where

- (V, E) is a finite directed graph, the arena of the game, with vertices V and edges $E \subseteq V \times V$, such that $\forall v \in V, \exists v' \in V : e = (v, v') \in E$ (no deadlock),
- V_1, V_2 form a partition of V such that V_i is the set of vertices controlled by player $i \in \{1, 2\}$,
- $r = (r_1, r_2)$ is the weight function such that $r_i : E \rightarrow \mathbb{Q}$ associates a rational reward with each edge for player i ,
- $\text{Payoff} = (\text{Payoff}_1, \text{Payoff}_2)$ is the payoff function where $\text{Payoff}_i : V^\omega \rightarrow \mathbb{R}$ is the payoff function of player i such that Payoff_i is defined starting from r_i .

When an initial vertex $v_0 \in V$ is fixed, we call $(\mathcal{G}, v_0)$ an *initialized weighted game*. A *play* of $(\mathcal{G}, v_0)$ is an infinite sequence $\rho = \rho_0 \rho_1 \dots \in V^\omega$ such that $\rho_0 = v_0$ and $(\rho_i, \rho_{i+1}) \in E$ for all $i \in \mathbb{N}$. *Histories* of $(\mathcal{G}, v_0)$ are finite sequences $h = h_0 \dots h_n \in V^+$ defined in the same way. A *prefix* (resp. *suffix*) of a play ρ is a finite sequence $\rho_0 \dots \rho_n$ (resp. infinite sequence $\rho_n \rho_{n+1} \dots$) denoted by $\rho_{\leq n}$ (resp. $\rho_{\geq n}$). The *length* of $\rho_{\leq n}$ is the number n of its edges. For a play $\rho \in V^\omega$ and a player $i \in \{1, 2\}$, we denote by $r_i(\rho) = r_i(\rho_0, \rho_1) r_i(\rho_1, \rho_2) \dots$ the sequence of player i weights along ρ . The *payoff* of ρ for player i is given by $\text{Payoff}_i(\rho)$, and $\text{Payoff}(\rho) = (\text{Payoff}_1(\rho), \text{Payoff}_2(\rho))$ is the payoff of ρ in $\mathcal{G}$.

A *strategy* σ for player $i \in \{1, 2\}$ is a function $\sigma : V^* V_i \rightarrow V$ assigning to each history $hv \in V^* V_i$ a vertex $v' = \sigma(hv)$ such that $(v, v') \in E$. We denote by Σ_i the set of strategies of player i . A strategy σ for player i is *positional* if $\sigma(h) = \sigma(h')$ for all histories h, h' ending with the same vertex (σ only depends on the last vertex of the history). In particular, a positional strategy is a function $\sigma : V_i \rightarrow V$ (instead of $\sigma : V^* V_i \rightarrow V$). A strategy σ is a *finite-memory* strategy if it needs only finite memory of the history (recorded by a finite strategy automaton).

Given a strategy $\sigma \in \Sigma_i$ with $i \in \{1, 2\}$, we say that a play ρ of $\mathcal{G}$ is *consistent* with σ if $\rho_{k+1} = \sigma(\rho_{\leq k})$ for all $k \in \mathbb{N}$ such that $\rho_k \in V_i$. A *strategy profile* of $\mathcal{G}$ is a pair (σ_1, σ_2) of strategies, with $\sigma_i \in \Sigma_i$ for each $i \in \{1, 2\}$. Given an initial vertex v_0 , such a strategy profile determines a unique play

of $(\mathcal{G}, v_0)$ that is consistent with both strategies. This play is called the *outcome* of (σ_1, σ_2) and is denoted by $\langle \sigma_1, \sigma_2 \rangle_{v_0}$. For a history $hv \in V^*V$, and a strategy profile (σ_1, σ_2) , we denote by $\langle \sigma_1|_h, \sigma_2 \rangle_v$ the outcome of $(\sigma_1|_h, \sigma_2)$ in the initialized game $(\mathcal{G}, v)$, where $\sigma_1|_h$ is the strategy defined by $\sigma_1|_h(h'v') = \sigma_1(hh'v')$ for all histories $h'v' \in V^*V_1$ that begin with v . The outcome $\langle \sigma_1, \sigma_2|_h \rangle_v$ is defined similarly. A player *deviates* from a strategy (resp. from a play) if he does not carefully follow this strategy (resp. play). We say that a strategy profile (σ_1, σ_2) is a positional (resp. finite-memory) strategy profile if σ_1 and σ_2 are positional (resp. finite-memory) strategies.

In this paper, we focus on several well-known payoff functions (see for instance [10]).

Definition 2. Given a weighted game $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$, we define the payoff function Payoff as one of the measures in $\{\text{Inf}, \text{Sup}, \text{LimInf}, \text{LimSup}, \text{InfMP}, \text{SupMP}, \text{Disc}^\lambda\}$ for $\lambda \in]0, 1[$, where for all $i \in \{1, 2\}$ and $\rho \in V^\omega$:

- $\text{Inf}_i(\rho) = \inf_{n \in \mathbb{N}} r_i(\rho_n, \rho_{n+1})$,
- $\text{Sup}_i(\rho) = \sup_{n \in \mathbb{N}} r_i(\rho_n, \rho_{n+1})$,
- $\text{LimInf}_i(\rho) = \liminf_{n \rightarrow \infty} r_i(\rho_n, \rho_{n+1})$,
- $\text{LimSup}_i(\rho) = \limsup_{n \rightarrow \infty} r_i(\rho_n, \rho_{n+1})$,
- $\text{InfMP}_i(\rho) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} r_i(\rho_k, \rho_{k+1})$,
- $\text{SupMP}_i(\rho) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} r_i(\rho_k, \rho_{k+1})$,
- $\text{Disc}_i^\lambda(\rho) = (1 - \lambda) \cdot \sum_{n=0}^{\infty} \lambda^n r_i(\rho_n, \rho_{n+1})$,

We also call these games Payoff weighted games.

2.2 Equilibria

We now recall the concept of Nash equilibrium and secure equilibrium. In this aim we need to fix two lexicographic orders on $\mathbb{R}^2$: a lexicographic order $\preccurlyeq_1$ w.r.t. the first component and a lexicographic order $\preccurlyeq_2$ w.r.t. the second component such that for all $(x_1, x_2), (x'_1, x'_2) \in \mathbb{R}^2$,

$$\begin{aligned} (x_1, x_2) \preccurlyeq_1 (x'_1, x'_2) &\text{ iff } (x_1 < x'_1) \vee (x_1 = x'_1 \wedge x_2 \geq x'_2), \\ (x_1, x_2) \preccurlyeq_2 (x'_1, x'_2) &\text{ iff } (x_2 < x'_2) \vee (x_2 = x'_2 \wedge x_1 \geq x'_1). \end{aligned}$$

Notice that $(\mathbb{R}^2, \preccurlyeq_1)$ and $(\mathbb{R}^2, \preccurlyeq_2)$ are totally ordered sets.

A strategy profile (σ_1, σ_2) in an initialized weighted game $(\mathcal{G}, v_0)$ is a Nash equilibrium if player 1 (resp. player 2) has no incentive to deviate unilaterally from σ_1 (resp. σ_2), since he cannot strictly increase his payoff when using σ'_1 (resp. σ'_2) instead of σ_1 (resp. σ_2). The notion of secure equilibrium is stronger in the sense that player i has no incentive to deviate from σ_i with respect to the order $\preccurlyeq_i$ (instead of the usual order $\leq$ on his payoffs).

