

A quest for algorithmically random infinite structures

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Abstract

The last two decades have witnessed significant advances in the investigation of algorithmic randomness, such as Martin-Löf randomness, of infinite strings. In spite of much work, research on randomness of infinite strings has excluded the investigation of algorithmic randomness for infinite algebraic structures. The main obstacle in introducing algorithmic randomness for infinite structures is that many classes of infinite structures lack measure. More precisely, it is unclear how one would define a meaningful measure through which it would be possible to introduce algorithmic randomness for infinite structures. In this paper, we overcome this obstacle by proposing a limited amount of finiteness conditions on various classes of infinite structures. These conditions will enable us to introduce measure and, as a consequence, reason about algorithmic randomness. Our classes include finitely generated universal algebras, connected graphs and trees of bounded degree, and monoids. For all these classes one can introduce algorithmic randomness concepts and prove existence of random structures. In particular, we prove that Martin-Löf random universal algebras, graphs, trees, and monoids exist. In the case of trees we show a stronger result that Martin-Löf random computably enumerable trees exist.

Categories and Subject Descriptors Theory of computation [Models of computation]: computability

General Terms Theory

Keywords Algorithmic randomness, Martin-Löf randomness, finitely generated universal algebra, graphs and trees of bounded degree, computably enumerable sets, the halting problem.

1. Introduction

1.1 Motivation

The last two decades have witnessed significant advances in the investigation of algorithmic randomness of infinite strings. Monographs by Downey and Hirschfeldt [4] and by Nies [12], and the textbooks by Calude [2] and Li and Vitanyi [9] on the topic of randomness and computability, give an account of the most recent research activities in the area. The modern history of this subject is fascinating and goes back to the work of Kolmogorov [8], Martin-

Löf [10], Chaitin [3], and later Schnorr [14] [15] and Levin [16]. Various notions of algorithmic randomness and related concepts have been introduced and investigated through Martin-Löf tests, Schnorr tests, prefix free complexity, K -triviality, martingales, Solovay and other reducibilities [4] [12]. Brattka, Miller and Nies have recently found connections between algorithmic randomness of reals and analysis through differentiability [1].

For this paper, the main notion of algorithmic randomness will be Martin-Löf randomness, that we write ML-randomness for short. This concept is central in defining randomness in the setting of infinite binary strings. An important ingredient here is the natural measure present in the Cantor space of infinite binary strings. Our definition of algorithmic randomness for infinite structures will also be based on Martin-Löf tests considered on appropriate measurable spaces.

Every infinite binary string α can be viewed as the infinite successor structure $(\omega; S, P)$ with unary predicate P , where $S(i) = i + 1$ and P is true at n if and only if $\alpha(n) = 1$. In this respect, algorithmic randomness of strings can be identified with algorithmic randomness of specific infinite structures of the type $(\omega; S, P)$. However, this view does not provide a satisfactory answer to the questions of the following kind. What is an algorithmically random infinite tree, graph, monoid, or generally, a universal algebra?

In spite of much work, research on randomness of infinite strings has excluded the investigation of algorithmic randomness for infinite algebraic structures of the types mentioned above. The main obstacle in introducing algorithmic randomness for infinite structures, like the ones we mentioned, is that these classes of structures lack measure. More precisely, it is unclear how one would define a meaningful measure through which it would be possible to introduce algorithmic randomness for infinite structures. In this paper, we overcome this obstacle by proposing a limited amount of finiteness conditions on various classes of infinite structures. These conditions will enable us to introduce measure and, as a consequence, reason about algorithmic randomness, such as ML-randomness in these classes of structures. Our classes include finitely generated universal algebras, connected graphs and trees of bounded degree, d -ary trees, and monoids. Through our framework, we believe, one can define and investigate algorithmic randomness for traditional mathematical structures such as finitely generated groups and rings.

What algebraic, computability-theoretic and logical properties should we expect from an algorithmically random infinite structure? Firstly, we would like randomness to be a property of the isomorphism type of the structure rather than a property of some of its isomorphic copies. This is a natural requirement as structures are typically identified up to isomorphisms. One might refer to this as *absoluteness property* of algorithmic randomness. Secondly, we would like random structures to be in abundance, the continuum, just like in the case of random infinite binary strings. This represents randomness as a property of collective, the idea that goes

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back to von Mises [11]. Thirdly, no computable structure can be counted random because in computable structures all the atomic predicates, including the equality relation, and operations are effectively calculable. Thus, by the absoluteness property no random structure can be isomorphic to a computable one. The last but not least, any effective attempt to formally describe the isomorphism type of a random structure or an infinite part of it should fail. This, of course, needs more discussions and various formalisations. For instance, by a description one might mean a function f from ω into the set of first order sentences such that almost all $f(i)$ are quantifier free formula. One might also allow to use effective disjunctions of the type $\exists \bar{x}(\bigvee_{i \in \omega} L_i(\bar{x}, \bar{a}_i))$ where L is an atomic predicate or its negation. This represents the idea of selection rule that again goes back to von Mises [11]. As an example, the order of rational numbers or the order of ω both have obvious formal descriptions. The first is described by a finite list of axioms (linear order plus density axioms) and the second is described as an effective disjunction stating that each element is a finite distance apart from 0. So, one would not expect these two linearly ordered sets to be counted as random structures. This work is partly supported by Marsden Fund of Royal Society of New Zealand.

1.2 Randomness via strings

One obvious yet, in a sense, naive way to introduce randomness for infinite structures is to identify structures with infinite strings that code the atomic diagrams of the structures. Let $\mathcal{A} = (A; P_0^{n_0}, \dots, P_k^{n_k})$ be an infinite relational structure with domain A being ω . Let $c_n : \omega \rightarrow \omega^n$ be a bijective computable coding of n -tuples. As each predicate P is a $\{0, 1\}$ -valued function, we form the following string $\alpha_{\mathcal{A}}$:

$$P_0^{n_0} c_{n_0}(0) \dots P_k^{n_k} c_{n_k}(0) P_0^{n_0} c_{n_0}(1) \dots P_k^{n_k} c_{n_k}(1) \dots$$

Clearly, the string $\alpha_{\mathcal{A}}$ depends on the enumeration of the domain and the codings c_n of the tuples.

To avoid notation and to save space, we consider graphs and also omit the definition of ML-randomness for strings. We will, however, formally define ML-randomness in Section 3 in the setting of algebras. Thus, in the case of graphs, $\mathcal{G} = (V; E)$, where V is the vertex set and E the set of unordered pairs of integers (edges), we have a computable enumeration $c : \omega \rightarrow \omega^{[2]}$ of all unordered pairs. The binary string $\alpha_{\mathcal{G}}$ has 1 at position i iff $c(i)$ is an edge of $\mathcal{G}$.

Definition 1.1. Call the structure $\mathcal{A}$ *string-random* if the string $\alpha_{\mathcal{A}}$ is ML-random.

Let $i_1, \dots, i_k$ be positions of α . If $\alpha(i_1) \dots \alpha(i_k)$ equals a binary word w then we say that $(i_1, \dots, i_k)$ attains w on α . We need the following intuitively obvious property of ML-random strings.

Lemma 1.2. *If α is ML-random then for any string w of length k and any computable injective function $f : \omega \rightarrow \omega^k$ such for all $x \in \omega$ the coordinates of $f(x)$ are all pairwise distinct, there exist infinitely many x such that $f(x)$ attains the value w on α .* $\square$

Consider the Fraïssé limit $\mathcal{F}$ of the class of finite graphs; $\mathcal{F}$ is called the random graph [7]. We refer to this as *model-theoretic randomness*. The first order theory of $\mathcal{F}$ is $\aleph_0$ -categorical. The isomorphism type of $\mathcal{F}$ is characterised by the following statement. For any finite set X and its partition $X = Y_1 \cup Y_2$ where $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ there is a z such that $E(y_1, z)$ for all $y_1 \in Y_1$ and $\neg E(y_2, z)$ for all $y_2 \in Y_2$. The theorem below uses the lemma above. A. Nies has independently observed the same result [13].

Theorem 1.3. *If a graph $\mathcal{G}$ is string-random then it is random model-theoretically.*

Proof. Suppose that $\alpha_{\mathcal{G}}$ is a Martin-Löf random string. Select $x_1, \dots, x_n, y_1, \dots, y_m$ pairwise distinct elements. We want to show that there exists a z such that z is adjacent to $x_1, \dots, x_n$ and not adjacent to each $y_1, \dots, y_m$. Consider the mapping $f : \omega \rightarrow \omega^{n+m}$ such that

$$f(z) = (c^{-1}(x_1, z), \dots, c^{-1}(x_n, z), c^{-1}(y_1, z), \dots, c^{-1}(y_m, z))$$

for all $i \in \omega$, where $c : \omega \rightarrow \omega^{[2]}$ is the computable bijection mentioned above. By Lemma 1.2, there are infinitely many z such that $f(z)$ attains the value $1^n 0^m$. These all z are desired witnesses as follows from the definition of $\alpha_{\mathcal{G}}$. $\square$

The above theorem imply the next corollary. For the proof, see [7] for instance.

