

A type theory for productive coprogramming via guarded recursion

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Abstract

To ensure consistency and decidability of type checking, proof assistants impose a requirement of productivity on corecursive definitions. In this paper we investigate a type-based alternative to the existing syntactic productivity checks of Coq and Agda, using a combination of guarded recursion and quantification over clocks. This approach was developed by Atkey and McBride in the simply typed setting, here we extend it to a calculus with dependent types. Building on previous work on the topos-of-trees model we construct a model of the calculus using a family of presheaf toposes, each of which can be seen as a multi-dimensional version of the topos-of-trees. As part of the model construction we must solve the coherence problem for modelling dependent types in locally cartesian closed categories simultaneously in a whole family of locally cartesian closed categories. We do this by embedding all the categories in a large one and applying a recent approach to the coherence problem due to Streicher and Voevodsky.

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1. Introduction

Recursive type equations are ubiquitous in semantics of programming languages. Inductive types and coinductive types are initial and final solutions to covariant type equations, and solutions to contravariant and mixed variance type equations are needed for modelling the untyped lambda calculus and programming languages with advanced type systems.

Guarded recursion [17] is a new approach to solving recursive type equations, offering unique solutions to all equations of the form $X \cong F(\blacktriangleright X)$ where $\blacktriangleright$ is a modal type operator pronounced “later”. When combining guarded recursion with dependent types, one obtains a very powerful type system in which one can, for ex-

ample, construct models of advanced programming languages [7], see also [4]. The models obtained are synthetic variants of step-indexing models [3], but unlike these offer a type system for expressing the model constructions. Guarded recursion has also been used to model higher order reactive programming [14].

When considering covariant type equations, guarded recursive types are a third alternative to inductive and coinductive types. In this paper, we study the relationship between guarded recursive types and coinductive types. We show that the latter can be encoded from the former, and that using a combination of the two suggests a way of building the productivity checks of corecursive definitions into the type system of proof assistants.

Consider, for example, a type of streams of natural numbers $S(\mathbb{N})$. This is a coinductive type satisfying $S(\mathbb{N}) \cong \mathbb{N} \times S(\mathbb{N})$. To ensure consistency and decidability of type checking, proof assistants require corecursive definitions of elements of type $S(\mathbb{N})$ to be productive, i.e., the n ’th element of the stream should be computable in finite time. For example, the definition $xs \stackrel{\text{def}}{=} xs$ is not productive, but $zeros \stackrel{\text{def}}{=} 0 :: zeros$ is (here $::$ is the cons operation). Coq and Agda enforce productivity by checking that all occurrences of the recursion variable are directly underneath a constructor. The syntactic checks do not interact well with higher-order programming, and this has led to research on how to “code around” these limitations, e.g., [9].

Consider on the other hand the guarded recursive solution to the same type equation: $Str_{GR} \cong \mathbb{N} \times \blacktriangleright Str_{GR}$. The type $\blacktriangleright Str_{GR}$ should be thought of as typing streams that are only available one time-step from now, and thus we can think of Str_{GR} as a type of streams where the tail takes one time-step to compute. To define elements of Str_{GR} recursively we use a guarded fixpoint operator $\text{fix} : (\blacktriangleright X \rightarrow X) \rightarrow X$, satisfying the fixed point equation $f(\text{next}(\text{fix}(f))) = \text{fix}(f)$, where $\text{next} : X \rightarrow \blacktriangleright X$ is an operation that makes anything available now also available in the future. Using fix we can define $zeros$ as $\text{fix}(\lambda z : \blacktriangleright Str_{GR}. 0 :: z)$, but $\text{fix}(\lambda xs : Str_{GR}. xs)$ is not defined. In fact, the type $\blacktriangleright Str_{GR} \rightarrow Str_{GR}$ exactly captures productive recursive stream definitions. Since elements of coinductive types are sometimes called codata, we refer to this approach as productive coprogramming via guarded recursion. Guarded recursion can be seen as a light-weight alternative to sized types [1, 18].

To give some intuition for guarded recursion we recall the topos of trees model.

1.1 The topos of trees

The topos of trees is the category of contravariant presheaves over ω , the ordered set of natural numbers. In elementary terms, this means that a type is modelled as a family of sets and maps

$$X_1 \xleftarrow{r_1^X} X_2 \xleftarrow{r_2^X} X_3 \xleftarrow{r_3^X} \dots \quad (1)$$

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Intuitively, the set X_n describes the types as it looks if we have n computation steps to reason about it. For example, the type Str_{GR} is modelled as

$$\mathbb{N} \xleftarrow{\pi} \mathbb{N}^2 \xleftarrow{\pi} \mathbb{N}^3 \xleftarrow{\pi} \mathbb{N}^4 \xleftarrow{\pi} \dots \quad (2)$$

since in n computation steps we can at most access the n first elements of a stream.

The operation $\blacktriangleright$ maps X of (1) to

$$1 \xleftarrow{\quad} X_1 \xleftarrow{r_1^X} X_2 \xleftarrow{r_2^X} X_3 \xleftarrow{r_3^X} \dots$$

and since products are computed pointwise, the object Str_{GR} does indeed satisfy $Str_{GR} \cong \mathbb{N} \times \blacktriangleright Str_{GR}$.