Definition 3. Let $(\mathcal{G}, v_0)$ be an initialized weighted game. A strategy profile (σ_1, σ_2) with $\sigma_i \in \Sigma_i$, $i \in \{1, 2\}$, is a Nash equilibrium in $(\mathcal{G}, v_0)$ if, for each strategy $\sigma'_i \in \Sigma_i$, $i \in \{1, 2\}$,

$$\begin{aligned} \text{Payoff}_1(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) &\leq \text{Payoff}_1(\langle \sigma_1, \sigma_2 \rangle_{v_0}), \\ \text{Payoff}_2(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) &\leq \text{Payoff}_2(\langle \sigma_1, \sigma_2 \rangle_{v_0}). \end{aligned}$$

It is a secure equilibrium in $(\mathcal{G}, v_0)$ if, for each strategy $\sigma'_i \in \Sigma_i$, $i \in \{1, 2\}$,

$$\begin{aligned} \text{Payoff}(\langle \sigma'_1, \sigma_2 \rangle_{v_0}) &\preccurlyeq_1 \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}), \\ \text{Payoff}(\langle \sigma_1, \sigma'_2 \rangle_{v_0}) &\preccurlyeq_2 \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}). \end{aligned}$$

Note that the notion of secure equilibrium is a refinement of that of Nash equilibrium. In a Nash equilibrium, each player only cares about his own payoff (he maximizes it), whereas in a secure equilibrium, he cares about the payoff of both players (he maximizes his payoff and then minimizes the payoff of the other player).

With the notation of Definition 3, σ'_1 is called a *profitable deviation* for player 1 w.r.t. (σ_1, σ_2) if we have $\text{Payoff}_1(\langle \sigma_1, \sigma_2 \rangle_{v_0}) < (\text{resp. } \preccurlyeq_1) \text{Payoff}_1(\langle \sigma'_1, \sigma_2 \rangle_{v_0})$. Profitable deviations for player 2 are defined similarly. With these terms, we can say that (σ_1, σ_2) is a Nash equilibrium (resp. secure equilibrium) if no player has a profitable deviation w.r.t. (σ_1, σ_2) for the relation $<$ (resp. $\preccurlyeq_i$).

Example 1. Consider the initialized InfMP weighted game $(\mathcal{G}, v_0)$ depicted in Figure 1. Circle (resp. square) vertices are player 1 (resp. player 2) vertices, and the weights $(0, 0)$ are not specified. In this game, both players have two positional strategies, respectively denoted by σ_i and σ'_i for each player i . These strategies are depicted in Figure 1.

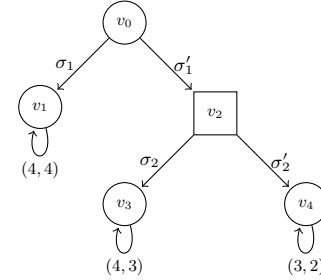


Figure 1. A simple weighted game

The strategy profiles (σ_1, σ'_2) and (σ'_1, σ_2) are secure equilibria in $(\mathcal{G}, v_0)$. Indeed, for the first one, only player 1 has control on the play $\langle \sigma_1, \sigma'_2 \rangle_{v_0}$ with payoff $(4, 4)$, and he decreases his payoff if he plays strategy σ'_1 instead of σ_1 $((3, 2) \preccurlyeq_1 (4, 4))$. For the second one, both players have control on the play $\langle \sigma'_1, \sigma_2 \rangle_{v_0}$ with payoff $(4, 3)$. If player 1 plays σ_1 instead of σ'_1 , he keeps the same payoff but increases the payoff of player 2 $((4, 4) \preccurlyeq_2 (4, 3))$. If player 2 plays σ'_2 instead of σ_2 , he decreases his payoff $((3, 2) \preccurlyeq_2 (4, 3))$. Hence in both examples no player has a profitable deviation, and these strategy profiles are secure equilibria (and thus Nash equilibria) in $(\mathcal{G}, v_0)$. However the strategy profile (σ_1, σ_2) is a Nash equilibrium which is not a secure equilibrium. Indeed, σ'_1 is a profitable deviation for player 1 since by playing σ'_1 instead of σ_1 , he keeps his own payoff but can decrease the payoff of player 2. Finally, the strategy profile (σ'_1, σ'_2) is neither a secure equilibrium nor a Nash equilibrium.

2.3 Problems and Main Theorems

To conclude Section 2, we state the three problems studied in the paper and our results. Let $(\mathcal{G}, v_0)$ be an initialized two-player non zero-sum weighted game.

Problem 1. Does there exist a secure equilibrium in $(\mathcal{G}, v_0)$? If so, does there exist a finite-memory one?

Problem 2. What is the complexity of constructing a (finite-memory) secure equilibrium in $(\mathcal{G}, v_0)$ if one exists?

Problem 3. Given two thresholds $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$, is it decidable whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$, i.e. $\mu_i \leq \text{Payoff}_i(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu_i$ for $i \in \{1, 2\}$?

As expected, if no restriction is given on the payoff function used in weighted games, the answer to Problem 1 is negative. An example of a weighted game with no Nash equilibrium (and thus with no secure equilibrium) is given in [6]. In this paper, we solve the stated problems for weighted games with the payoff functions given in Definition 2. Our solutions are as follows (see also Tables 1-2):

Theorem 1. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game. Then $(\mathcal{G}, v_0)$ has a secure equilibrium with memory size¹ bounded by $|V| + 2$ (resp. $|V| \cdot |E|^2 + 3$) for payoff functions InfMP, SupMP, LimInf, LimSup and Disc^λ (resp. Inf and Sup).*

Theorem 2. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game. Then one can compute a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ in pseudo-polynomial time (resp. polynomial time) for payoff functions InfMP, SupMP and Disc^λ (resp. LimInf, LimSup, Inf and Sup).*

Theorem 3. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game and $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Then one can decide in $\text{NP} \cap \text{co-NP}$ (resp. in P) whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$ for payoff functions InfMP and SupMP (resp. LimInf, LimSup, Inf and Sup).*

	Memory size	Construction time
InfMP, SupMP	$ V + 2$	pseudo-polynomial
LimInf, LimSup	$ V + 2$	polynomial
Inf, Sup	$ V \cdot E ^2 + 3$	polynomial
Disc ^λ	$ V + 2$	pseudo-polynomial

Table 1. Memory size and construction time of a finite-memory secure equilibrium (see Theorems 1-2)

	Complexity
InfMP, SupMP	$\text{NP} \cap \text{co-NP}$
LimInf, LimSup	P
Inf, Sup	P
Disc ^λ	?

Table 2. Complexity of Problem 3 (see Theorem 3)

3. Lexicographic Payoff Games and Equilibria

To solve Problems 1-3, we follow the approach proposed in [6] to solve the first problem for Nash (instead of secure) equilibria. The general idea is the following one. Given an initialized (non zero-sum) weighted game $(\mathcal{G}, v_0)$, we derive two well-chosen two-player (zero-sum) games $\mathcal{G}^{\preceq_1}$ and $\mathcal{G}^{\preceq_2}$, and under adequate hypotheses, we obtain properties about secure equilibria in $(\mathcal{G}, v_0)$ through determinacy results and characterization of the optimal strategies of $\mathcal{G}^{\preceq_1}$ and $\mathcal{G}^{\preceq_2}$.

In this section, we first introduce the games $\mathcal{G}^{\preceq_1}$ and $\mathcal{G}^{\preceq_2}$. Then for each studied problem, we propose a general framework (i.e. adequate general hypotheses on $(\mathcal{G}, v_0)$, $\mathcal{G}^{\preceq_1}$ and $\mathcal{G}^{\preceq_2}$) under which we are able to solve it (Propositions 1-3). Later in Sections 4-5, we prove that these hypotheses are

satisfied for most of the weighted games with the payoff functions of Definition 2. We will thus be able to prove Theorems 1-3 as a consequence of Propositions 1-3.

3.1 Lexicographic Payoff Games

Let $\mathcal{G}$ be a weighted game as in Definition 1. We associate with $\mathcal{G}$ two zero-sum games $\mathcal{G}^1$ and $\mathcal{G}^2$, one for each player, that are respectively equipped with the lexicographic order $\preceq_1$ and $\preceq_2$ used to compare payoffs of these games.