Corollary 1.4. *The first order theory of any string-random structure is $\aleph_0$ -categorical and is decidable.* $\square$

All of the above, as it is well-known [7], implies the following:

1. All graphs $\mathcal{G}$ for which the strings $\alpha_{\mathcal{G}}$ are random are isomorphic.
2. These graphs are isomorphic to computable graphs.
3. The isomorphism types of these graphs are described by the following extension axioms:

$$\forall x_1 \dots x_n \forall y_1 \dots y_m \exists z (\& x_i E(x_i, z) \& \& x_j \neg E(y_j, z)),$$

where $n, m \geq 1$ [7].

The last three facts defy the intuitive notion of infinite algorithmically random structure that we postulated above. Indeed, firstly we would like to have continuum algorithmically random graphs just like there are continuum random strings. Secondly, randomness can not imply computability. Thirdly, randomness should not imply a description of random objects, in our case the description of the isomorphism type of the structure. Therefore, introducing randomness for structures fails if one codes their atomic diagrams by strings and declares random structures to be the ones represented by random strings. This observation suggests that we need to take another approach to define algorithmic randomness for infinite structures.

We mention that Kolmogorov randomness has been investigated for finite graphs. For instance, H. Buhrman, et al. [6] introduced Kolmogorov random finite graphs and studied some of their statistical and algebraic properties. Interestingly, their approach is based on connecting finite graphs $\mathcal{G}$ with finite strings $\alpha_{\mathcal{G}}$ (as we explained above), and this turned out to be the right and fruitful idea. Importantly, as opposed to our setting, finiteness of graphs made this connection work. One example of their result is that almost all Kolmogorov random finite graphs have no nontrivial automorphisms.

1.3 Contribution of this paper

- The main contribution of our paper is that we provide a framework for reasoning about algorithmic randomness for various classes of infinite structures. The framework, we think, can be applied to define algorithmic randomness for standard mathematical structures such as groups and rings. On technical level our contribution is as follows:
- In Section 3, we define Martin-Löf randomness for finitely generated universal algebras. We prove in Theorem 3.5 that ML-randomness does not depend on the generators. We show that ML-random algebras have measure 1, hence have the cardinality of the continuum. In Theorem 3.9 we construct a random algebra that is computable in the Halting problem.

- In Section 4.1, we extend framework for algebras to the class of connected graphs of bounded degree. As for algebras, we prove that there are continuum ML-random graphs in this class. We show in Theorem 4.3 that there are ML-random graphs computable in the Halting problem.
- In Section 4.2 we investigate ML-randomness for the class of trees such that each node of the tree has at most d children. Theorem 4.5 constructs an ML-random computably enumerable tree. This significantly strengthens Theorems 3.9 and 4.3 that we mentioned above. There is a philosophical reason to search for computably enumerable ML-random structures. In the setting of reals for that matter, Chaitin's Ω -numbers are ML-random yet left computably enumerable. Hence, it is important to know if ML-random structures exist, as in the case of trees, that behave like Ω -numbers, and hence can still be approximated in a computability-theoretic way.
- We apply our framework to the class of finitely generated monoids in Section 4.3. This shows that our framework can be applied to some traditional mathematical structures.

2. A measure for algebra class

An *algebra* $\mathcal{A}$ is a tuple $(A; f_1, \dots, f_n, c_1, \dots, c_m)$, where A is a nonempty set called the *domain* of $\mathcal{A}$, each f_i is a *total function* $A^{k_i} \rightarrow A$ called a *atomic operation* of arity k_i , and each c_j is a *distinguished element* (or a *constant*) of $\mathcal{A}$. The *signature* of $\mathcal{A}$ is the sequence $f_1, \dots, f_n, c_1, \dots, c_m$ of symbols representing the operations and constants. Denote this signature by Γ_m , where m indicates the number of constants. The signature also specifies the arity k_i of f_i . Note that we used the sequence $f_1, \dots, f_n, c_1, \dots, c_m$ in two ways: one as representing operations and elements of the algebra and the other as a sequence of symbols. Which of these meanings is used in any particular context will be clear from the context. In this paper, we always assume that the signature Γ_m contains at least two unary operations symbols or a function symbol of arity greater than 1. For a background on universal algebras, see [5].

2.1 Algebras and their heights

Let V be a set of variables. The *terms* of the signature Γ_m is defined inductively as follows. Each constant symbol and each variable is a term. If $t_1, \dots, t_{k_i}$ are terms, then so is the expression $f_i(t_1, \dots, t_{k_i})$. We call a term a *ground term* if it contains no variables. The set of all ground terms is denoted by T_G .

The set T_G of ground terms is turned into an algebra of signature Γ_m as follows. The interpretation of each c_i is c_i . For each function symbol f_i and tuple of ground terms $(t_1, \dots, t_{k_i})$, set the value of f_i on this tuple be the ground term $f_i(t_1, \dots, t_{k_i})$. The resulting algebra is called a *term algebra* and we denote it by $\mathcal{T}_G$. Sometimes the algebra $\mathcal{T}_G$ is also called *absolutely free algebra* generated by the constants $c_1, \dots, c_m$. The term algebra $\mathcal{T}_G$ has the following properties [5]: (1) The algebra $\mathcal{T}_G$ is generated by its constants and (2) Each algebra of signature Γ_m generated by its constants is a homomorphic image of $\mathcal{T}_G$.

Let $\mathcal{A} = (A; f_1, \dots, f_n, a_1, \dots, a_m)$ be an algebra of signature Γ_m . We extend a function $s : V \rightarrow A$, which we think of as an interpretation of the variables in A to the *interpretation* $i(t)$ of terms t in $\mathcal{A}$ by induction as follows. First, for each a variable x , let $i(x) = s(x)$, and for each constant symbol c_j , let $i(c_j) = c_j$ (where c_j on the right is the value of the constant in $\mathcal{A}$). In the inductive step, for each basic operation f_j , let $i(f_j(t_1, \dots, t_{k_j})) = f_j(i(t_1), \dots, i(t_{k_j}))$. Note that f_j on the left side of this equality is the symbol f_j from the signature, while f_j on the right side represents the algebra operation (that we also denoted by f_j). The value of $i(t)$ depends on s . Therefore, if $t \in T_G$

then $i(t)$ does not depend on s . So, we use the notation $t_{\mathcal{A}}$ for the values of the ground terms t in the algebra $\mathcal{A}$. Note that if $\mathcal{A}$ is the term algebra $\mathcal{T}_G$, then for this algebra we clearly have $i(t) = t$ for all $t \in T_G$.

Definition 2.1. We say that the algebra $\mathcal{A}$ is *c-generated* if every element of $\mathcal{A}$ is the interpretation of some ground term; in other words, if every element of $\mathcal{A}$ can be obtained from its constants by some chain of basic operations of $\mathcal{A}$.

Thus, *c-generated* algebras $\mathcal{A}$ are such that for every element $a \in A$ there is a ground term t for which $t_{\mathcal{A}} = a$. We call t a *representation* of a in the algebra $\mathcal{A}$. Note that the element a might have more than one representations, (or in fact, infinitely many) ground terms t such that $t_{\mathcal{A}} = a$. Every *c-generated* algebra is finitely generated. However, there are finitely generated algebras of signature Γ_m that are not *c-generated*.

We fix the signature $\Gamma_m = \langle f_1, \dots, f_n, c_1, \dots, c_m \rangle$. Let t be a ground term of the signature. The *height* $h(t)$ of a ground term t is defined by induction as follows. If $t = c_i$ is a constant then $h(t) = 0$. If t is of the form $f_i(t_1, \dots, t_{k_i})$, then $h(t) = \max\{h(t_1), \dots, h(t_{k_i})\} + 1$. If an algebra $\mathcal{A}$ is *c-generated* then every element a there is a ground term t such that $t_{\mathcal{A}} = a$. Hence, with the element a of $\mathcal{A}$ we can associate its height defined as follows:

Definition 2.2. For a *c-generated* algebra $\mathcal{A}$ and an element a of $\mathcal{A}$, the *height* $h(a)$ of the element a is the minimal height among the heights of all the ground terms representing a . The *height* of the algebra $\mathcal{A}$ is the supremum of all the heights of its elements.