1.2 A calculus with universal quantification over clocks

Guarded recursive types are not a replacement for coinductive types. Sometimes the time-steps get in the way. For example, the head of the tail of a stream has type $\blacktriangleright \mathbb{N}$. In the topos of trees, the endomaps on Str_{GR} correspond to the causal stream functions, i.e., those for which the n first elements of the output stream only depend on the n first elements of the input stream.

Atkey and McBride [5] suggest a simply typed calculus that allows one to conveniently switch between guarded recursive types and coinductive types, using the former whenever constructing streams, and using the latter when taking them apart. The calculus types terms in a context of free clock variables ranged over by $\kappa, \kappa' \dots$ each corresponding to a different notion of time step. For each κ , there is a $\blacktriangleright^\kappa$ and fix^κ etc. The calculus includes universal quantification over clocks in types, i.e., if A is a type, so is $\forall \kappa. A$.

The key result relating guarded types to coinductive types is that the latter can be encoded using the former. Precisely, they show that $\nu X. F(X) \stackrel{\text{def}}{=} \forall \kappa. \text{Fix}^\kappa X. F(\blacktriangleright^\kappa X)$ is a coinductive type, if $\text{Fix}^\kappa X. F(\blacktriangleright^\kappa X) \cong F(\blacktriangleright^\kappa (\text{Fix}^\kappa X. F(\blacktriangleright^\kappa X)))$. The result holds for F strictly positive and κ not appearing in F , and is proved with respect to a denotational model.

For these results to be applicable in proof assistants, they must be extended to a type theory with dependent types. We do that here, and give a model using a family of categories $\text{GR}[\Delta]$ indexed by clock contexts Δ each a multi-dimensional version of the topos of trees. Types and terms in the context of a single clock variable are modelled in the topos of trees, and those in empty clock context in the category of sets. Universal quantification over clocks is modelled by taking limits. Note that the limit of the sequence (2) is the set of streams.

Each $\text{GR}[\Delta]$ is locally cartesian closed, and so to model dependent type theory, we must address the coherence problem, i.e., the problem that pullback, which models substitution, usually does not commute strictly with constructions such as dependent products and is not associative. To do this, we follow a recent approach by Voevodsky and Streicher using universes, but since we have not just one locally cartesian closed category but a whole family, we must first embed all the categories in a single one in a structure preserving way.

The focus of the paper is on presenting the calculus and the model construction. Atkey and McBride give a series of examples illustrating the usefulness of their calculus, including an encoding of stream processors due to Ghani et al. [10]. Since their calculus is essentially subsumed by the one presented here, all these examples can be transferred to our type theory. We shall therefore not argue further for the usefulness of the type theory, but only give a few examples, including one (Example 2) that requires dependent types and therefore cannot be expressed in the calculus of Atkey and McBride.

$$\begin{array}{c} \frac{X = 0, 1, 2}{\Delta; \Gamma \vdash X : \text{Type}} \quad \frac{\Delta; \Gamma \vdash A : \text{Type} \quad \Delta; \Gamma \vdash B : \text{Type}}{\Delta; \Gamma \vdash A \times B : \text{Type}} \\ \frac{\Delta; \Gamma \vdash A : \text{Type} \quad \Delta; \Gamma \vdash B : \text{Type}}{\Delta; \Gamma \vdash A + B : \text{Type}} \quad \frac{\Delta; \Gamma, x : A \vdash B : \text{Type}}{\Delta; \Gamma \vdash \prod x : A. B : \text{Type}} \\ \frac{\Delta; \Gamma, x : A \vdash B : \text{Type}}{\Delta; \Gamma \vdash \sum x : A. B : \text{Type}} \quad \frac{\Delta; \Gamma, x : A \vdash B : \text{Type}}{\Delta; \Gamma \vdash Wx : A. B : \text{Type}} \end{array}$$

Figure 1. Types of a basic dependent type theory with clock contexts

1.3 Structure of the paper

We extend the simply typed calculus of Atkey and McBride with dependent types and universes in Section 2 and prove correct the encoding of coinductive types using guarded recursive types in Section 3. As a special case we show how to encode M-types using guarded recursion (Corollary 2). We construct the family of categories $\text{GR}[\Delta]$ in Section 4, and in Section 5 we show how to construct universes in these from Grothendieck universes.

The category GR subsuming all the categories $\text{GR}[\Delta]$ is presented in Section 6, and Section 7 constructs the universes needed both for solving the coherence problem and to model the universes in the type theory. The interpretation of the calculus in the model is presented in Section 8.

2. Syntax

We now describe the calculus in details. To accomodate quantification over clocks the calculus has a context of clock variables. We assume given a countably infinite set CV of clock variables, and we use κ, κ' etc to range over this set. A clock context is a finite list Δ of clock variables without repetition. For every clock context Δ there are judgements of well-formed types and contexts over Δ , as well as typing judgements for terms. These rules include basic types like 0 (the empty type), 1 (unit type), 2 (a boolean type), binary products and sums, and formation of dependent sums and products and W-types, the type theoretic version of inductive types. We do not consider identity types in this paper. We consider an equality theory as corresponding to models in locally cartesian closed categories with W-types [16]. The rules are standard and for reasons of space we just include the type formation rules, see Figure 1. We write $A \rightarrow B$ (and sometimes B^A) for $\prod x : A. B$ when x is not free in B .