Definition 4. *From a weighted game $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$ as in Definition 1, we derive a zero-sum lexicographic payoff game $\mathcal{G}^{\preceq_1}$ of the form $(V, V_1, V_2, E, r, \text{Payoff}, \preceq_1)$.*

In the zero-sum game $\mathcal{G}^{\preceq_1}$, the two players have antagonistic goals. For each play $\rho \in V^\omega$, player 1 receives the payoff $\text{Payoff}(\rho)$ that he wants to maximize w.r.t. the lexicographic order $\preceq_1$, while player 2 pays the cost $\text{Payoff}(\rho)$ that he wants to minimize w.r.t. $\preceq_1$. When Payoff is one among the payoff functions of Definition 2, we say that $\mathcal{G}^{\preceq_1}$ is a **Payoff lexicographic payoff game**.

We also consider a dual lexicographic payoff game $\mathcal{G}^{\preceq_2} = (V, V_2, V_1, E, r, \text{Payoff}, \preceq_2)$ where the roles of the two players are exchanged and the used lexicographic order is $\preceq_2$. In this game player 2 wants to maximize $\text{Payoff}(\rho)$ w.r.t. $\preceq_2$, while player 1 wants to minimize it.

Given a lexicographic payoff game $\mathcal{G}^{\preceq_1}$, we define for every vertex $v \in V$, the *upper value* $\overline{\text{Val}}(v) = \inf_{\sigma_2 \in \Sigma_2} \sup_{\sigma_1 \in \Sigma_1} \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v)$, and the *lower value* $\underline{\text{Val}}(v) = \sup_{\sigma_1 \in \Sigma_1} \inf_{\sigma_2 \in \Sigma_2} \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_v)$. The game $\mathcal{G}^{\preceq_1}$ is *determined* if, for every $v \in V$, we have $\overline{\text{Val}}(v) = \underline{\text{Val}}(v)$. We also say that $\mathcal{G}^{\preceq_1}$ has a *value* from v , and we write $\text{Val}(v) = \overline{\text{Val}}(v) = \underline{\text{Val}}(v)$. Moreover, we say that $\sigma_1^* \in \Sigma_1$ is an *optimal strategy* for player 1 and vertex v if, for each strategy $\sigma_2 \in \Sigma_2$, we have $\underline{\text{Val}}(v) \preceq_1 \text{Payoff}(\langle \sigma_1^*, \sigma_2 \rangle_v)$. Similarly, $\sigma_2^* \in \Sigma_2$ is an optimal strategy for player 2 and vertex v if, for each strategy $\sigma_1 \in \Sigma_1$, we have $\text{Payoff}(\langle \sigma_1, \sigma_2^* \rangle_v) \preceq_1 \overline{\text{Val}}(v)$. Finally, we say that $\mathcal{G}^{\preceq_1}$ is *positionally-determined* if it is determined and has positional optimal strategies for both players and all vertices. Additionally, we say that $\mathcal{G}^{\preceq_1}$ is *uniformly-determined* if the positional optimal strategies σ_1^*, σ_2^* can be chosen globally independently of vertex v . We call these strategies *uniform*.

The next properties are other important ingredients to solve Problems 1-3.

Definition 5 ([6]). *Let $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$ be a weighted game. The payoff function Payoff_i , $i \in \{1, 2\}$, is prefix-linear in $\mathcal{G}$ if, for every vertex $v \in V$ and history $hv \in V^+$, there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ such that, for every play $\rho \in V^\omega$ whose first vertex is v , we have :*

$$\text{Payoff}_i(h\rho) = a + b \cdot \text{Payoff}_i(\rho).$$

Moreover, Payoff_i is prefix-independent if $\text{Payoff}_i(h\rho) = \text{Payoff}_i(\rho)$.

Remark 1. Clearly the two components of the payoff functions LimInf, LimSup, InfMP and SupMP are prefix-independent. For payoff function Disc^λ, the components are not prefix-independent but rather prefix-linear (see [6]). Finally it is easy to see that neither Inf nor Sup functions have prefix-linear components.

¹ The memory size is the number of states of the strategy automaton.

² Remind that the infimum and supremum functions are applied on a set of payoffs lexicographically ordered with $\preceq_1$.

3.2 Existence of Secure Equilibria

We can now state the general framework under which Problem 1 is positively solved. It is composed of all initialized weighted games $(\mathcal{G}, v_0)$ such that for each $i \in \{1, 2\}$, lexicographic payoff games $\mathcal{G}^{\preceq i}$ are uniformly-determined, and payoff functions Payoff_i are prefix-linear.

Proposition 1. *Let $\mathcal{G} = (V, V_1, V_2, E, r, \text{Payoff})$ be a weighted game and v_0 be an initial vertex. If for each $i \in \{1, 2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preceq i}$ is uniformly-determined, then there exists a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ with memory size bounded by $|V| + 2$.*

This proposition is an adaptation to secure equilibria in two-player games of a theorem³ given in [6, 12] for the existence of Nash equilibria in multi-player games. The main difference is that we here need to work with (two-dimensional) lexicographic payoff games instead of classical (one-dimensional, non lexicographic) zero-sum quantitative games. The proof of this proposition is similar to the one given in [6, 12]. Let σ_1^* (resp. σ_{-1}^*) be a uniform optimal strategy of player 1 (resp. player 2) in $\mathcal{G}^{\preceq 1}$, and similarly let σ_2^* (resp. σ_{-2}^*) in $\mathcal{G}^{\preceq 2}$. The secure equilibrium in $(\mathcal{G}, v_0)$ is constructed as follows: player i plays according to his strategy σ_i^* in $\mathcal{G}^{\preceq i}$, and punishes the other player j if he is the first player to deviate from his strategy σ_j^* in $\mathcal{G}^{\preceq j}$, by playing according to σ_{-j}^* in $\mathcal{G}^{\preceq j}$.⁴

Notice that we could prove a result similar to Proposition 1 such that the hypotheses are replaced by the following ones: $(\mathcal{G}, v_0)$ is an initialized weighted game such that for each $i \in \{1, 2\}$, $\mathcal{G}^{\preceq i}$ is determined (less restrictive than uniformly-determined) and Payoff_i is prefix-independent (more restrictive than prefix-linear). The same kind of result is proved in [12] in the case of Nash equilibria; its proof can be easily adapted to the case of secure equilibria as done for the proof of Proposition 1.

3.3 Construction of Secure Equilibria

We have proposed a general framework to solve Problem 1: a finite-memory secure equilibrium can be constructed from the uniform optimal strategies in $\mathcal{G}^{\preceq 1}$ and $\mathcal{G}^{\preceq 2}$. Hence, under the additional hypothesis that one can compute those strategies, Problem 2 is also solved.

Proposition 2. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game such that for each $i \in \{1, 2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preceq i}$ is uniformly-determined with computable uniform optimal strategies. If such strategies can be computed in $\mathbb{C}$ time for both players, then a finite-memory secure equilibrium in $(\mathcal{G}, v_0)$ can also be constructed in $\mathbb{C}$ time.*

3.4 Constrained Existence of Secure Equilibria

Let us turn to Problem 3. First notice that contrarily to Problem 2, strategy profiles for solving this problem require infinite memory, as shown by Example 2.

Example 2. Consider the initialized InfMP weighted game $(\mathcal{G}, v_0)$ of Figure 2. Take thresholds $\mu = (1, 1)$ and $\nu = (+\infty, +\infty)$.

As explained in [24, Proof of Lemma 7], any strategy profile that is finite-memory produces an ultimately periodic

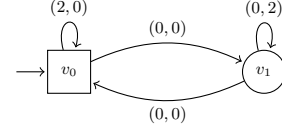


Figure 2. An weighted game that shows the need of infinite memory for Problem 3

outcome ρ such that $\text{Payoff}_1(\rho) + \text{Payoff}_2(\rho) < 1$, thus not satisfying threshold $(1, 1)$. However there exists an infinite path ρ^* in $(\mathcal{G}, v_0)$ with payoff equal to $(1, 1)$: ρ^* visits n times vertex v_0 and then n times vertex v_1 , and repeats this forever with increasing values of n .

Let us define a strategy profile (σ_1, σ_2) such that its outcome is equal to ρ^* , and as soon as one player deviates from ρ^* , the other player punishes him by always choosing an edge with weights $(0, 0)$. This strategy profile is a secure equilibrium since one verifies that the player who first deviates receives a payoff of 0 instead of 1.