We note that for any ground term t of height n and any $i \leq n$ there exists an algebra $\mathcal{A}$ such that the height of the element $t_{\mathcal{A}}$ is i . If $\mathcal{A}$ is *c-generated* algebra, then every element a of it has a height. It is clear that a *c-generated* algebra is finite iff there exists an n such that all elements of $\mathcal{A}$ have height at most n . So, infinite *c-generated* algebras are exactly those with infinite height ω .

From now on we use the following notation borrowed from the theory of formal languages. Given signature Γ_m , denote the class of all *c-generated* finite algebras by Γ_m^* , and the class of *c-generated* infinite algebras by Γ_m^ω .

2.2 Proper partial algebras

Let $\mathcal{A}$ be a *c-generated* algebra of the signature Γ_m . For each $n \in \omega$, define the following subset of $\mathcal{A}$:

$$A[n] = \{a \in A \mid h(a) \leq n\}.$$

Each k_i -ary basic operation f_i of $\mathcal{A}$ gives rise to a *partial operation* $f_{i,n}$ on $A[n]$ defined as follows. For all $a_1, \dots, a_{k_i}$ from $A[n]$ the value of $f_{i,n}(a_1, \dots, a_{k_i})$ equals $f_i(a_1, \dots, a_{k_i})$ if $h(a_i) < n$ for $i = 1, \dots, k_i$; and $f_{i,n}(a_1, \dots, a_{k_i})$ is undefined otherwise. Thus, we have the *partial algebra* $\mathcal{A}[n]$ obtained by restricting $\mathcal{A}$ to all elements of height at most n in $\mathcal{A}$. For instance, the domain of $\mathcal{A}[0]$ is the set $\{c_1, \dots, c_m\}$ of all (values of) constants. We call the partial algebra $\mathcal{A}[n]$, the *n-th slice* of $\mathcal{A}$.

Definition 2.3. We say that two *c-generated* algebras $\mathcal{A}$ and $\mathcal{B}$ *agree at n*, if the partial algebras $\mathcal{A}[n]$ and $\mathcal{B}[n]$ are isomorphic.

For instance, two *c-generated* algebras $\mathcal{A}$ and $\mathcal{B}$ agree at level 0 if and only if for all constants c_i and c_j we have $c_i = c_j$ in $\mathcal{A}$ if and only if $c_i = c_j$ in $\mathcal{B}$. The following is an easy lemma that characterises isomorphic *c-generated* algebras in terms of n -th slices of algebras. The lemma states that the isomorphism between *c-generated* algebras is a Π_1^0 -condition. For the proof one can use König's lemma.

Lemma 2.4. Two *c-generated* algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if they agree at n for all n . $\square$

The next two lemmas show that the class Γ_m^* of all finite c -generated algebras is dense in the class Γ_m^ω .

Lemma 2.5. *Let $\mathcal{A}$ be an infinite c -generated algebra. For each $n \geq 0$ there exists a finite algebra $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ agree at level n .*

Proof. Consider the partial algebra $\mathcal{A}[n]$. We define a finite c -generated algebra $\mathcal{B}$ as follows. The domain of $\mathcal{B}$ is $A[n] \cup \{s\}$, where s is a new element not present in $A[n]$. For each tuple $(a_1, \dots, a_{k_i})$ of $A[n]$ and each basic (partial) operation $f_{i,n}$ such that $f_{i,n}(a_1, \dots, a_{k_i})$ is not defined in $\mathcal{A}[n]$, declare the value of $f_{i,n}$ on the tuple $(a_1, \dots, a_{k_i})$ be s . Also, declare the value of $f_{i,n}$ on any tuple that contains the new element s to be equal to s . In all other tuples $(a_1, \dots, a_{k_i})$ keep the value $f_{i,n}(a_1, \dots, a_{k_i})$ unchanged. It is easy to see that $\mathcal{B}$ defined is a finite c -generated algebra that agrees with $\mathcal{A}$ at level n . $\square$

Lemma 2.6. *If $\mathcal{B}$ is a finite c -generated algebra of height n , then there is an infinite c -generated algebra $\mathcal{A}$ such that $\mathcal{A}$ and $\mathcal{B}$ agree at level n .*

Proof. Let us fix an element b of the algebra $\mathcal{B}$ such that the height of b is n . With each ground term $t \in T_G$, we associate a b -reduced term $r_b(t)$ defined as follows:

1. Each (value of the) constant c of the algebra $\mathcal{B}$ is b -reduced term.
2. Consider a ground term $t = f(t_1, \dots, t_k)$. Assume that we have defined $r_b(t_1), \dots, r_b(t_k)$. Then if each of the b -reduced terms $r_b(t_1), \dots, r_b(t_k)$ belongs to $\mathcal{B}$ and b does not belong to $\{r_b(t_1), \dots, r_b(t_k)\}$ then declare the b -reduced term $r_b(t)$ be the value of $f(r_b(t_1), \dots, r_b(t_k))$ in the algebra $\mathcal{B}$. Otherwise, the b -reduced term $r_b(t)$ is the expression $f(r_b(t_1), \dots, r_b(t_k))$.

It is clear that the set of all b -reduced terms is infinite. The definition above also determines the interpretations of the basic operation symbols f (from the signature Γ_m) on all b -reduced terms. This defines an algebra $\mathcal{A}$. It is not hard to see that the algebra $\mathcal{A}$ is c -generated and agrees with $\mathcal{B}$ at level n . $\square$

We now single out the n -slices of c -generated algebras in the following definition.

Definition 2.7. We call a finite partial algebra $\mathcal{B}$ *proper* if it is an n -th slice of some algebra $\mathcal{A}$ from Γ_m^ω . We call n the *height* of the proper algebra $\mathcal{B}$.

Note that for any proper partial algebra $\mathcal{B}$ of height n and a ground term t with $h(t) \leq n$, the value t_B of the ground term t in $\mathcal{B}$ exists. A syntactic characterization of proper partial algebras is given in the next lemma whose proof follows from the definitions:

Lemma 2.8. *A partial algebra $\mathcal{B}$ is proper iff there is an n such that $\mathcal{B}$ satisfies the following properties:*

1. For all $t \in T_G$ with $h(t) \leq n$, the value t_B is defined;
2. For all atomic operations f_i (of arity k_i) and all tuples $b_1, \dots, b_{k_i}$ from $\mathcal{B}$ if $h(b_1) < n, \dots, h(b_{k_i}) < n$ then $f_i(b_1, \dots, b_{k_i})$ is defined, and if there exists a b_i such that $h(b_i) = n$ then $f(b_1, \dots, b_{k_i})$ is undefined. $\square$

For instance, a proper algebra of height 0 has its domain $\{c_1, \dots, c_m\}$, where some of the constants c_i might be equal to c_j , such that the functions f_i are undefined on any k_i tuple from the domain.

2.3 Viewing Γ_m^ω as paths in a tree

Fix signature Γ_m . There are finitely many non-isomorphic proper algebras of height n . Let $r_m(n)$ be the number of non-isomorphic proper partial algebras of height n . For instance, $r_m(0)$ is the number of equivalence relations on the set $\{c_1, \dots, c_m\}$. It is obvious that $r_m(n) < r_m(n+1)$ for all $n \in \omega$. Note that the function $n \rightarrow r_m(n)$ is computable. Below we show that the class Γ_m^ω of all infinite c -generated algebras of the signature Γ_m can be viewed as paths through a finitely branching tree $\mathcal{T}_m$. This allows us to introduce topology, measure and metric into the set Γ_m^ω .

We formally define the tree $\mathcal{T}_m$ as follows. The root of $\mathcal{T}_m$ is the empty set. This is level -1 of $\mathcal{T}_m$. The nodes of the tree $\mathcal{T}_m$ at level $n \geq 0$ are proper partial algebras of height n . There are exactly $r_m(n)$ of them. Let $\mathcal{B}$ be a proper partial algebra of height n . Its successor on the tree $\mathcal{T}_m$ is any proper partial algebra $\mathcal{C}$ of height $n+1$ such that $\mathcal{B}$ and $\mathcal{C}$ agree at level n .

Lemma 2.9 (Computable Tree Lemma). *The tree $\mathcal{T}_m$ satisfies the following properties:*

1. Given any node x of $\mathcal{T}_m$, we can effectively compute the proper partial algebra $\mathcal{B}_x$ associated with the node x . We identify the nodes x of $\mathcal{T}_m$ and the proper partial algebras $\mathcal{B}_x$.
2. For each x in $\mathcal{T}_m$, the partial algebra $\mathcal{B}_x$ has an immediate successor. Moreover, we can compute the number of immediate successors of x .
3. For each path $\eta = \mathcal{B}_0, \mathcal{B}_1, \dots$ in $\mathcal{T}_m$ we have: $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \dots$. Thus, the union of this chain determines the algebra $\mathcal{B}_\eta = \bigcup_i \mathcal{B}_i \in \Gamma_m^\omega$.
4. The mapping $\eta \rightarrow \mathcal{B}_\eta$ is a bijection between all the infinite paths of $\mathcal{T}_m$ and the class Γ_m^ω . $\square$

2.4 Topology, measure and metric

Using the tree $\mathcal{T}_m$ we can introduce the topology and measure into the class Γ_m^ω .