Figures 2 and 3 describe an extension of dependent type theory with guarded recursion and universal quantification over clocks. We consider universal quantification over clocks a binding construction and thus include (implicitly) α -equalities of universally quantified types and Λ -abstraction. We define the set of free clock variables $\text{fc}(t) \subset \text{CV}$ of terms, types and contexts in the obvious way. We say a term or type is closed if it can be typed in an empty context Γ (the clock context needs not be empty).

We say that two terms t, u of same type in the same context are *provably equal* if they are in the least congruence relation generated by the equalities presented here.

2.1 Guarded recursion

The type constructor $\blacktriangleright^\kappa$ is an applicative functor in the sense of McBride and Paterson [15]. Note in particular that this means that there is a term

$$\Delta; \Gamma, f : A \rightarrow B \vdash \lambda x : \blacktriangleright^\kappa A. (\text{next}^\kappa f) \circledast^\kappa x : \blacktriangleright^\kappa A \rightarrow \blacktriangleright^\kappa B$$

wellformed types			
$\Delta, \kappa; \Gamma \vdash A: \text{Type}$	$\kappa \notin \text{fc}(\Gamma)$	$\Delta; \Gamma \vdash A: \text{Type}$	$\kappa \in \Delta$
$\Delta; \Gamma \vdash \forall \kappa. A: \text{Type}$		$\Delta; \Gamma \vdash \blacktriangleright^\kappa A: \text{Type}$	
Typing rules			
$\Delta; \Gamma \vdash t: A$	$\Delta; \Gamma \vdash t: \blacktriangleright^\kappa(A \rightarrow B)$	$\Delta; \Gamma \vdash u: \blacktriangleright^\kappa A$	$\Delta; \Gamma, x: \blacktriangleright^\kappa A \vdash t: A$
$\Delta; \Gamma \vdash \text{next}^\kappa(t): \blacktriangleright^\kappa A$	$\Delta; \Gamma \vdash t \circledast^\kappa u: \blacktriangleright^\kappa B$		$\Delta; \Gamma \vdash \text{fix}^\kappa x.t: A$
$\kappa \notin \text{fc}(\Gamma)$	$\Delta, \kappa; \Gamma \vdash A: \text{Type}$	$\Delta, \kappa'; \Gamma, \Gamma' \vdash t: \forall \kappa. A$	$\Delta, \kappa; \Gamma \vdash t: A \quad \kappa \notin \text{fc}(\Gamma)$
$\Delta, \kappa'; \Gamma, \Gamma' \vdash t[\kappa']: A[\kappa'/\kappa]$			$\Delta; \Gamma \vdash \Lambda \kappa. t: \forall \kappa. A$

Figure 2. An extension of dependent type theory with guarded recursion and universal quantification over clocks.

$$\begin{aligned}
(\Lambda \kappa. t)[\kappa'] &= t[\kappa'/\kappa] \\
\Lambda \kappa. (t[\kappa]) &= t & \kappa \notin \text{fc}(t) \\
\text{next}^\kappa(\lambda x. x) \otimes^\kappa u &= u \\
\text{next}^\kappa(\circ) \otimes^\kappa s \otimes^\kappa t \otimes^\kappa u &= s \otimes^\kappa (t \otimes^\kappa u) \\
\text{next}^\kappa(t) \otimes^\kappa \text{next}^\kappa(u) &= \text{next}^\kappa(t(u)) \\
u \otimes^\kappa \text{next}^\kappa(t) &= \text{next}^\kappa(\lambda g. g(t)) \otimes^\kappa u \\
t[\text{next}^\kappa(\text{fix}^\kappa x. t)/x] &= \text{fix}^\kappa x. t \\
t[\text{next}^\kappa(u)/x] &= u \\
u &= \text{fix}^\kappa x. t
\end{aligned}$$

Figure 3. Equational rules for an extension of dependent type theory with guarded recursion and universal quantification over clocks. In the fourth equality, $\circ$ is function composition.

which behaves functorially (i.e., preserves identities and commutes with function composition). We shall write $\blacktriangleright^\kappa f: \blacktriangleright^\kappa A \rightarrow \blacktriangleright^\kappa B$, whenever $f: A \rightarrow B$. On the other hand, $\blacktriangleright^\kappa$ is not a monad. Indeed, in the model to be described later, there is no map $\blacktriangleright^\kappa \blacktriangleright^\kappa 0 \rightarrow \blacktriangleright^\kappa 0$.

There are two ways of distributing $\blacktriangleright^\kappa$ over function types. The first one is

$$\blacktriangleright^\kappa(A \rightarrow B) \rightarrow \blacktriangleright^\kappa A \rightarrow \blacktriangleright^\kappa B \quad (3)$$

obtained by currying $\otimes$. This mapping is an isomorphism in the model, but not in the calculus. The other one is

$$d: \blacktriangleright^\kappa(\prod x: A. B) \rightarrow \prod x: A. \blacktriangleright^\kappa B \quad (4)$$

defined as $\lambda f: \blacktriangleright^\kappa \prod x: A. B. \lambda x: A. \blacktriangleright^\kappa(\text{ev}_x)(f)$ where ev_x is $\lambda g: (\prod x: A. B). g(x)$. In the case of $\kappa \notin \text{fc}(A)$ this is an isomorphism in the model. Note that it satisfies $d(\text{next}^\kappa f) = \lambda x: A. \text{next}^\kappa(f(x))$.