Therefore there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ that satisfies $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$, but it cannot be finite-memory.

Let us now show that Problem 3 is decidable in the same framework as for Problem 1, with the additional hypotheses that one can compute values in $\mathcal{G}^{\preceq i}$, for $i \in \{1, 2\}$, and the next problem is decidable.

Problem 4. *Let $G = (V, E, v_0, r, \text{Val})$ be a finite directed graph with an initial vertex v_0 , a weight function $r = (r_1, r_2)$ with $r_i : E \rightarrow \mathbb{Q}$, and a value function $\text{Val} = (\text{Val}^1, \text{Val}^2)$, with $\text{Val}^i : V \rightarrow \mathbb{Q}^2$. Let $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Is it decidable whether there exists an infinite path ρ in G starting in v_0 such that*

- $\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_{\geq k})$, and
- $\mu \leq \text{Payoff}(\rho) \leq \nu$?

Proposition 3. *Let $(\mathcal{G}, v_0)$ be an initialized weighted game and $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Suppose that*

- for each $i \in \{1, 2\}$, Payoff_i is prefix-linear, and $\mathcal{G}^{\preceq i}$ is uniformly-determined with computable values,
- Problem 4 is decidable for the graph G constructed from $\mathcal{G}$ such that both functions Val^i , $i \in \{1, 2\}$, are constructed from the values $\text{Val}^i(v)$, $v \in V$, in $\mathcal{G}^{\preceq i}$.

One can decide whether there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}, v_0)$ such that $\mu \leq \text{Payoff}(\langle \sigma_1, \sigma_2 \rangle_{v_0}) \leq \nu$.

The justification is based on the next lemma that characterizes the outcomes of secure equilibria.

Lemma 1. *Let ρ be a play in a weighted game $(\mathcal{G}, v_0)$. Then ρ is the outcome of a secure equilibrium in $(\mathcal{G}, v_0)$ iff*

$$\forall k \geq 0, \forall i \in \{1, 2\}, \quad \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_{\geq k}). \quad (1)$$

Indeed, an algorithm for Problem 3 is as follows. First compute $\text{Val}^i(v)$ for each vertex v of $\mathcal{G}^{\preceq i}$, $i \in \{1, 2\}$. Then construct the graph $G = (V, E, v_0, r, \text{Val})$. Finally test whether there exists a path ρ in G as in Problem 4.⁵

³In Theorem 10 of [6], one hypothesis is missing: determinacy must be uniform determinacy.

⁴Notice that the proof of Proposition 1 also holds for payoff functions that mix different measures for the two players, like for example InfMP₁ for player 1 and Sup₂ for player 2.

⁵Again, as for Propositions 1-2, Proposition 3 also holds for distinct payoff functions that mix different measures for the two players.

4. Determinacy of Lexicographic Games

In Section 3, we have established strong links between secure equilibria in weighted games $(\mathcal{G}, v_0)$ and determinacy of the two games $\mathcal{G}^{\preceq_1}$, $\mathcal{G}^{\preceq_2}$. We have shown how to solve Problems 1-3 under some general hypotheses (see Proposition 1-3). In this section, we study the determinacy of lexicographic payoff games for the payoff functions of Definition 2. We show that these games are uniformly-determined, except for the **Inf** and **Sup** payoffs for which they are only positionally-determined. We also study the complexity of computing values and optimal strategies for these games. Besides being important ingredients to solve Problems 1-3, these results are also very interesting on their own right. Our results are the following ones (we state them only for $\mathcal{G}^{\preceq_1}$) (see also Table 3):

Theorem 4. *Let $\mathcal{G}^{\preceq_1}$ be a lexicographic payoff game. Then $\mathcal{G}^{\preceq_1}$ is uniformly-determined (resp. positionally-determined) for payoff functions **InfMP**, **SupMP**, **LimInf**, **LimSup** and Disc^λ (resp. **Inf** and **Sup**).*

Theorem 5. *Let $\mathcal{G}^{\preceq_1}$ be a lexicographic payoff game.*

1. *Let v be a vertex of $\mathcal{G}^{\preceq_1}$ and $(\alpha, \beta) \in \mathbb{Q}^2$ be a pair of rationals. Deciding whether $(\alpha, \beta) \preceq_1 \text{Val}(v)$ is in $\text{NP} \cap \text{co-NP}$ for payoff functions **InfMP**, **SupMP** and Disc^λ , and P-complete for payoff functions **LimInf**, **LimSup**, **Inf** and **Sup**.*
2. *The (rational) value of each vertex of $\mathcal{G}^{\preceq_1}$ can be computed in pseudo-polynomial time for payoff functions **InfMP**, **SupMP** and Disc^λ , and in polynomial time for payoff functions **LimInf**, **LimSup**, **Inf** and **Sup**.*
3. *Uniform (resp. positional) optimal strategies for both players can be computed in pseudo-polynomial time for payoff functions **InfMP**, **SupMP** and Disc^λ , and in polynomial time for payoff functions **LimInf** and **LimSup** (resp. **Inf** and **Sup**).*

Theorem 4 states that every **Inf** lexicographic payoff game is positionally-determined. Example 3 shows that there is no hope to have uniform determinacy.

Example 3. Consider the **Inf** lexicographic payoff game depicted in Figure 3. This game is positionally-determined, each vertex has value $(2, 0)$ except vertex v_4 that has value $(3, 1)$. Let us show that it is not uniformly-determined. To guarantee value $(2, 0)$ for v_0 , player 1 has to use the positional strategy σ_1^* such that $\sigma_1^*(v_4) = v_3$. Indeed, this is the only way to have $(2, 0) \preceq_1 \text{Inf}((\sigma_1^*, \sigma_2)_{v_0})$ for all strategy σ_2 of player 2. However, with this strategy, player 1 cannot guarantee value $(3, 1)$ for v_4 .

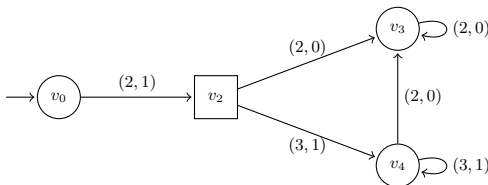


Figure 3. An **Inf** lexicographic payoff game that is not uniformly-determined.

Before giving the proofs of Theorems 4-5, notice that we are now able to prove Theorems 1-2 for the payoff functions **InfMP**, **SupMP**, **LimInf**, **LimSup** and Disc^λ . Indeed

the hypotheses of Propositions 1-2 are satisfied as the two components of these payoff functions are prefix-linear (see Remark 1) and the games $\mathcal{G}^{\preceq_i}$, $i \in \{1, 2\}$, are uniformly-determined with computable uniform optimal strategies (see Theorems 4-5).

For **Inf** and **Sup** payoffs, the proof of Theorem 1 cannot be based on Proposition 1 since none of its hypotheses is satisfied. Indeed, the payoff function **Inf** is not prefix-linear (see Remark 1), and there exist **Inf** lexicographic payoff games that are not uniformly-determined (see Example 3). However we get Theorems 1-2 for **Inf** (resp. **Sup**) payoffs thanks to a polynomial reduction to **LimInf** (resp. **LimSup**) payoffs⁶ and the fact that Theorems 1-2 hold for the latter payoffs.

In the next subsections, we prove Theorems 4-5 for the payoff functions **InfMP** and Disc^λ only. Proofs for **SupMP** payoff are adaptations of proofs for **InfMP** payoff, and proofs for the other payoffs are based on adequate reductions to classical ω -regular games.

4.1 InfMP Lexicographic Payoff Games

Theorems 4-5 for **InfMP** payoffs are non trivial extensions to dimension 2 of well-known results about (one-dimensional) mean-payoff games [13, 25]. In [2], similar results are proved for a class of games called lexicographic mean-payoff games. It should be noted that this class is different from our class of **InfMP** lexicographic payoff games. Indeed, the authors use a lexicographic order $\preceq'_1$ on $\mathbb{R}^2$ different from ours: $(x_1, x_2) \preceq'_1 (x'_1, x'_2)$ iff $(x_1 < x'_1) \vee (x_1 = x'_1 \wedge x_2 \leq x'_2)$, and most importantly the payoff of a play ρ is computed as

$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (r_1(\rho_k, \rho_{k+1}), r_2(\rho_k, \rho_{k+1}))$ at the level of two-dimensional vectors and with the $\liminf$ using order $\preceq'_1$ (whereas we proceed component wise). An example showing that the two definitions are not equivalent is given in [2]⁷.