Definition 2.10 (Topology). Let $\mathcal{B}$ be a proper partial algebra of height n . The **cone** of $\mathcal{B}$ is:

$$\text{Cone}(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \in \Gamma_m^\omega, \text{ and } \mathcal{A} \text{ and } \mathcal{B} \text{ agree at } n\}.$$

Declare the cones $\text{Cone}(\mathcal{B})$ to be the *base open sets* of the topology on Γ_m^ω . We refer to the proper partial algebra $\mathcal{B}$ as the *base of the cone*.

The measure μ_m of the cone $\text{Cone}(\mathcal{B}_x)$, where $x \in \mathcal{T}_m$, is defined by induction as follows.

Definition 2.11 (Measure). The measure of the cone based at the root is 1. Assume that the measure $\mu_m(\text{Cone}(\mathcal{B}_x))$, where $\mathcal{B}_x$ is a proper partial algebra of height n , has been defined. Let e_x be the number of proper partial algebras of height $n+1$ that agree with $\mathcal{B}_x$ at level n . Then for any immediate successor y of x the measure of $\text{Cone}(\mathcal{B}_y)$ is $\mu_m(\text{Cone}(\mathcal{B}_y)) = \mu_m(\text{Cone}(\mathcal{B}_x)) \mid e_x$.

For this definition we note that $e_x \geq 2$ because the signature contains at least two unary function symbols or at least one operation symbol of arity at least 1. As expected, we can also introduce metric into the set $\Gamma_m^* \cup \Gamma_m^\omega$ as follows:

Definition 2.12 (Metric). For two c -generated algebras $\mathcal{A}$ and $\mathcal{B}$, let n be the maximal level at which $\mathcal{A}$ and $\mathcal{B}$ agree. Let $\mathcal{C}$ be the n -th slice of $\mathcal{A}$ (hence of $\mathcal{B}$). The distance $d(\mathcal{A}, \mathcal{B})$ between the algebras is then defined as follows: $d(\mathcal{A}, \mathcal{B}) = \mu_m(\text{Cone}(\mathcal{C}))$.

The next lemma shows that the distance d determines a metric in Γ_m^ω . Note, we identify algebras up to isomorphism. So, an isomorphism maps values of constants c in one algebra to the values of the same constants c in the other algebra.

Lemma 2.13. *The function d is a metric in the space $\Gamma_m^* \cup \Gamma_m^\omega$ of all c -generated algebras.*

Proof. Lemma 2.4 shows that an algebra $\mathcal{A}$ is isomorphic to an algebra $\mathcal{B}$ if and only if $d(\mathcal{A}, \mathcal{B}) = 0$. It is obvious that $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$. So, we need to show that the triangle inequality $d(\mathcal{A}, \mathcal{B}) \leq d(\mathcal{A}, \mathcal{C}) + d(\mathcal{C}, \mathcal{B})$ holds for all algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

If two of these three algebras are isomorphic then the triangle inequality obviously holds. So, assume that the algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are pairwise not isomorphic. Let $n(\mathcal{A}, \mathcal{B})$ be the maximal level at which $\mathcal{A}$ and $\mathcal{B}$ agree. Similarly, consider $n(\mathcal{A}, \mathcal{C})$ and $n(\mathcal{C}, \mathcal{B})$.

If $n(\mathcal{A}, \mathcal{B}) > n(\mathcal{A}, \mathcal{C})$ then clearly the triangle inequality holds. If $n(\mathcal{A}, \mathcal{B}) \leq n(\mathcal{A}, \mathcal{C})$ then the $n(\mathcal{A}, \mathcal{B})$ -th slice of $\mathcal{A}$ is isomorphic to the $n(\mathcal{A}, \mathcal{B})$ -th slice of $\mathcal{C}$. This implies that $n(\mathcal{C}, \mathcal{B}) = n(\mathcal{A}, \mathcal{B})$. Therefore, $d(\mathcal{A}, \mathcal{B}) \leq d(\mathcal{A}, \mathcal{C}) + d(\mathcal{C}, \mathcal{B})$. $\square$

The following proposition follows from the lemmas and definitions above.

Proposition 2.14. *The metric space $\mathcal{M} = (\Gamma_m^* \cup \Gamma_m^\omega, d)$ has the following properties: (1) $\mathcal{M}$ is compact; (2) The countable set Γ_m^* is countable and dense in $\mathcal{M}$; (3) Finite unions of cones form clopen sets in the topology; (4) The set of all μ_m -measurable sets is a σ -algebra. $\square$*

3. ML-randomness for algebras

The set-up above allows us to formally define ML-random algebras. We prove that randomness in c -generated algebras does not depend on the generators. This resembles the fact that randomness of reals is independent on the base of representations [2]. We also give examples of non ML-random algebras.

3.1 Basic definitions

We say that a class $C \subseteq \Gamma_m^\omega$ of algebras is a Σ_1^0 -class if there is computably enumerable (c.e.) sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of proper partial algebras such that $C = \bigcup_{i \geq 1} \text{Cone}(\mathcal{B}_i)$. Computable enumerability of $\mathcal{B}_0, \mathcal{B}_1, \dots$ implies that given i we can compute the open diagram of $\mathcal{B}_i$; In particular, we can compute the cardinality and the atomic partial operations of $\mathcal{B}_i$. We now start with the following definitions from algorithmic randomness are applied to our setting.

Definition 3.1. A class $C \subseteq \Gamma_m^\omega$ is a Σ_1^0 -class if there exists computably enumerable (c.e.) sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of proper partial algebras such that:

$$C = \text{Cone}(\mathcal{B}_0) \cup \text{Cone}(\mathcal{B}_1) \cup \dots$$

For the next definition, recall that $r_m(n)$ is the number of partial proper algebras of height n in the signature Γ_m . We use the measure μ given in Definition 2.10.

Definition 3.2. Consider the class Γ_m^ω of all infinite c -generated algebras.

1. A *Martin-Löf test* is a uniformly c.e. sequence $\{G_n\}_{n \geq 1}$ of Σ_1^0 -classes in Γ_m^ω such that $G_{n+1} \subset G_n$ and $\mu_m(G_n) < 1/r_m(n)$ for all $n \geq 1$.
2. A c -generated algebra $\mathcal{A}$ *fails the Martin-Löf test* $\{G_n\}_{n \geq 1}$ if $\mathcal{A}$ belongs to $\bigcap_n G_n$. Otherwise, we say that the algebra $\mathcal{A}$ *passes the test*.
3. A c -generated algebra $\mathcal{A}$ is *ML-random* if it passes every Martin-Löf test.

We refer to Martin-Löf tests as ML-tests. It turns out that there exists a *universal* ML-test in the sense that passing that test is equivalent to passing all ML-tests. Formally, an ML-test $\{U_n\}_{n \geq 1}$ is *universal* if for any ML-test $\{G_m\}_{m \geq 1}$ it is the case that $\bigcap_m G_m \subseteq \bigcap_n U_n$. A construction of a universal ML-test is easy. Indeed, enumerate all ML-tests $\{G_k^e\}_{k \geq 1}$, where $e \geq 1$, uniformly on e and k , and set $U_n = \bigcup_e G_{n+e+1}^e$. It is not hard to see that $\{U_n\}_{n \geq 1}$ is an ML-test and for any ML-test $\{G_m\}_{m \geq 1}$ we have $\bigcap_m G_m \subseteq \bigcap_n U_n$. Therefore, to prove that a c -generated algebra $\mathcal{A}$ is ML-random it suffices to show that $\mathcal{A}$ passes the universal ML-test $\{U_n\}_{n \geq 1}$. Note the class of c -generated not ML-random algebras has μ_m -measure 0. Thus, we have the following simple corollary:

Corollary 3.3. *The number of ML-random algebras is continuum.*

Proof. The class of all c -generated algebras $\mathcal{A}$ that are not ML-random is a class of μ_m -measure 0. Hence, the number of ML-random algebras is continuum. $\square$

3.2 Generator independence

Here we show that ML-randomness does not depend on generators. This will prove that ML-randomness is an isomorphism invariant property. More formally, we prove that $\mathcal{A}$ is ML-random with respect to one set of generators if and only if $\mathcal{A}$ is ML-random with respect to any other set of generators in the class of all finitely generated algebras.