We say that a term $f: A \rightarrow B$ is *contractive*, if there is a κ and a term $g: \blacktriangleright^\kappa A \rightarrow B$ such that $f = g \circ \text{next}^\kappa$. Note that the equational theory states that contractive terms $f: A \rightarrow A$ have unique fixed points.

2.2 Clock quantification

Clock abstraction $\Lambda \kappa. t$ restricts the clock κ to t , and clock application $t[\kappa']$ instantiates the restricted clock. The purpose of the assumptions in the rule for clock application is to avoid synchronisation of clocks; the clock κ' that t is instantiated with is not allowed to already occur in A , nor any variable that A depends upon. On

the other hand κ' can appear in t and in the variables that t depends on, hence the two components of the context for t .

In categorical terms, universal quantification over clocks is a right adjoint as we now explain. First we construct categories of types and terms: if $\Delta; \Gamma$ is a valid context, we can consider the category whose objects are types A such that $\Delta; \Gamma \vdash A: \text{Type}$ and morphisms are terms $\Delta; \Gamma \vdash t: A \rightarrow B$ considered up to provable equality. We shall call this category the slice over $(\Delta; \Gamma)$. For $\kappa \in \Delta$, the type former $\blacktriangleright^\kappa$ extends to an endofunctor on the slice over $(\Delta; \Gamma)$ as explained above, and one can show that next^κ defines a natural transformation. The following lemma introduces weakening functors between slices.

Lemma 1. *Judgements of wellformed types, contexts and typing judgements are closed under weakening in both clock contexts and variable contexts, e.g., if $\Delta; \Gamma \vdash t: A$ is a valid typing judgement and $\Delta, \Delta'; \Gamma, \Gamma'$ is wellformed, then also $\Delta, \Delta'; \Gamma, \Gamma' \vdash t: A$ is a valid typing judgement.*

In particular, there is a weakening functor from the slice over $(\Delta; \Gamma)$ to the slice over $(\Delta, \kappa; \Gamma)$ whenever $\kappa \notin \Delta$. On the other hand, if $\kappa \notin \Delta$ we can define a functor $\forall \kappa$ going the other way, whose action on morphisms maps $\Delta, \kappa; \Gamma \vdash t: A \rightarrow B$ to

$$\forall \kappa(t) \stackrel{\text{def}}{=} \lambda x: \forall \kappa. A. \Lambda \kappa. t(x[\kappa]): \forall \kappa. A \rightarrow \forall \kappa. B$$

Proposition 1. *Let $\Delta; \Gamma$ be a well-formed context, and let $\Delta, \kappa; \Gamma \vdash A: \text{Type}$. There is a natural bijection between terms (considered up to provable equivalence) of the form $\Delta, \kappa; \Gamma \vdash t: A$ and those of the form $\Delta; \Gamma \vdash s: \forall \kappa. A$. In categorical terms, the functor $\forall \kappa$ from the slice over $(\Delta, \kappa; \Gamma)$ to the slice over $(\Delta; \Gamma)$ is right adjoint to the weakening functor.*

The correspondence maps $\Delta, \kappa; \Gamma \vdash t: A$ to $\Lambda \kappa. t$ and $\Delta; \Gamma \vdash s: \forall \kappa. A$ to $s[\kappa]$. The unit of the adjunction is

$$\lambda x: A. \Lambda \kappa. x: A \rightarrow \forall \kappa. A \quad (\kappa \notin \text{fc}(A))$$

and the counit is

$$\text{ev}_\kappa \stackrel{\text{def}}{=} \lambda x: \forall \kappa. A. x[\kappa].$$

Naturality of ev_κ is $\text{ev}_\kappa \circ \forall \kappa(f) = f \circ \text{ev}_\kappa$.

2.3 Type isomorphisms

We say that two types A, B well-formed in the same context $\Delta; \Gamma$ are isomorphic, if they are isomorphic as objects of the slice over $(\Delta; \Gamma)$, i.e., if there are terms $f: A \rightarrow B$ and $g: B \rightarrow A$ both well-typed in context $\Delta; \Gamma$ such that both compositions are provably equal to the identity. We write $A \cong B$ to mean that A and B are isomorphic. For example, the following three type

isomorphisms can be constructed in the calculus:

$$\begin{aligned} \forall \kappa. A \times B &\cong \forall \kappa. A \times \forall \kappa. B \\ \forall \kappa. \prod x: A. B &\cong \prod x: A. \forall \kappa. B & (\kappa \notin \text{fc}(A)) \\ \forall \kappa. \forall \kappa'. A &\cong \forall \kappa'. \forall \kappa. A \end{aligned}$$

We extend the type theory with the type isomorphisms of Figure 4. The justification for these is that they hold in the model (Theorem 3). Most of these are generalisations of the type equalities of [5], but will not be identities in our model. Precisely, for all the type isomorphisms listed, the term from left to right can be defined canonically in our calculus, but we add the term going the other way, plus two equalities of terms, stating that the added term is an inverse to the existing one.

$$\begin{aligned} \forall \kappa. A &\cong A & (\kappa \notin \text{fc}(A)) \\ (\forall \kappa. A) + (\forall \kappa. B) &\cong \forall \kappa. (A + B) \\ \sum x: A. \forall \kappa. B &\cong \forall \kappa. \sum x: A. B & (\kappa \notin \text{fc}(A)) \\ \blacktriangleright^{\kappa'} \forall \kappa. A &\cong \forall \kappa. \blacktriangleright^{\kappa'} A & (\kappa \neq \kappa') \\ \forall \kappa. A &\cong \forall \kappa. \blacktriangleright^{\kappa} A \end{aligned}$$

Figure 4. Type isomorphisms. The directions from left to right are definable in the calculus

The first of these states that the unit of the adjunction of Proposition 1 is an isomorphism. Thus, we obtain the following corollary to Proposition 1.