The proof that we propose for Theorem 4 is inspired from the proof given in [2], however with several non trivial adaptations. Before giving it, let us first make a remark and introduce some useful notions.

Remark 2. With an adequate shifting of the weights, we can assume w.l.o.g. that $\mathcal{G}^{\preceq_1}$ is a lexicographic payoff game with a weight function r with $r_i : E \rightarrow \mathbb{N}$ (instead of $r_i : E \rightarrow \mathbb{Q}$) for each $i \in \{1, 2\}$.

Let us briefly recall the two well-known classes of mean-payoff games and finite cycle-forming games [1, 13, 25]. *Mean-payoff games* are games as in Definition 4 except that functions r and $\text{Payoff} = \text{InfMP}$ are one-dimensional (instead of two-dimensional). The comparison of two payoffs is thus done with usual order $\leq$ instead of $\preceq_1$. It is a classical result that mean-payoff games are uniformly-determined. Mean-payoff games are usually studied by considering the finite duration versions of these games, such that a play is stopped as soon as some vertex is repeated. More precisely, the arena of a *finite cycle-forming game* $\mathcal{F}$ is a finite directed graph (V, E) such that the vertices are partitioned into V_1 and V_2 , and the edges are labeled by weights in $\mathbb{Q}$. The game starts in some initial vertex v_0 , and players play in a turn-based manner until some vertex v is visited for the second time. At this point, the game stops, a history hc called $\mathcal{F}$ -play, has been constructed by both players with c a cycle from v

⁶ We remember in each vertex the minimal (resp. maximal) weight seen along histories reaching this vertex

⁷ This example is given in the ArXiv version of [2], following discussions with one of the authors.

	determinacy	$(\alpha, \beta) \preceq_1 \text{Val}(v)$?	values	optimal strategies
InfMP, SupMP	uniform	NP $\cap$ co-NP	pseudo-polynomial	pseudo-polynomial
LimInf, LimSup	uniform	P-complete	polynomial	polynomial
Inf, Sup	positional	P-complete	polynomial	polynomial
Disc ^{λ}	uniform	NP $\cap$ co-NP	pseudo-polynomial	pseudo-polynomial

Table 3. Obtained results for lexicographic games (see Theorems 4-5)

to v . A payoff is then associated with hc depending on the edge weights of cycle c . As for lexicographic payoff games, the notions of value and optimal strategies can be defined for mean-payoff games and finite cycle-forming games. The next proposition is proved in [1, Section 7.2].

Proposition 4. *A finite cycle-forming game $\mathcal{F}$ such that the payoff of an $\mathcal{F}$ -play depends only on the vertices that appear in the formed cycle (modulo cyclic permutations) is uniformly-determined.*

In the sequel we also need the next lemma about the cycle decomposition of an infinite path in an arena (V, E) . Given a path ρ , we consider its *cycle decomposition* into a multiset of simple cycles as follows.⁸ We push successively vertices $\rho_0, \rho_1, \dots$ onto a stack. Whenever we push a vertex ρ_m equal to a vertex ρ_n in the stack (i.e. a simple cycle $\rho_n \dots \rho_m$ is formed), we remove it from the stack (i.e. we remove all the vertices above ρ_n , but not ρ_n) and add it to the cycle decomposition multiset of ρ . Notice that at any moment, the stack contains the vertices of a simple path, thus at most $|V|$ vertices.

Lemma 2. *Let (V, E) be an arena. For each infinite path ρ in (V, E) and its cycle decomposition, the next properties hold. Let $\mathcal{F}$ be a finite cycle-forming game with arena (V, E) . If c is a cycle in the cycle decomposition multiset of ρ , then c appears in an $\mathcal{F}$ -play hc in the game $\mathcal{F}$, for some history h starting in ρ_0 and composed of edges of ρ .*

Let us now proceed with the proof of Theorem 4 for InfMP payoff. Let $\mathcal{G}^{\preceq_1} = (V, V_1, V_2, E, r, \text{InfMP}, \preceq_1)$ be an InfMP lexicographic payoff game with natural weights (see Remark 2). For each $i \in \{1, 2\}$, we denote by $|r_i| = \max\{r_i(e) \mid e \in E\}$ the maximal weight for component i , and by $|R| = \max\{|r_1|, |r_2|\}$ the maximal weight of the game.

We derive from $\mathcal{G}^{\preceq_1}$ a mean-payoff game $\mathcal{M} = (V, V_1, V_2, E, r_1, \text{InfMP}_1)$ such that the weights are limited to the first component r_1 of r . As such a game is uniformly-determined [13, 25], each vertex v has a value $\text{Val}(v)$ in $\mathcal{M}$. Notice that it is not possible for player 1 to move to a vertex with a higher value, and player 1 will never choose to move to a vertex with a lower value. Similarly, player 2 cannot move to a vertex with a lower value and will never decide to move to a vertex with a higher value.

Thus, for each value l , we can consider the subarena⁹ of (V, E) restricted to V^l being the set of all vertices $v \in V$ such that $\text{Val}(v) = l$. We denote by $\mathcal{G}_{[l]}^{\preceq_1}$ the lexicographic subgame of $\mathcal{G}^{\preceq_1}$ restricted to V^l . We also consider a finite cycle-forming game denoted by $\mathcal{F}^l$ restricted to V^l and with the following payoff function **Cost**. From a given initial vertex v_0 in $\mathcal{F}^l$, the two players play until a cycle c is formed. Let

$\rho = \rho_0 \dots \rho_m \dots \rho_n$ be the constructed $\mathcal{F}^l$ -play such that $v_0 = \rho_0$ and $c = \rho_m \dots \rho_n$ with $\rho_n = \rho_m$. We denote by $\text{MP}_i(c) = \frac{1}{n-m} \sum_{k=m}^{n-1} r_i(\rho_k, \rho_{k+1})$ the mean-payoff of the cycle c according to component i . We define:

$$\text{Cost}(\rho) = \begin{cases} -1 & \text{if } \text{MP}_1(c) > l \\ |r_2| + 1 & \text{if } \text{MP}_1(c) < l \\ \text{MP}_2(c) & \text{if } \text{MP}_1(c) = l. \end{cases}$$

In the finite cycle-forming game $\mathcal{F}^l$, player 1 wants to *minimize* the payoff while player 2 wants to *maximize* it.¹⁰ Recall that $0 \leq \text{MP}_2(c) \leq |r_2|$ since the weights are supposed positive (see Remark 2) and $|r_2|$ denotes the maximal weight for the second component.

By Proposition 4, the finite cycle-forming game $\mathcal{F}^l$ is uniformly-determined. We have the next lemma.

Lemma 3. *Let v be a vertex in $\mathcal{F}^l$ with value¹¹ $\beta^l(v)$, then $0 \leq \beta^l(v) \leq |r_2|$.*

Lemma 4 relates values in game $\mathcal{F}^l$ to values in game $\mathcal{G}_{[l]}^{\preceq_1}$ (notice that it is not the whole game $\mathcal{G}^{\preceq_1}$), as well as for optimal strategies. It is the last step to get Theorem 4.

Lemma 4. 1. *Let v be a vertex such that its value in $\mathcal{M}$ is $\text{Val}(v) = l$ and its value in $\mathcal{F}^l$ is $\beta^l(v)$. Then its value in $\mathcal{G}_{[l]}^{\preceq_1}$ is $(\text{Val}(v), \beta^l(v))$.*

2. *A uniform optimal strategy in $\mathcal{F}^l$ is optimal in $\mathcal{G}_{[l]}^{\preceq_1}$.*

Proof. As the game $\mathcal{F}^l$ is uniformly-determined, a uniform optimal strategy τ_1^* (resp. τ_2^*) exists for player 1 (resp. player 2) in $\mathcal{F}^l$. To prove the lemma, we show that for each $v \in V^l$, strategy τ_1^* ensures at least¹² the value $(\text{Val}(v), \beta^l(v))$ in $\mathcal{G}_{[l]}^{\preceq_1}$, and τ_2^* ensures at most $(\text{Val}(v), \beta^l(v))$ in $\mathcal{G}_{[l]}^{\preceq_1}$. We only prove this claim for player 1 (it is in the same vein for player 2).