We start with several simple definitions from model theory. Let $\mathcal{A}$ be an algebra of the signature Γ_m . An *inessential expansion* of algebra $\mathcal{A}$ in signature Γ_m is an algebra of the form $(\mathcal{A}, b_1, \dots, b_k)$, where $b_1, \dots, b_k$ are arbitrary chosen elements of $\mathcal{A}$. Thus, inessential expansions are just algebras over an extended signature Γ_{m+k} .

Definition 3.4. An algebra $\mathcal{A}$ of signature Γ_m is *finitely generated* if some inessential expansion $(\mathcal{A}, b_1, \dots, b_k)$ of $\mathcal{A}$ is a c -generated algebra.

Of course there are algebras over Γ_m not generated by the constants, but that have inessential expansions that make the algebras c -generated over the extended (by constants) signature. Thus, an algebra $\mathcal{A}$ of signature Γ_m is finitely generated iff there are finitely many elements $b_1, \dots, b_k$ such that the following holds. For each $a \in \mathcal{A}$ there is a ground term t over the signature Γ_{m+k} , where the new constants $c_{m+1}, \dots, c_{m+k}$ are interpreted as $b_1, \dots, b_m$ respectively, such that the value of t in the expanded algebra equals a .

Theorem 3.5 (Generator independence). *Martin-Löf randomness for algebras is independent of the generators.*

Proof. The proof idea is simple but some care should be taken with calculations. Let $\bar{a} = a_1, \dots, a_m$ and $\bar{b} = b_1, \dots, b_k$ be two sets of generators of algebra $\mathcal{A}$. We can assume that the signature of $\mathcal{A}$ contains no constant symbols. Thus, we have $(\mathcal{A}, \bar{a}) \in \Gamma_m^\omega$ and $(\mathcal{A}, \bar{b}) \in \Gamma_k^\omega$. Our goal is to show that $(\mathcal{A}, \bar{a})$ is ML-random if and only if $(\mathcal{A}, \bar{b})$ is ML-random.

There exist ground terms $t_1, \dots, t_k$ and $q_1, \dots, q_m$ over Γ_m and Γ_k , respectively, such that $t_i(\bar{a}) = b_i$ and $q_j(\bar{b}) = a_j$ for these terms, where $i = 1, \dots, k$, and $j = 1, \dots, m$. Call these equalities the *base equalities*. We set

$$\begin{aligned} j_0 &= 2 \cdot \max\{h(t_i) + h(q_j) \mid i = \overline{1, k}, j = \overline{1, m}, \\ &\quad k_0 = \max\{h(t_i) \mid i = \overline{1, k}\} \text{ and} \\ &\quad m_0 = \max\{h(q_j) \mid j = \overline{1, m}\}. \end{aligned}$$

If $(\mathcal{D}, b_1, \dots, b_k)$ from Γ_k^ω satisfies the base equalities then the algebra $(\mathcal{D}, q_1(\bar{b}), \dots, q_m(\bar{b}))$ belongs to Γ_m^ω . Denote this mapping

$(\mathcal{D}, b_1, \dots, b_k) \rightarrow (\mathcal{D}, q_1(\bar{b}), \dots, q_m(\bar{b}))$ by α . This is a partial map from Γ_k^ω to Γ_m^ω .

Let S_k be the class of all proper partial algebras of signature Γ_k in which the base equalities hold. Set $[S_k] = \bigcup_{\mathcal{B} \in S_k} C(\mathcal{B})$. Similarly, consider the set $[S_m]$ of algebras of signature Γ_m in which the base equalities hold. Both $[S_k]$ and $[S_m]$ are open Σ_1^0 -sets and have non-zero measure with measures x_k and x_m , respectively. The function α defined above is a bijection from $[S_k]$ to $[S_m]$. Below we identify the algebras from the class $[S_k]$ with the algebras from $[S_m]$ via α .

Let $\mathcal{B} = (B; \bar{b}) \in S_k$ and let $t > j_0$ be the height of $\mathcal{B}$. With $\mathcal{B}$, associate a proper partial algebra $\mathcal{B}'$ of signature Γ_m with domain $B' = \{x \in B \mid h(x) \leq t - k_0\}$ and $a_1 = q_1(\bar{b}), \dots, a_m = q_m(\bar{b})$. Clearly, if $\mathcal{B}$ and $\mathcal{C}$ are isomorphic then $\mathcal{B}'$ and $\mathcal{C}'$ are also isomorphic. Similarly, we map proper partial algebras $\mathcal{B}'$ of height $t - k_0$ in signature Γ_m into proper partial algebras $\mathcal{B}''$ of height $t - k_0 - m_0$ in signature Γ_k . Note that $\mathcal{B}''$ satisfies the base equalities. The mappings $\mathcal{B} \rightarrow \mathcal{B}'$ and $\mathcal{B}' \rightarrow \mathcal{B}''$ satisfy the following properties:

1. If $\mathcal{B}'$ is a proper partial algebra of height $t - k_0$ in Γ_m satisfying the base equalities then there is a proper partial algebra $\mathcal{B}$ such that $\mathcal{B} \rightarrow \mathcal{B}'$.
2. $r'_k(t - k_0 - m_0) \leq r'_m(t - k_0) \leq r'_k(t)$, where $r'_k(i)$ and $r'_m(j)$ are the numbers of proper partial algebras of height i in signatures Γ_k and Γ_m , respectively, satisfying the base equalities.
3. $C(\mathcal{B}) \subseteq C(\mathcal{B}') \subseteq C(\mathcal{B}'')$.

Parts (1) and (2) are clear. Part (3) uses the map α .

Let $\{G_n\}_{n \geq 1}$ be an ML-test that fails $(\mathcal{A}, b_1, \dots, b_k)$. We transform this test into an ML-test that fails $(\mathcal{A}, a_1, \dots, a_m)$. This will prove the theorem.

We can assume that the base of every cone in G_n satisfies the base equalities for all n . Hence, $C(\mathcal{B}) \subseteq [S_k]$ for all cones $C(\mathcal{B})$ in G_n . Therefore, we will assume that $\bigcup G_n \subseteq [S_k]$. For each cone $C(\mathcal{B}) \subseteq G_n$, the measure $\mu_{[S_k]}(C(\mathcal{B}))$ of $C(\mathcal{B})$ in the set $[S_k]$ is $\mu_{[S_k]}(C(\mathcal{B})) = (1/x_k) \cdot \mu_k(C(\mathcal{B}))$. Hence, the measure $\mu_{[S_k]}(G_n)$ of G_n in $[S_k]$ is bounded $(1/x_k) \cdot (1/2^n)$. Thus, for the ML-test $\{G_n\}_{n \geq 1}$ we have: (a) $G_n \subseteq [S_k]$, (b) $\mu_{[S_k]}(G_n) \leq (1/x_k) \cdot (1/r_m(n))$, and (c) $G_n \supseteq G_{n+1}$.

For a given $\epsilon > 0$ proceed as follows. First, find a level $n(\epsilon)$ such that $\mu(\mathcal{B}'') < \epsilon$ for all proper partial algebras $\mathcal{B}$ of height $n(\epsilon)$. Secondly, let $\delta(\epsilon)$ be the minimal measure of all the cones $C(\mathcal{B})$ among all proper partial algebras $\mathcal{B}$ of height $n(\epsilon)$. Thirdly, compute $k(\epsilon)$, the number of proper partial algebras of height $n(\epsilon)$. Finally, find the first $t(\epsilon)$ such that $\mu(G_{t(\epsilon)}) < \delta(\epsilon)/k(\epsilon)$. Choosing $G_{t(\epsilon)}$ in this way guarantees that for all cones $C(\mathcal{B})$ in $G_{t(\epsilon)}$ we have the height of $\mathcal{B}$ is at least $n(\epsilon)$ and also $\mu(\mathcal{B}'') < \epsilon/t(\epsilon)$.

Set $V_n = G_{t(1/2^n)}$ for $n \geq 1$. Clearly, this sequence $\{V_n\}_{n \geq 1}$ is a ML-test that fails $(\mathcal{A}, b_1, \dots, b_k)$. Since the sequence $\{V_n\}_{n \geq 1}$ is a Σ_1^0 -sequence, we can effectively write each V_n as a union of pairwise disjoint cones $V_n = \bigcup C(\mathcal{B})$. Now for each n set $V'_n = \bigcup C(\mathcal{B}')$ and $V''_n = \bigcup C(\mathcal{B}'')$. From the construction, it is not too hard to see that the sequence $\{V''_n\}_{n \geq 1}$ is a ML-test such that $\mu(V''_n) < 1/2^n$ for all n and the algebra $(\mathcal{A}, b_1, \dots, b_k)$ fails this test.