Corollary 1. *Let $\Delta; \vdash A$: Type be a closed type and let $\kappa \notin \Delta$. There is a bijective (up to provable equivalence) correspondence between terms $\Delta, \kappa; \vdash t: A$ and terms $\Delta; \vdash s: A$.*

The last isomorphism of Figure 4 is particularly important. Following [5] we name it force, and the axioms added state that force is inverse to the term

$$\lambda x: \forall \kappa. A. \Lambda \kappa. \text{next}^{\kappa}(x[\kappa]): \forall \kappa. A \rightarrow \forall \kappa. \blacktriangleright^{\kappa} A$$

The term force allows us to ignore time steps in universally quantified clocks. It is used in the encoding of coinductive types using guarded recursive types (proof of Theorem 2).

Atkey and McBride state one more type equality commuting inductive type formation over $\forall \kappa$ in special cases. Since the only inductive types considered in this paper are W-types, we restrict attention to these. To best describe the type isomorphism, recall that W-types are initial algebras for polynomial functors, in particular, the W-type $\Delta; \Gamma \vdash Wx: A. B$: Type can be captured up to isomorphism as the initial algebra for the functor $F(X) = \sum a: A. \prod b: B.X$.

Now, consider polynomials of two variables, i.e., of the form

$$F(X, Y) = \sum a: A. (\prod b: B_0.X) \times (\prod b: B_1.Y)$$

for some A, B_0, B_1 such that

$$\begin{aligned} \Delta; \Gamma \vdash A &: \text{Type} \\ \Delta; \Gamma, a: A \vdash B_0 &: \text{Type} \\ \Delta; \Gamma, a: A \vdash B_1 &: \text{Type} \end{aligned}$$

Whenever $\Delta; \Gamma \vdash Y$: Type, the initial algebra $\mu X. F(X, Y)$ exists for the functor $F(-, Y)$, since it is a W-Type. This also extends: if $\kappa \notin \Delta$ and $\Delta, \kappa; \Gamma \vdash Y$: Type, also the initial algebra $\mu X. F(X, Y)$ exists for the functor $F(-, Y)$ on the slice over $(\Delta, \kappa; \Gamma)$.

We add to the type theory inverses making the canonical map

$$\mu X. F(X, \forall \kappa. Y) \rightarrow \forall \kappa. \mu X. F(X, Y) \quad (5)$$

an isomorphism.

2.4 Universes

Following [8] we use universes to introduce guarded recursive types. Universes should contain the base types and be closed under type constructors including $\blacktriangleright^{\kappa}$, but of course only for κ in the current clock context. Since the collection of types in the universe thus depends on the current clock context, it does not make sense to talk about a single universe closed under all these constructions, rather we must consider a family of universes U_{Δ} indexed by clock contexts Δ . See also Remark 1 for a semantic motivation for indexing universes by clock contexts.

Figure 5 gives the rules for universes. The $\blacktriangleright^{\kappa}$ in the hypothesis for the rule for $\triangleright^{\kappa}$ is crucial for ensuring a large collection of *contractive functors* as we shall see in Section 3.

3. Corecursion via guarded recursion

In this section we show how to encode coinductive types via guarded recursive types extending the results of [5] to dependent types. We need to speak of *internal slice categories* (as opposed to the external ones of Section 2.2) and functors between them. For readability we write X rather than $\text{El}(X)$.

3.1 Internal slice categories

If $\Delta; \vdash I$: Type is a valid judgement, we introduce the internal slice category over I to be the category whose objects are closed terms of type $X: I \rightarrow U_{\Delta}$ and morphisms are equivalence classes of closed terms $f: \prod i: I. X(i) \rightarrow Y(i)$ considered up to provable equality. We write simply $f: X \rightarrow Y$ if it is clear from context that X and Y are objects of the internal slice category.

A functor between slice categories is a pair of closed terms

$$\begin{aligned} F_0: (I \rightarrow U_{\Delta}) &\rightarrow (J \rightarrow U_{\Delta'}) \\ F_1: \prod X, Y: U_{\Delta}. (\prod i: I. Y(i)^{X(i)}) &\rightarrow \prod j: J. F_0 Y(j)^{F_0 X(j)} \end{aligned}$$

preserving identity maps and composition up to provable equality. Note that F_0, F_1 must necessarily be typed in a clock context Δ'' such that $\Delta, \Delta' \subseteq \Delta''$. Which Δ'' we choose is irrelevant by Corollary 1. Following standard convention from category theory, we often write simply F for both F_0 and F_1 .