Let us fix strategy τ_1^* and consider any strategy σ_2 of player 2 in $\mathcal{G}_{[l]}^{\preceq_1}$. Let $v \in V^l$ and $\rho = \langle \tau_1^*, \sigma_2 \rangle_v$. Let $\rho_{\leq n}$ be any prefix of ρ . By Lemma 2, each simple cycle c of the cycle decomposition multiset of $\rho_{\leq n}$ appears in an $\mathcal{F}$ -play hc from v consistent with τ_1^* in the finite cycle-forming game $\mathcal{F}^l$. By Lemma 3, $\beta^l(v) \leq |r_2|$. Since τ_1^* is optimal in $\mathcal{F}^l$, it follows that either $\text{Cost}(hc) = -1$ or $\text{Cost}(hc) = \text{MP}_2(c) \leq \beta^l(v)$. Equivalently, with $\beta = \beta^l(v)$ we have

$$\text{either } \text{MP}_1(c) = l \text{ and } \text{MP}_2(c) \leq \beta \quad (2)$$

$$\text{or } \text{MP}_1(c) \geq \bar{l} > l. \quad (3)$$

For each prefix $\rho_{\leq n}$ of ρ of length n , let us denote by $J_1(n)$ (resp. $J_2(n)$) the sum of the lengths of all simple cycles in its cycle decomposition that satisfy Property (2)

⁸ We can similarly consider the cycle decomposition of finite paths.

⁹ The notion of subarena is classical. Recall that it requires that there is no deadlock.

¹⁰ Here the payoff is rather a cost as player 1 wants to minimize it.

¹¹ It is its value in $\mathcal{F}^l$, not to get confused with its value l in $\mathcal{M}$. We use superscript l in $\beta^l(v)$ to recall the underlying game $\mathcal{F}^l$.

¹² according to $\preceq_1$.

(resp. Property (3)). Recall that $n - J_1(n) - J_2(n) \leq |V|$ by definition of the cycle decomposition. First observe that $0 \leq \liminf_{n \rightarrow \infty} \left(\frac{J_i(n)}{n} \right) \leq \limsup_{n \rightarrow \infty} \left(\frac{J_i(n)}{n} \right) \leq 1$, for $i \in \{1, 2\}$, and $\lim_{n \rightarrow \infty} \frac{J_1(n) + J_2(n)}{n} = 1$, since $1 = \lim_{n \rightarrow \infty} \frac{n}{n} \geq \lim_{n \rightarrow \infty} \frac{J_1(n) + J_2(n)}{n} \geq \lim_{n \rightarrow \infty} \frac{n - |V|}{n} = 1$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{k=0}^{n-1} r_1(\rho_k, \rho_{k+1}) &\geq J_1(n) \cdot l + J_2(n) \cdot \bar{l}, \text{ and} \\ \sum_{k=0}^{n-1} r_2(\rho_k, \rho_{k+1}) &\leq J_1(n) \cdot \beta + J_2(n) \cdot |R| + |V| \cdot |R|. \end{aligned}$$

Let $\kappa = \liminf_{n \rightarrow \infty} \frac{J_2(n)}{n}$. We have either $\kappa > 0$ or $\kappa = 0$. Using standard additive properties of $\liminf$ and $\limsup$ (see for instance [10, Lemma 3]), $\text{InfMP}_1(\rho) \geq$

$$\begin{aligned} &\geq \liminf_{n \rightarrow \infty} \left(\frac{J_1(n)}{n} \cdot l + \frac{J_2(n)}{n} \cdot \bar{l} \right) \\ &\geq \liminf_{n \rightarrow \infty} \left(\frac{J_1(n) + J_2(n)}{n} \cdot l \right) + \liminf_{n \rightarrow \infty} \left(\frac{J_2(n)}{n} \cdot (\bar{l} - l) \right) \\ &= l + \kappa \cdot (\bar{l} - l). \end{aligned}$$

We can now prove that $(l, \beta) \preceq_1 \text{InfMP}(\rho)$. This will show that strategy τ_1^* ensures at least value (l, β) from vertex v in $\mathcal{G}_{[l]}^{\preceq_1}$ against any strategy of player 2.

If $\kappa > 0$, since $\bar{l} > l$, we have $\text{InfMP}_1(\rho) \geq l + \kappa \cdot (\bar{l} - l) > l$, and so $(l, \beta) \preceq_1 \text{InfMP}(\rho)$. If $\kappa = 0$, then $\text{InfMP}_1(\rho) \geq l$, and (recalling that $|R|, \beta \geq 0$), $\text{InfMP}_2(\rho) \leq$

$$\begin{aligned} &\leq \liminf_{n \rightarrow \infty} \left(\frac{J_1(n)}{n} \cdot \beta + \frac{J_2(n)}{n} \cdot |R| + \frac{|V|}{n} \cdot |R| \right) \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{J_2(n)}{n} |R| \right) - \liminf_{n \rightarrow \infty} \left(\frac{-J_1(n)}{n} \cdot \beta + \frac{-|V|}{n} \cdot |R| \right) \\ &\leq \liminf_{n \rightarrow \infty} \left(\frac{J_2(n)}{n} \right) \cdot |R| + \limsup_{n \rightarrow \infty} \left(\frac{J_1(n)}{n} \right) \cdot \beta \\ &\quad + \limsup_{n \rightarrow \infty} \left(\frac{|V|}{n} \cdot |R| \right) \\ &\leq 0 \cdot |R| + 1 \cdot \beta + 0 = \beta. \end{aligned}$$

In this case we thus also have $(l, \beta) \preceq_1 \text{InfMP}(\rho)$. $\square$

We can now complete the proof of Theorem 4.

Proof (of Theorem 4 with InfMP payoff). Let $\mathcal{G}^{\preceq_1}$ be an InfMP lexicographic payoff game. By Lemma 4, the lexicographic subgame $\mathcal{G}_{[l]}^{\preceq_1}$ is uniformly-determined, for each value l . Let $\sigma_1^*[l]$ (resp. $\sigma_2^*[l]$) be a uniform optimal strategy of player 1 (resp. player 2) in $\mathcal{G}_{[l]}^{\preceq_1}$. We construct a uniform strategy σ_1^* (resp. σ_2^*) of player 1 (resp. player 2) such that its restriction to V^l is equal to $\sigma_1^*[l]$ (resp. $\sigma_2^*[l]$). These strategies are well-defined since the different sets V^l form a partition of V . Let us prove that the values in game $\mathcal{G}^{\preceq_1}$ are the same as in the subgames $\mathcal{G}_{[l]}^{\preceq_1}$ and that strategies σ_1^*, σ_2^* are optimal. We only give the proof for player 1 (it is similar for player 2).

Let σ_2 be any strategy of player 2. Let $v \in V$ be such that its value in $\mathcal{M}$ is $\text{Val}(v) = l$ and its value in $\mathcal{F}^l$ is $\beta^l(v) = \beta$. By Lemma 4, its value in $\mathcal{G}_{[l]}^{\preceq_1}$ is equal to (l, β) . Let us show that $(l, \beta) \preceq_1 \text{InfMP}(\rho)$ with $\rho = \langle \sigma_1^*, \sigma_2 \rangle_v$.