The sequence $\{V'_n\}_{n \geq 1}$ is a Σ_1^0 -sequence in Γ_m^ω . Recall that we have the inclusion (3): $C(\mathcal{B}) \supseteq C(\mathcal{B}') \supseteq C(\mathcal{B}'')$. Moreover, $V'_n \supseteq V'_{n+1}$ for all n . Due to the inequality (2) above, we also have: $\mu_k(V_n) \leq \mu_m(V'_n) \leq \mu_k(V''_n)$. All these imply that the sequence $\{V''_n\}_{n \geq 1}$ is a ML-test that fails the algebra $(\mathcal{A}, a_1, \dots, a_m)$ from Γ_m^ω . This is what was required to be proved. $\square$

3.3 Randomness in the Halting set

An infinite algebra $\mathcal{A}$ is *computable* if it is isomorphic to an algebra with domain ω and whose all atomic operations are computable. Thus, computability is an isomorphism property of algebras. Let $\mathcal{A}$ be a c -generated infinite algebra and $h : \mathcal{T}_G \rightarrow \mathcal{A}$ be the homomorphism from the term algebra $\mathcal{T}_G$ onto $\mathcal{A}$. The *word problem* of $\mathcal{A}$ is the set:

$$WP(\mathcal{A}) = \{(t, q) \mid t, q \in \mathcal{T}_G \text{ \& } h(t) = h(q)\}.$$

The following is an easy exercise:

Proposition 3.6. *The algebra $\mathcal{A}$ is computable if and only if the set $WP(\mathcal{A})$ is decidable.* $\square$

As expected we have that no computable algebra is a random algebra.

Proposition 3.7. *If $\mathcal{A}$ is a computable infinite algebra then $\mathcal{A}$ is not ML-random.*

Proof. There exists an effective procedure that given n , computes the n^{th} -slice $[n]$ of the algebra $\mathcal{A}$. Indeed, this follows from the fact that given an element $a \in \mathcal{A}$, we can effectively find a ground term t such that $t_{\mathcal{A}} = a$. This allows us to compute the height of the element a . Hence, the sequence of cones $Cone(\mathcal{A}[0]), Cone(\mathcal{A}[1]), Cone(\mathcal{A}[2]), \dots$ forms a Martin-Löf test that the algebra $\mathcal{A}$ fails. $\square$

A natural class that contains the class of all computable algebras is the class of algebras computable in the Halting set. This class contains all finitely presented algebras. We denote the halting set by $\mathcal{H}$. Here is a definition.

Definition 3.8. An algebra $\mathcal{A}$ is $\mathcal{H}$ -computable if the word problem $WP(\mathcal{A})$ of $\mathcal{A}$ for the algebra is computable in $\mathcal{H}$.

Clearly, every computable algebra is $\mathcal{H}$ -computable. The next theorem shows that there are ML-random $\mathcal{H}$ -computable algebras. Thus, the proposition above can not be extended to $\mathcal{H}$ -computable algebras.

Theorem 3.9. *Martin-Löf random $\mathcal{H}$ -computable algebras exist.*

Proof. We build an $\mathcal{H}$ -computable random algebra $\mathcal{A}$ of signature $\Gamma_1 = (f, g, c)$ where f and g are unary functions and c is a constant. The algebra is generated by the (value of the) constant c . Let $\{U_n\}_{n \geq 1}$ be the universal ML-test for the class of finitely generated algebras of the signature Γ_1 . So, $\mu_1(U_n) < 1/r_1(n)$ for all $n \geq 1$. We want construct $\mathcal{A}$ such that $\mathcal{A} \notin \bigcap U_n$. For this it suffices to build $\mathcal{A}$ such that $\mathcal{A} \notin U_1$. Since $U_1 = C(\mathcal{B}_1) \cup C(\mathcal{B}_2) \cup \dots$ is the union of uniformly c.e. set of cones we will build $\mathcal{A}$ so that $\mathcal{A}$ avoids all the cones $C(\mathcal{B}_i)$ for all $i \geq 1$. Using the oracle $\mathcal{H}$, we assume that the sequence $C(\mathcal{B}_0) \cup C(\mathcal{B}_1) \cup \dots$ satisfies the following for the sequence $C(\mathcal{B}_1), C(\mathcal{B}_2), \dots$: (1) For all $i \neq j$ we have $C(\mathcal{B}_i) \cap C(\mathcal{B}_j) = \emptyset$, and (2) For all i , $height(C(\mathcal{B}_i)) \leq height(C(\mathcal{B}_{i+1}))$.

The algebra $\mathcal{A}$ is built by stages. At stage n , we define a proper partial algebra $\mathcal{A}_n$ so that $\mathcal{A}_n$ avoids the cone $C(\mathcal{B}_n)$ and $\mathcal{A}_{n-1} \subseteq \mathcal{A}_n$. The algebra $\mathcal{A}$ will then be the limit of this sequence, that is, $\mathcal{A} = \bigcup_n \mathcal{A}_n$. Here is our construction.

Stage I. Let t_1 be the last number m so that $\mathcal{B}_1, \dots, \mathcal{B}_m$ all have the same height, say h_1 . Let $\mathcal{A}_1$ be a proper partial algebra of height h_1 such that $\mathcal{A}_1 \notin \{\mathcal{B}_1, \dots, \mathcal{B}_{t_0}\}$ and $\mu_m(C(\mathcal{A}_1)) > \mu_m(C(\mathcal{A}_1) \cap (\bigcup_{i > t_1} C(\mathcal{B}_i)))$. Such partial algebra $\mathcal{A}_1$ exists as otherwise, $\mu_1(U_1) \geq 1/r_1(1)$.

Stage s+1. Let t_{s+1} be the first m such that $\mathcal{B}_{t_s+1}, \dots, \mathcal{B}_m$ all have the same height, say h_{s+1} . Set $\mathcal{A}_{s+1}$ to be a proper partial algebra such that $\mathcal{A}_{s+1}$ has height h_{s+1} , $\mathcal{A}_{s+1} \notin \{\mathcal{B}_{t_s+1}, \dots, \mathcal{B}_m\}$.

$\dots, \mathcal{B}_{t_{s+1}}\}, \mathcal{A}_s \subset \mathcal{A}_{s+1}$, and $\mu_m(C(\mathcal{A}_{s+1})) > \mu_m(C(\mathcal{A}_s) \cap (\cup_{i>t_{s+1}} C(\mathcal{B}_i)))$. Such proper partial algebra $\mathcal{A}_{s+1}$ exists as, otherwise, we would have a contradiction with the choice of $\mathcal{A}_s$.

Thus, the algebra $\mathcal{A} = \cup_{s \geq 1} \mathcal{A}_s$ is not in U_1 . So, $\mathcal{A}$ is ML-random. Note that at each stage we can select the partial algebra $\mathcal{A}_s$ computably but using the oracle for $\mathcal{H}$. Hence, the algebra $\mathcal{A}$ is $\mathcal{H}$ -computable. $\square$

An important class of algebras between the class of computable algebras and the class of $\mathcal{H}$ -computable algebras is the class of computably enumerable algebras. Here is a definition:

Definition 3.10. An algebra $\mathcal{A}$ is *computably enumerable* (c.e.) if the word problem $WP(\mathcal{A})$ for the algebra is a computably enumerable set.

Every computable algebra is c.e. and every c.e. algebra is $\mathcal{H}$ -computable. We don't know if a ML-random c.e. algebra exists.

4. ML-random graphs, trees, and monoids

In this section we consider graphs and trees. The goal of this section is to construct ML-random structures in the classes of graphs, trees, and monoids. The techniques are based on the framework built in the previous section for the class of universal algebras. The case of graphs is a direct application of the framework from the previous section. In the case of trees is interesting as we show the existence of computably enumerable ML-random trees. This is much stronger than just building ML-structures computable in the Halting set. The case of monoids deserves attention as (1) this class is closest to traditional structures of algebra, and (2) some work is needed to get the Tree Lemma 2.9 for monoids. This indicates applicability of our approach in the setting of standard infinite algebraic structures such as groups and rings. Also, as in Proposition 3.6, no computable structure from any of these classes is ML-random.

4.1 The case of graphs

A *graph* is a structure $(V; E)$, where V is the vertex set of the graph and E is the edge set which is a symmetric binary relation on V . Thus, self-loops, that is pairs of the form (v, v) , are allowed to be edges. The degree of a vertex v is the number of vertices adjacent to v . The graph is of *bounded degree* d , where d is a fixed integer, if the degree of every vertex is bounded by d . By a *tree* is we mean a connected graph with no cycles, in particular with no self-loops. A tree is d -ary if the degree of every node in the tree is bounded by d . For this section, we *always assume* that our graphs are connected and the graphs and trees are bounded of degree $d > 2$. Clearly, connected graphs and trees of bounded degree 1 and 2 are easily described.