For example, for any slice $(\Delta; I)$ we can define an endofunctor $\blacktriangleright^{\kappa}$ pointwise as $(\blacktriangleright^{\kappa} X)(i) = \triangleright^{\kappa}(\text{next}^{\kappa}(X(i)))$. If, moreover, $\kappa \notin \Delta$, we can construct the functor $\forall \kappa$ from the slice over $(\Delta, \kappa; I)$ to that over $(\Delta; I)$ which acts on objects by

$$(\forall \kappa. X)(i) = \forall \kappa. (\Lambda \kappa. X(i))$$

The universe inclusion $\text{in}_{\Delta', \Delta}: U_{\Delta'} \rightarrow U_{\Delta}$ induces a functor from the slice over $(\Delta'; I)$ to that over $(\Delta; I)$ which we shall also call $\text{in}_{\Delta', \Delta}$. We say that an endofunctor F on the slice over $(\Delta'; I)$ *extends* to $(\Delta; I)$ if there is an endofunctor G on the slice over $(\Delta; I)$ such that $G \circ \text{in}_{\Delta', \Delta} = \text{in}_{\Delta', \Delta} \circ F$. We shall often simply write F also for the extension of F , since in the examples we have in mind, these are often just the same (open) type expression.

3.2 Contractive functors

We say that a functor F is *contractive* if both F_0 and F_1 are contractive maps. For example, the endofunctor $\blacktriangleright^{\kappa}$ is contractive, as witnessed on objects by the composite

$$\blacktriangleright^{\kappa}(I \rightarrow U_{\Delta}) \xrightarrow{d} (I \rightarrow \blacktriangleright^{\kappa} U_{\Delta}) \xrightarrow{(I \rightarrow \triangleright^{\kappa})} (I \rightarrow U_{\Delta})$$

wellformed types	
$\frac{\Delta' \subseteq \Delta}{\Delta; \Gamma \vdash U_{\Delta'} : \text{Type}}$	$\frac{\Delta; \Gamma \vdash t : U_{\Delta'}}{\Delta; \Gamma \vdash \text{El}(t) : \text{Type}}$
Typing rules	
$\frac{X = 0, 1, 2}{\Delta; \Gamma \vdash X : U_{\Delta'}}$	$\frac{\Delta; \Gamma \vdash A : U_{\Delta'} \quad \Delta; \Gamma \vdash B : U_{\Delta'}}{\Delta; \Gamma \vdash A \times B : U_{\Delta'}}$
$\frac{\Delta; \Gamma \vdash A : U_{\Delta'} \quad \Delta; \Gamma \vdash B : U_{\Delta'}}{\Delta; \Gamma \vdash A + B : U_{\Delta'}}$	
$\frac{\Delta; \Gamma \vdash A : U_{\Delta'} \quad \Delta; \Gamma, x : \text{El}(A) \vdash B : U_{\Delta'}}{\Delta; \Gamma \vdash \prod x : A. B : U_{\Delta'}}$	
$\frac{\Delta; \Gamma \vdash A : U_{\Delta'} \quad \Delta; \Gamma, x : \text{El}(A) \vdash B : U_{\Delta'}}{\Delta; \Gamma \vdash \sum x : A. B : U_{\Delta'}}$	
$\frac{\Delta; \Gamma \vdash A : U_{\Delta'} \quad \Delta; \Gamma, x : \text{El}(A) \vdash B : U_{\Delta'}}{\Delta; \Gamma \vdash Wx : A. B : U_{\Delta'}}$	
$\frac{\Delta; \Gamma \vdash A : \blacktriangleright^{\kappa} U_{\Delta'} \quad \kappa \in \Delta'}{\Delta; \Gamma \vdash \blacktriangleright^{\kappa} A : U_{\Delta'}}$	$\frac{\Delta; \Gamma \vdash A : \forall \kappa. U_{\Delta', \kappa}}{\Delta; \Gamma \vdash \bar{\forall} \kappa. A : U_{\Delta'}}$
$\frac{\Delta; \Gamma \vdash A : U_{\Delta''} \quad \Delta'' \subseteq \Delta' \subseteq \Delta}{\Delta; \Gamma \vdash \text{in}_{\Delta'', \Delta'}(A) : U_{\Delta'}}$	
Equalities	
$\text{El}(X) \equiv X$	$X = 0, 1, 2$
$\text{El}(A \times B) \equiv \text{El}(A) \times \text{El}(B)$	
$\text{El}(A + B) \equiv \text{El}(A) + \text{El}(B)$	
$\text{El}(\blacktriangleright^{\kappa} \text{next}^{\kappa}(A)) \equiv \blacktriangleright^{\kappa} \text{El}(A)$	
$\text{El}(\bar{\forall} \kappa. (\Lambda \kappa. A)) \equiv \forall \kappa. \text{El}(A)$	
$\text{El}(\sum x : A. B) \equiv \sum x : \text{El}(A). \text{El}(B)$	
$\text{El}(\prod x : A. B) \equiv \prod x : \text{El}(A). \text{El}(B)$	
$\text{El}(Wx : A. B) \equiv Wx : \text{El}(A). \text{El}(B)$	
$\text{El}(\text{in}_{\Delta'', \Delta'}(A)) \equiv \text{El}(A)$	
$\text{in}_{\Delta'', \Delta'}(\text{in}_{\Delta''', \Delta'}(A)) \equiv \text{in}_{\Delta''', \Delta'}(A)$	
$\text{in}_{\Delta', \Delta'}(A) \equiv A$	

Figure 5. Rules for universes

where the first map is the d of (4), and the action on morphisms is proved contractive similarly. Since $f \circ g$ is contractive whenever f or g is, we deduce the following lemma.