Clearly, play ρ eventually stays inside some set V^L such that $L \geq l$ since there is a finite number of such sets. Indeed, by definition of σ_1^* , only player 2 can force the play to leave V^{l_1} to go to another V^{l_2} , and only for values $l_2 > l_1$. Suppose first that $L = l$, i.e., the whole play ρ stays in V^l . Then by Lemma 4, we get $(l, \beta) \preceq_1 \text{InfMP}(\rho)$. Suppose now that $L > l$, and let $n > 0$ such that $\rho' = \rho_{\geq n} \in (V^L)^\omega$. As the function InfMP is prefix-independent, $\text{InfMP}(\rho') = \text{InfMP}(\rho)$, and we can limit our study to ρ' . By definition of σ_1^* and Lemma 4, we have $(\text{Val}(\rho_n), \beta^L(\rho_n)) \preceq_1 \text{InfMP}(\rho')$. It follows by definition of $\preceq_1$ that $\text{InfMP}_1(\rho') \geq \text{Val}(\rho_n) = L > l$. Consequently, $(l, \beta) \preceq_1 \text{InfMP}(\rho')$. $\square$

Finally, the proof of Theorem 5 for InfMP payoff is inspired by a proof given in [2]: it uses a reduction of InfMP lexicographic payoff games to mean-payoff games for optimal strategies, and then applies classical results on mean-payoff games [13, 25].

4.2 Disc^λ Lexicographic Payoff Games

The proof of Theorem 4 for Disc^λ lexicographic payoff games follows from the next simple idea: player 1 first tries to maximize his payoff limited to the first component and then to minimize his payoff limited to the second component. He can use a uniform strategy to achieve this goal because at each step, the game reduces to a classical one-dimensional *discounted game* [25] known to be uniformly-determined. Moreover computing the value of the vertices in a discounted game is in $\text{NP} \cap \text{co-NP}$ [25].

Let us now go into the details. The proof of Theorem 4 for Disc^λ payoff will be obtained as a consequence of Lemmas 5-6 given below.

Let $\lambda \in]0, 1[$ and $\mathcal{G}^{\preceq_1}$ be a Disc^λ lexicographic payoff game. We derive from $\mathcal{G}^{\preceq_1}$ the discounted game $\mathcal{D} = (V, V_1, V_2, E, r_1, \text{Disc}_1^\lambda)$ such that the weights are limited to the first component r_1 of r , and player 1 wants to maximize the payoff Disc_1^λ while player 2 wants to minimize it. By [25], this game is uniformly-determined, and the value of each vertex $v \in V$ and uniform strategies can be defined by the following system of equations: $\text{Val}(v) =$

$$\begin{cases} \max_{(v,v') \in E} \{(1-\lambda) \cdot r_1(v, v') + \lambda \cdot \text{Val}(v')\} & \text{if } v \in V_1, \\ \min_{(v,v') \in E} \{(1-\lambda) \cdot r_1(v, v') + \lambda \cdot \text{Val}(v')\} & \text{if } v \in V_2. \end{cases} \quad (4)$$

We denote by E' the set of all optimal edges of E , i.e. edges realizing the maximum (resp. minimum) for player 1 (resp. player 2) in system (4).

The next lemma states that against an optimal strategy of player 1 in $\mathcal{D}$, player 2 should rather choose edges in E' .

Lemma 5. 1. Let σ_1^* be a uniform optimal strategy of player 1 in $\mathcal{D}$, and let σ_2 be a strategy of player 2 in $\mathcal{D}$. Let v be a vertex in V with value $\text{Val}(v)$ in $\mathcal{D}$. If $\rho = \langle \sigma_1^*, \sigma_2 \rangle_v$ contains two vertices ρ_l, ρ_{l+1} with $(\rho_l, \rho_{l+1}) \in E \setminus E'$, then $\text{Disc}_1^\lambda(\langle \sigma_1^*, \sigma_2 \rangle_v) > \text{Val}(v)$.
2. Similarly let σ_2^* be a uniform optimal strategy of player 2, and σ_1 be a strategy of player 1. If $\rho = \langle \sigma_1, \sigma_2^* \rangle_v$ contains two vertices ρ_l, ρ_{l+1} with $(\rho_l, \rho_{l+1}) \in E \setminus E'$, then $\text{Disc}_1^\lambda(\langle \sigma_1, \sigma_2^* \rangle_v) < \text{Val}(v)$.

Proof. We only give the proof for the first statement, the proof being similar for the second one. Notice that player 1 only uses edges in E' . Suppose that player 2 follows a strategy σ_2 such that $\rho = \langle \sigma_1^*, \sigma_2 \rangle_v$ uses at least one edge in $E \setminus E'$. We consider the smallest index $l \geq 0$ such that $\rho_l \in V_2$ and $(\rho_l, \rho_{l+1}) \in E \setminus E'$.

On one hand, by (4), we have $\text{Val}(v) = (1 - \lambda) \cdot \sum_{k=0}^{l-1} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^l \cdot \text{Val}(\rho_l)$ (*). On the other hand, by optimality of σ_1^* , we have that $\text{Disc}_1^\lambda((\sigma_1^*, \sigma_2)_v) \geq (1 - \lambda) \cdot \sum_{k=0}^{l-1} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^{l+1} \cdot \text{Val}(\rho_{l+1})$. Then by definition of l , we have $\text{Disc}_1^\lambda((\sigma_1^*, \sigma_2)_v) > (1 - \lambda) \cdot \sum_{k=0}^{l-1} \lambda^k \cdot r_1(\rho_k, \rho_{k+1}) + \lambda^l \cdot \text{Val}(\rho_l)$. By this last inequality and (*), we can conclude that $\text{Disc}_1^\lambda((\sigma_1^*, \sigma_2)_v) > \text{Val}(v)$. $\square$

We have shown that if both players play optimally in $\mathcal{D}$, then the produced outcome uses edges of E' only. We now consider the discounted game $\mathcal{D}' = (V, V_1, V_2, E', r_2, \text{Disc}_2^\lambda)$ such that the edges are limited to E' and the weight function is now r_2 . Another difference is that player 1 wants to *minimize* his payoff (now seen as a cost) while player 2 wants to *maximize* it.

The following lemma describes values and optimal strategies of the lexicographic payoff game $\mathcal{G}^{\leq 1}$ in relation with the ones of discounted games $\mathcal{D}$ and $\mathcal{D}'$. Theorem 4 for Disc^λ payoff is a consequence of this lemma.

Lemma 6. 1. Let v be a vertex. If v has value $\text{Val}(v)$ in $\mathcal{D}$ and $\beta(v)$ in $\mathcal{D}'$, then v has value $(\text{Val}(v), \beta(v))$ in $\mathcal{G}^{\leq 1}$.
2. A uniform optimal strategy in $\mathcal{D}'$ is optimal in $\mathcal{G}^{\leq 1}$.

Proof. Let τ_1^* be a uniform optimal strategy for player 1 in $\mathcal{D}'$, and σ_2 be a strategy of player 2 in $\mathcal{G}^{\leq 1}$. Suppose vertex v has value $\text{Val}(v)$ in $\mathcal{D}$ and $\beta(v)$ in $\mathcal{D}'$. Let $\rho = \langle \tau_1^*, \sigma_2 \rangle_v$. We are going to show that $(\text{Val}(v), \beta(v)) \preceq_1 \text{Disc}^\lambda(\rho)$.

As τ_1^* is a strategy in $\mathcal{D}'$, it is an optimal strategy in $\mathcal{D}$. Therefore $\text{Disc}_1^\lambda(\rho) \geq \text{Val}(v)$. If this inequality is strict, then $(\text{Val}(v), \beta(v)) \preceq_1 \text{Disc}^\lambda(\rho)$. Let us thus suppose that $\text{Disc}_1^\lambda(\rho) = \text{Val}(v)$. By Lemma 5, the outcome ρ uses edges of E' only and is then an outcome in $\mathcal{D}'$. By optimality of τ_1^* in $\mathcal{D}'$, it follows that $\text{Disc}_2^\lambda(\rho) \leq \beta(v)$, and thus $(\text{Val}(v), \beta(v)) \preceq_1 \text{Disc}^\lambda(\rho)$.