Let $\mathcal{G}$ be an infinite graph. Fix an initial vertex, say c . For the meantime the signature of the graph contains the symbol c . This constant plays the same role played by constants in the case of algebras. For $n \in \omega$, let $D_{\mathcal{G},n}(c)$ be the collection of all the vertices in $\mathcal{G}$ that are at distance at most n from c . We call the graphs $D_{\mathcal{G},n}(c)$ the n -neighbourhoods of c . Since d is fixed, there are obviously at most finitely many isomorphism types of all the n -neighbourhoods of the constant c . Just like in Section 2.3, we formally define the following tree $\mathcal{T}$. The root of $\mathcal{T}$ is the empty set. This is level -1 of the tree. The nodes of the tree $\mathcal{T}$ at level $n \geq 0$ are the isomorphism types of the n -neighbourhoods of c . The function that maps n to the number (of the isomorphism types) of the graphs $D_{\mathcal{G},n}$ is computable. Let $\mathcal{G}$ be (an isomorphism type of) a graph at level n of the tree $\mathcal{T}$. Its successors on the tree $\mathcal{T}$ is any $(n+1)$ -neighbourhood $\mathcal{G}'$ such that $\mathcal{G} \subset \mathcal{G}'$ where the constants in $\mathcal{G}$ and $\mathcal{G}'$ are identified. For the tree $\mathcal{T}$ we have the following properties as expressed in the Tree Lemma 2.9:

1. Given any node x of $\mathcal{T}$, we can effectively compute the graph $\mathcal{G}_x$ associated with x . We identify the nodes x and the graphs $\mathcal{G}_x$.
2. For every x in $\mathcal{T}$, we can compute the number of immediate successors of x . Note that the number of successor nodes can be 0;
3. For each path $\eta = \mathcal{B}_0, \mathcal{B}_1, \dots$ in $\mathcal{T}$ we have the chain: $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \dots$. The union $\mathcal{G}_\eta = \cup_i \mathcal{G}_i \in \Gamma_m^\omega$ is a connected graph of bounded degree d .
4. The mapping $\eta \rightarrow \mathcal{G}_\eta$ is a bijection between of all the infinite paths of $\mathcal{T}$ and all infinite connected graphs of bounded degree d .

Having the tree $\mathcal{T}$, we can transform all the definitions (such measure, cones, topology, ML-tests) and theorems of the previous two sections to the class of all connected graphs of bounded degree d . For instance, Corollary 3.3 can be proved word-by-word:

Corollary 4.1. *The number of ML-random connected graphs is continuum.* $\square$

The tree $\mathcal{T}$ constructed depends on the initially selected vertex c . Using the same technique as in the proof of Theorem 3.5, we have the following result:

Theorem 4.2. *The ML-randomness for connected graphs of bounded degree d is independent of the constant c selected. Moreover, ML-randomness is an isomorphism invariant property in the class of all connected graphs of bounded degree d .* $\square$

Just like in Theorem 3.9, we can also prove the following result. The proof follows a line similar to the proof of Theorem 3.9:

Theorem 4.3. *Martin-Löf random $\mathcal{H}$ -computable graphs of bounded degree d exist.* $\square$

4.2 The case of trees

The case of trees is interesting because Theorem 3.9 and Theorem 4.3 can be made much stronger in the setting of trees. Namely, we prove that there are ML-random computably enumerable (c.e.) trees. We recast the definition of c.e. algebras for graphs (and hence trees).

Let E be an equivalence relation on ω . We say that a relation $Edge \subseteq \omega^2$ *respects* E if for all x_1, y_1, x_2, y_2 such that $(x_1, x_2) \in E$ and $(y_1, y_2) \in E$ we have $Edge(x_1, y_1)$ if and only if $Edge(x_2, y_2)$. In other words, the $Edge$ relation does not depend on the representatives of the E -equivalence classes. Denote by $[x]$ the equivalence class of x . If $Edge$ respects E then we can naturally define the structure $(\omega/E; Edge)$, where ω/E is the quotient set and $Edge$ is the relation on ω/E induced by the original relation $Edge$. Namely, for all E -equivalence classes $[x], [y]$ we set $Edge([x], [y])$ if and only if $Edge(x, y)$ holds. Here is the definition of a c.e. graph:

Definition 4.4. A graph $\mathcal{G}$ is *computably enumerable* (c.e.) if there exists a computably enumerable equivalence relation E on ω and a symmetric binary relation $Edge$ that respects E on ω such that the graph $\mathcal{G}$ is isomorphic to the graph $(\omega/E; Edge)$.

In the theorem below we construct an ML-random computable enumerable d -ary tree. Firstly, as in the case of graphs we select one node c in a given d -ary tree. We can view c as the root of the tree. Since we have the constant c , one can naturally define the heights of finite trees. Secondly, we need to define a finitely branching computable tree $\mathcal{T}$, just like what we did for algebras and graphs, such that there is a one-to-one correspondence between the infinite d -ary trees and paths through the tree $\mathcal{T}$. Formally, define the following tree $\mathcal{T}$. The root of $\mathcal{T}$ is the empty set. This

is level -1 of $\mathcal{T}$. The nodes of the tree $\mathcal{T}$ at level $n \geq 0$ are the isomorphism types of finite d -ary trees of height n . The function mapping n to the number (of the isomorphism types) of trees of height n is computable. Let $\mathcal{X}$ be (an isomorphism type of) a tree at level n of $\mathcal{T}$. Its successor on the tree $\mathcal{T}$ is any tree $\mathcal{X}'$ of height $n+1$ such that $\mathcal{X} \subset \mathcal{X}'$, where the constants in $\mathcal{X}$ and $\mathcal{X}'$ are identified.

For the tree $\mathcal{T}$, as in the Tree Lemma 2.9, we have the following properties:

1. Given any node v of $\mathcal{T}$, we can effectively compute the tree $\mathcal{X}_v$ associated with the node v . We identify the nodes v of $\mathcal{T}$ and the trees $\mathcal{X}_v$.
2. For every node v in $\mathcal{T}$, we can compute the number of immediate successors of v . Note that the number of successor nodes is greater than 1.
3. For each path $\eta = v_0, v_1, \dots$ in $\mathcal{T}$ we have the chain: $\mathcal{X}_{v_0} \subset \mathcal{X}_{v_1} \subset \dots$. The union of this chain is the d -ary tree $\mathcal{X}_\eta = \bigcup_i \mathcal{X}_{v_i}$.
4. The mapping $\eta \rightarrow \mathcal{X}_\eta$ is a bijection between all the infinite paths of $\mathcal{T}$ and all infinite d -ary trees.

Based on $\mathcal{T}$, it is easy to define ML-randomness for d -ary trees just like we did for finitely generated algebras and graphs. As for algebras and graphs, the definition of ML-random tree is independent of the constant, there are uncountably many ML-random trees, and there are $\mathcal{H}$ -computable ML-random trees. Here is our main theorem that significantly strengthens the last fact:

Theorem 4.5. *There exist Martin L f random computably enumerable d -ary trees.*

Proof. We need several technical concepts. Let I be a finite set, E be an equivalence relation on I and $Edge$ be a binary relation respecting E such that the structure $\mathcal{X} = (I/E; Edge)$ is a tree. The root of this structure $\mathcal{X}$ is the constant c (or more formally, the E -equivalence class $[c]$ of c). Let $[x_1], [x_2], [x_3]$ be pairwise distinct nodes of the tree $\mathcal{X}$ such that $[x_3]$ is a leaf, $([x_2], [x_3]) \in Edge$ and $([x_1], [x_2]) \in Edge$. Let E' be the least equivalence relation that contains E and the pair (x_1, x_3) . Note that all equivalence classes of E apart from $[x_1]$ and $[x_3]$ stay unchanged and E' collapses the E -equivalence classes $[x_1]$ and $[x_3]$ into one equivalence class $[x_1] \cup [x_3]$. The $Edge$ relation still respects E' ; Moreover, the resulting structure $\mathcal{X}' = (I/E'; Edge)$ is still a tree with one less leaf. We call the construction that builds $\mathcal{X}'$ from $\mathcal{X}$ a *reduction*, and denote this by $\mathcal{X} \triangleright \mathcal{X}'$. We note that $\mathcal{X}'$ can now be viewed as a subtree of the original tree $\mathcal{X}$. Obviously, there could be more than one reduction from the original tree $\mathcal{X}$. Note that for the following lemma we identify finite trees up to isomorphism. The proof of the lemma is quite clear:

Lemma 4.6. *For the tree $\mathcal{X}$ and any of its subtrees $\mathcal{Y}$ of height at least 2 there exists a sequence of reductions $\mathcal{X}_1 \triangleright \mathcal{X}_2 \triangleright \dots \mathcal{X}_{n-1} \triangleright \mathcal{X}_n$ such that $\mathcal{X}_1 = \mathcal{X}$ and $\mathcal{X}_n = \mathcal{Y}$.* $\square$

Let $\{U_n\}_{n \geq 1}$ be a universal ML-test in for the class of all d -ary trees. Thus, we have $\mu(U_n) < 1/r(n)$ for all $n \in \omega$, where $r(n)$ is the number of non-isomorphic d -ary trees of height n and μ is the measure defined by the tree $\mathcal{T}$. We want construct a tree T such that $T \notin \bigcap_{n \geq 1} U_n$. For this it suffices to guarantee that $T \notin U_2$. Note that U_2 is a uniformly c.e. set of cones $U_2 = C(\mathcal{X}_1) \cup C(\mathcal{X}_2) \cup \dots$, where each $\mathcal{X}_i$ is a finite tree. Hence, it suffices to construct T so that $T \notin C(\mathcal{X}_i)$ for all $i \geq 1$. We work with U_2 (instead of U_1 as in algebra case) because we apply Lemma 4.6 in which reductions do not produce trees of height less than 2.

Using the tree $\mathcal{T}$, we employ the natural height-lexicographic order on all finite trees $\mathcal{X}$. Our construction of T is by stages. At stages s we build $T_s = (I_s/E_s; Edge_s)$ such that T_{s-1} is embedded into T_s as a subtree. The tree constructed will be the limit of the trees T_s . The stage s will guarantee that T constructed will not belong to the cone $C(\mathcal{X}_{s-1})$.

Stage 0. At this stage, set T_0 to be height-lexicographically the first tree of height 2.

Stage $s+1$. Let $T_s = (I_s/E_s; Edge_s)$ be the tree constructed at stage s such that I_s is an initial segment of ω , E_s is an equivalence relation on I_s , $Edge_s$ is a binary relation that respects E_s . Let h_s be the height of the tree T_s . Consider the cone $C(\mathcal{X}_s)$.

Case 1: The cones $C(T_s)$ and $C(\mathcal{X}_s)$ have the empty intersection. In this case we extend T_s to the first successor of T_s in the tree $\mathcal{T}$. This defines the tree $T_{s+1} = (I_{s+1}/E_{s+1}; Edge_{s+1})$ of height $h_s + 1$.

Case 2: The cone $C(\mathcal{X}_s)$ is properly contained in the cone $C(T_s)$. Find the largest subtree $\mathcal{Y}$ of T_s , with respect to height-lexicographic order, such that $\mathcal{Y}$ has a successor $\mathcal{Y}'$ for which $C(\mathcal{Y}) \cap (C(\mathcal{X}_1) \cup \dots \cup C(\mathcal{X}_s)) = \emptyset$. Reduce T_s to $\mathcal{Y}$, then extend $\mathcal{Y}$ to $\mathcal{Y}'$, and build T_{s+1} to be isomorphic to $\mathcal{Y}'$. This amounts to extending the initial segment I_s to another initial segment I_{s+1} , extending the equivalence relation E_s to E_{s+1} , and $Edge_s$ to $Edge_{s+1}$ that respects E_{s+1} so that $\mathcal{Y}'$ is isomorphic to $T_{s+1} = (I_{s+1}/E_{s+1}; Edge_{s+1})$.

Case 3: The cone $C(\mathcal{X}_s)$ contains the cone $C(T_s)$. So, here we view $\mathcal{X}_s$ as a subtree of T_s . Find the largest subtree $\mathcal{Y}$ of $\mathcal{X}_s$, with respect to height-lexicographic order, such that $\mathcal{Y}$ has a successor $\mathcal{Y}'$ for which $C(\mathcal{Y}) \cap (C(\mathcal{X}_1) \cup \dots \cup C(\mathcal{X}_s)) = \emptyset$. Reduce T_s to $\mathcal{Y}$, then extend $\mathcal{Y}$ to $\mathcal{Y}'$, and build T_{s+1} to be isomorphic to $\mathcal{Y}'$. This amounts to extending the initial segment I_s to another initial segment I_{s+1} , extending the equivalence relation E_s to E_{s+1} , and $Edge_s$ to $Edge_{s+1}$ that respects E_{s+1} so that $\mathcal{Y}'$ is isomorphic to $T_{s+1} = (I_{s+1}/E_{s+1}; Edge_{s+1})$.

Note that because of the constraint $\mu(U_1) \leq 1/r(2)$, at any stage s , we will always be able to find $\mathcal{Y}'$ desired as in Cases 2 and 3. Also, for each n there must exist a stage s such that all the trees $\mathcal{X}_k$, with $k \geq s$ will have height higher than n . Hence, from stage s the subtree of T_s of height n will never change. Reduction process is a c.e. process, hence the tree T that we have constructed is a c.e. tree. The tree T is ML-random as, by construction, it passes the universal ML-test. $\square$

4.3 The case of monoids

Recall that a *monoid* is a structure $\mathcal{M} = (M; \circ, e)$ such that $\circ$ is an associative binary operation on M a non empty set M and e is the identity element. We consider monoids with two generators x and y . As monoids are universal algebras all the concepts of universal algebra are applied here such as partial monoids, proper partial monoids, and heights. For instance, a proper partial monoid $\mathcal{M} = (M; \circ, e, x, y)$ of height n is determined by the following properties:

1. For each $m \in M$ there is a ground term t of height at most n such that $t_{\mathcal{M}} = m$.
2. For all $m_1, m_2 \in M$ if $h(m_1) < n$ and $h(m_2) < n$ then $m_1 \circ m_2$ is defined.
3. For all $m_1, m_2 \in M$ if either $h(m_1) = n$ or $h(m_2) = n$ then $m_1 \circ m_2$ is undefined.

We need to build a computable tree $\mathcal{T}$ such that nodes are proper partial monoids. Moreover, we need to guarantee that there is a bijection between the paths of $\mathcal{T}$ and two generated monoids. In order to build such a tree we need the following lemma:

Lemma 4.7. *Let $\mathcal{M}$ be a proper partial monoid of height n . There are at least two non-isomorphic infinite two generated monoids that extend $\mathcal{M}$. In particular, $\mathcal{M}$ has at least two non-isomorphic proper partial monoid extensions of the same height.* $\square$

Proof. Monoid $\mathcal{M}_1$ is build as follows. Let u be such that $u \notin M$. Set $M_1 = M \cup \{u\}^*$. For all $x, y \in M_1$ define $x \circ y$ as follows. If $x, y \in M$ and $x \circ y$ was defined then preserve the value; if $x \circ y$ was undefined then set $x \circ y = u$. If $x \in \{u\}^*$ and $y \in M$ then $x \circ y = y \circ x = x$. If both $x, y \in \{u\}^*$ then $x \circ y = xy$.

To construct $\mathcal{M}_2$ we set $M_2 = M \cup \{u_1, u_2\} \cup \{a_1, a_2\}^*$, where $u_1, u_2, a_1, a_2 \notin M$. On M_2 , we define $\circ$ as follows. For all $x, y \in M$ we preserve $x \circ y$ if this was already defined. Otherwise we set $x \circ y = u_1$. Set $u_1 \circ u_1 = u_2, u_1 \circ u_2 = a_1$, and $u_2 \circ u_1 = a_2$. The rest is defined similar as above. For instance, If $x \in \{a_1, a_2\}^*$ and $y \in M$ then $x \circ y = y \circ x = x$. If both $x, y \in \{a_1, a_2\}^*$ then $x \circ y = xy$. $\square$

The result above allows us to transfer the Tree Lemma 2.9 and build the tree $\mathcal{T}$ for the class of two generated monoids. Thus, we can carry the proofs of the results for ML-random universal algebras to monoids setting and obtain the the following:

Theorem 4.8. *The ML-randomness for finitely generated monoids is independent on the generators. Hence, ML-randomness is an isomorphism invariant property in the class of all finitely generated monoids. Moreover, ML-random $\mathcal{H}$ -computable finitely generated monoids exist.* $\square$

There are several interesting questions remain open. For instance, we do not know if there exist computably enumerable ML-random universal algebras, connected graphs of bounded degree, two generated monoids. We conjecture that such structures exist. It would be interesting to construct ML-random two generated groups; however, the tree $\mathcal{T}$ that is needed to represent two generated groups is not computable. So, one needs to be careful in building ML-random two generated groups. We also conjecture that no finitely presented random universal algebra exists.

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