Lemma 2. *For any functor F between slice categories in context Δ , and for any $\kappa \in \Delta$, the composite functor $F \circ \blacktriangleright^{\kappa}$ is contractive.*

The next theorem shows the interest in contractive functors.

Theorem 1 ([8]). *Let F be a contractive endofunctor on a slice. Then F has a fixed point $\text{Fix}^{\kappa} X.F(X)$ (up to identity) which is both an initial algebra and a final coalgebra for F . Moreover*

$\text{Fix}^{\kappa} X.F(X)$ is the unique up-to-identity fixed point, and also unique up to isomorphism among the up-to-iso fixed points.

Proof. Since F_0 is contractive it has a unique fixed point. Given any fixed point up to isomorphism $f : FX \cong X$ and an algebra $g : FY \rightarrow Y$ the map $\lambda h : X \rightarrow Y. g \circ Fh \circ f^{-1}$ is contractive and thus has a unique fixed point, which is a unique algebra map from f to g . Thus, any up-to-iso fixed point for F is an initial algebra and thus all up-to-iso fixed point are isomorphic. The case of final coalgebras is similar. $\square$

Note that from the proof we can construct a term

$$\text{fold} : \prod Z : I \rightarrow U_{\Delta}. (FZ \rightarrow Z) \rightarrow (\text{Fix}^{\kappa} X.F(X)) \rightarrow Z$$

giving the unique map from $\text{Fix}^{\kappa} X.F(X)$ to any other algebra.

Example 1. *In the slice over $(\kappa, 1)$, the endofunctor given by $F(X) = \mathbb{N} \times \blacktriangleright^{\kappa} X$ is contractive and thus has a unique fixed point Str^{κ} , which is at the same time an initial algebra and final coalgebra for F . Note that even if we want to talk about types rather than codes of types, we get*

$$\begin{aligned} \text{El}(\text{Str}^{\kappa}) &\equiv \text{El}(\mathbb{N} \times \blacktriangleright^{\kappa} (\text{next}^{\kappa}(\text{Str}^{\kappa}))) \\ &\equiv \mathbb{N} \times \blacktriangleright^{\kappa} \text{El}(\text{Str}^{\kappa}) \end{aligned}$$

Example 2. *We now consider a more advanced example involving dependent types. In the slice category over $\mathbb{N}$ (empty clock context), Altenkirch and Morris [2] define the functor*

$$F(X)(n) = \text{Fin}(n) + X(n) \times X(n) + X(n+1)$$

where Fin is defined by induction on n :

$$\text{Fin}(0) = 1$$

$$\text{Fin}(n+1) = \text{Fin}(n) + 1.$$

as interpreted in set theory, the initial algebra Lam for F has as $\text{Lam}(n)$ the set of lambda terms with n free variables. The final coalgebra for ILam has $\text{ILam}(n)$ the set of possibly infinite lambda terms with n free variables.

We can also consider the type Lam^{κ} defined as the fixed point of $F \circ \blacktriangleright^{\kappa}$, i.e., satisfying

$$\text{Lam}^{\kappa}(n) = \text{Fin}(n) + \blacktriangleright^{\kappa} \text{Lam}^{\kappa}(n) \times \blacktriangleright^{\kappa} \text{Lam}^{\kappa}(n) + \blacktriangleright^{\kappa} \text{Lam}^{\kappa}(n+1)$$

Since $\lambda n : \mathbb{N}. \text{Lam}^{\kappa}(n+1)$ carries an algebra for $F \circ \blacktriangleright^{\kappa}$, and since Lam^{κ} is an initial algebra, we obtain an inclusion $i : \text{Lam}^{\kappa}(n) \rightarrow \text{Lam}^{\kappa}(n+1)$ corresponding to weakening. We can use this to construct the element

$$l = \text{fix}^{\kappa} x : \blacktriangleright^{\kappa} \text{Lam}^{\kappa}(n). \text{in}_{\lambda}(\blacktriangleright^{\kappa}(i)(x))$$

where $\text{in}_{\lambda} : \blacktriangleright^{\kappa} \text{Lam}^{\kappa}(n+1) \rightarrow \text{Lam}^{\kappa}(n)$ is the inclusion. The element l corresponds to the infinite lambda term $\lambda x. \lambda x. \lambda x. \dots$

3.3 Encoding coinductive types

We can now state the main theorem of this section, relating guarded recursive types to coinductive types.

Definition 1. *Let F be an endofunctor on the slice over $(\Delta; I)$, and let $\kappa \notin \Delta$. We say that F commutes with $\forall \kappa$, if F extends to $(\Delta, \kappa; I)$ (as in Section 3.1) and the canonical map $F(\forall \kappa. X) \rightarrow \forall \kappa. F(X)$ is an isomorphism for all X .*

Theorem 2. *Let F be an endofunctor on a slice over $(\Delta; I)$ and let $\kappa \notin \Delta$. If F commutes with $\forall \kappa$, then*

$$\nu X. F(X) \stackrel{\text{def}}{=} \forall \kappa. \text{Fix}^{\kappa} X. F(\blacktriangleright^{\kappa} X)$$

carries a final coalgebra structure for F . Moreover, there is a term

$$\text{unfold} : \prod Z : I \rightarrow U_{\Delta}. (Z \rightarrow FZ) \rightarrow Z \rightarrow \nu X. F(X)$$

computing the unique map from any other coalgebra.