With τ_2^* a uniform optimal strategy for player 2 in $\mathcal{D}'$ and σ_1 a strategy of player 1 in $\mathcal{G}^{\leq 1}$, we show similarly that $\text{Disc}^\lambda(\langle \sigma_1, \tau_2^* \rangle_v) \preceq_1 (\text{Val}(v), \beta(v))$. $\square$

Finally, the proof of Theorem 5 for Disc^λ payoff is obtained by using classical complexity results on the discounted games $\mathcal{D}$ and $\mathcal{D}'$ [25].

5. Study of Problems 3 and 4

The aim of this section is to prove Theorem 3 about the constrained existence of a secure equilibrium in such games (see Problem 3). Thanks to Proposition 3 that provides a general framework for solving Problem 3, in view of the results of Section 4, it remains to study Problem 4 about the constraint existence of a path in a graph with weights and values. Hence we first study the latter problem. We then derive a proof of Theorem 3 for InfMP, SupMP, LimInf, LimSup, Inf and Sup payoffs. We leave the discounted case open. However we show that a solution in this case would provide a solution to an open problem mentioned in [11] itself related to other difficult open problems in mathematics [3].

5.1 Solution for Problem 4

Problem 4 is decidable for InfMP, SupMP, LimInf and LimSup payoffs. The approach that we develop is inspired by proof techniques proposed in [23] for InfMP payoff.

Theorem 6. Let $G = (V, E, v_0, r, \text{Val})$ be a finite directed graph with an initial vertex v_0 , a weight function r , and a

value function Val . Let $\mu, \nu \in (\mathbb{Q} \cup \{\pm\infty\})^2$ be two thresholds. Then for Payoff = InfMP, SupMP, LimInf and LimSup, one can decide in polynomial time whether there exists an infinite path ρ in G such that $\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_{\geq k})$, and $\mu \leq \text{Payoff}(\rho) \leq \nu$.

Since each Payoff_i is prefix-independent in Theorem 6 (see Remark 1), condition

$$\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho_{\geq k})$$

can be replaced by the simpler condition

$$\forall k \geq 0, \forall i \in \{1, 2\}, \text{Val}^i(\rho_k) \preceq_i \text{Payoff}(\rho).$$

Hence, it is possible to get a proof of Theorem 6 with the next lemma.

Lemma 7. One can decide in polynomial time whether there exists an infinite path ρ in G such that for each $i \in \{1, 2\}$, $\mu_i \sim_i \text{Payoff}_i(\rho) \sim'_i \nu_i$ with $\sim_i, \sim'_i \in \{<, \leq\}$.

We now have the required material to prove Theorem 3 for all the payoffs except Disc^λ payoff. We begin with InfMP, SupMP, LimInf and LimSup payoffs (since Inf and Sup payoffs require a separate proof). An algorithm to solve Problem 3 consists in: (1) computing $\text{Val}^i(v)$ for each vertex v of $\mathcal{G}^{\leq i}$, $i \in \{1, 2\}$; (2) constructing from $(\mathcal{G}, v_0)$ the graph $G = (V, E, v_0, r, \text{Val})$; and (3) applying the algorithm of Theorem 6. For each payoff InfMP, SupMP, LimInf and LimSup, steps (2)-(3) can be done in polynomial time by Theorem 6. For LimInf and LimSup payoffs, step (1) can also be done in polynomial time by Theorem 5. One can prove that it is in $\text{NP} \cap \text{co-NP}$ for InfMP and SupMP payoffs.

For Inf and Sup payoffs, Theorem 3 is proved thanks to the same reduction to LimInf and LimSup weighted games, as done for the proof of Theorems 1-2.

5.2 Particular Case of Disc^λ Payoff

In this section, we consider Disc^λ weighted games. We are going to show that Problem 3 for these games is related to the next problem whose decidability is unknown [11]. Moreover the latter problem is related to other hard open problems in diverse mathematical fields according to [3].

Problem 5. Given three rational numbers a, b and t , and a rational discount factor $\lambda \in]0, 1[$, does there exist an infinite sequence $w = w_0 w_1 \dots \in \{a, b\}^\omega$ such that $\sum_{k=0}^\infty w_k \lambda^k$ is equal to t ?

Let us provide the next reduction of Problem 5 to Problem 3 for Disc^λ payoff. Let $a, b, t \in \mathbb{Q}$ and a rational factor $\lambda \in]0, 1[$. We consider the Disc^λ weighted game $(\mathcal{G}^{a,b}, v_0)$ played on the arena depicted in Figure 4, where c is a rational number such that $c > \max\{|a|, |b|\}$.¹³ We also consider the thresholds $\mu = ((1-\lambda) \cdot t, -(1-\lambda) \cdot t)$ and $\nu = (+\infty, +\infty)$. Notice that threshold ν imposes no constraint. We want to show the next proposition:

Proposition 5. There exists a sequence $w \in \{a, b\}^\omega$ such that $\sum_{k=0}^\infty w_k \lambda^k = t$ iff there exists a secure equilibrium in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ such that $\mu \leq \text{Disc}^\lambda(\rho) \leq \nu$.

We need the next lemma where $\mathcal{A}$ denotes the subarena of $\mathcal{G}^{a,b}$ restricted to the set of vertices $\{v_0, v_1\}$.

¹³ Notice that $(\mathcal{G}^{a,b}, v_0)$ have mutiple edges between two vertices. Adequate intermediate vertices should be added to respect Definition 1.

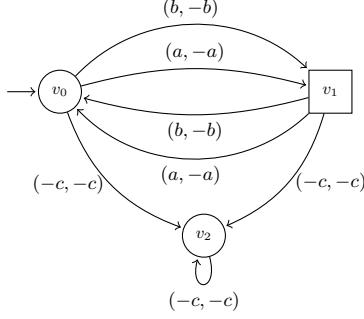


Figure 4. The Disc^λ weighted game $\mathcal{G}^{a,b}$

Lemma 8. For all infinite paths ρ of $\mathcal{G}^{a,b}$ starting in v_0 , ρ is a path in $\mathcal{A}$ iff there exists a secure equilibrium (σ_1, σ_2) in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ .

By definition of the weight function of $\mathcal{G}^{a,b}$, we have $\text{Disc}_1^\lambda(\rho) = -\text{Disc}_2^\lambda(\rho)$. Therefore, by Lemma 8, there exists a sequence $w \in \{a, b\}^\omega$ such that $\sum_{k=0}^\infty w_k \lambda^k = t$ iff there exists a secure equilibrium in $(\mathcal{G}^{a,b}, v_0)$ with outcome ρ such that $\mu \leq \text{Disc}^\lambda(\rho) (\leq \nu)$. Therefore, Proposition 5 holds, and we have shown that Problem 5 reduces to Problem 3 for Disc^λ payoff.

6. Conclusion

In this paper, we have studied Problems 1-3 about secure equilibria in two-player weighted games, and proposed three general frameworks in which we can solve them. We have proved that weighted games with a classical payoff function like InfMP , SupMP , LimInf , LimSup , Inf , Sup and Disc^λ , all fall in these frameworks, except the Disc^λ payoff for Problem 3 (see Tables 1-2). We have shown that this particular problem is linked to another open problem [3, 11].

Our approach was inspired by the recent work [6] that states the existence of Nash equilibria in a large class of multi-player weighted games. As in [6], we have considered two particular zero-sum games $\mathcal{G}^{\leq 1}$, $\mathcal{G}^{\leq 2}$ associated with the initial game $\mathcal{G}$; we have proved that they are uniformly-determined and studied their complexity for all the classical payoffs (see Table 3). These results are very interesting on their own right, and to the best of our knowledge, were not studied in the literature, except for a variant of the InfMP payoff studied in [2]. For InfMP payoff, our proofs were inspired by techniques developed in [2, 23], however with far from trivial adaptations.

Independently of our paper, in [21], the authors give general hypotheses on multi-player weighted games that guarantee the existence of a secure equilibrium. These hypotheses are satisfied by the weighted games studied in this paper with LimInf , LimSup , Inf , Sup and Disc^λ payoffs (but not with InfMP and SupMP payoffs). Thus for these payoffs, Problem 1 is also positively solved thanks to the results proved in [21] (however with no indication about the existence of a secure equilibrium that is finite-memory).

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