By the type isomorphisms of Section 2.3, dependent and binary sums and products commute with universal quantification over clocks. In particular, the functors F of Examples 1 and 2 commute with $\forall\kappa$. Thus $\forall\kappa.Str^\kappa$ is a coinductive type of streams, and $\forall\kappa.Lam^\kappa$ is a coinductive type of infinite lambda terms. The term $\Lambda\kappa.l: \forall\kappa.Lam^\kappa(0)$ is the element of this type corresponding to $\lambda x.\lambda x.\lambda x.\dots$

All polynomial functors commute with $\forall\kappa$ and since M-types by definition are final coalgebras for these we get the following.

Corollary 2. *The rule for forming M-types*

$$\frac{\Delta; \Gamma \vdash A: U_{\Delta'} \quad \Delta; \Gamma, x: El(A) \vdash B: U_{\Delta'}}{\Delta; \Gamma \vdash Mx: A.B: U_{\Delta'}}$$

can be encoded as

$$Mx: A.B \stackrel{\text{def}}{=} \forall\kappa.Fix^\kappa X.(\sum x: A.(B \rightarrow \blacktriangleright^\kappa X)) \quad \kappa \notin \Delta$$

Proof of Theorem 2. For readability we define

$$\begin{aligned} Y &\stackrel{\text{def}}{=} Fix^\kappa X.F(\blacktriangleright^\kappa X) \\ Z &\stackrel{\text{def}}{=} \forall\kappa.Y \end{aligned}$$

Thus $Y \equiv F(\blacktriangleright^\kappa Y)$, and $F(Z) \cong Z$ by the following isomorphism

$$Z \equiv \forall\kappa.F(\blacktriangleright^\kappa Y) \xrightarrow{\cong} F(\forall\kappa.\blacktriangleright^\kappa Y) \xrightarrow{F(\text{force})} F(Z)$$

We show that given a coalgebra $f: X \rightarrow FX$ for F , the correspondence between maps $h: X \rightarrow Y$ and maps $\bar{h}: X \rightarrow Z$ of Proposition 1 restricts to a bijective correspondence between coalgebra maps $\bar{h}$

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \\ \bar{h} \downarrow & & \downarrow F(\bar{h}) \\ Z & \xrightarrow{\text{unfold}_Z} & F(Z) \end{array} \quad (6)$$

in the slice over $(\Delta; I)$ and coalgebra maps h :

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \xrightarrow{F(\text{next}^\kappa)} F(\blacktriangleright^\kappa X) \\ h \downarrow & & \downarrow F(\blacktriangleright^\kappa h) \\ Y & \xrightarrow{\text{unfold}_Y} & F(\blacktriangleright^\kappa Y) \end{array} \quad (7)$$

in the slice over $(\Delta, \kappa; I)$ (unfold_Y is simply the identity). The theorem then follows from Y being a final coalgebra.

Suppose first that (7) commutes. Consider the map

$$\phi_X: \forall\kappa.F(\blacktriangleright^\kappa X) \rightarrow F(\forall\kappa.X)$$

defined by composing the $F(\text{force})$ with the commutativity of F and $\blacktriangleright^\kappa$. The following commutes

$$\begin{array}{ccccc} \forall\kappa.X & \xrightarrow{\forall\kappa(F(\text{next}^\kappa) \circ f)} & \forall\kappa.F(\blacktriangleright^\kappa X) & \xrightarrow{\phi_X} & F(\forall\kappa.X) \\ \downarrow \forall\kappa(h) & & \downarrow \forall\kappa(F(\blacktriangleright^\kappa h)) & & \downarrow F(\forall\kappa(h)) \\ \forall\kappa.Y & \xrightarrow{\forall\kappa(\text{unfold}_Y)} & \forall\kappa.F(\blacktriangleright^\kappa Y) & \xrightarrow{\phi_Y} & F(\forall\kappa.Y) \end{array}$$

Now, the top line of this diagram is isomorphic to f via the isomorphism $X \cong \forall\kappa.X$, and thus we obtain (6).

Suppose now that (6) commutes. Let $d_X: F(\forall\kappa.X) \rightarrow \forall\kappa.F(X)$ denote the canonical isomorphism. Note first that

$$\begin{aligned} &ev_\kappa \circ \text{unfold}_Z^{-1} \\ &= ev_\kappa \circ \forall\kappa(\text{unfold}_Y^{-1}) \circ d_{\blacktriangleright^\kappa Y} \circ F(\forall\kappa(\text{next}^\kappa)) \\ &= ev_\kappa \circ \forall\kappa(\text{unfold}_Y^{-1}) \circ \forall\kappa(F(\text{next}^\kappa)) \circ d_Y \\ &= \text{unfold}_Y^{-1} \circ F(\text{next}^\kappa) \circ ev_\kappa \circ d_Y \\ &= \text{unfold}_Y^{-1} \circ F(\text{next}^\kappa) \circ F(ev_\kappa) \end{aligned}$$

so $\text{unfold}_Y \circ ev_\kappa = F(\text{next}^\kappa) \circ F(ev_\kappa) \circ \text{unfold}_Z$. So now,

$$\begin{aligned} \text{unfold}_Y \circ h &= \text{unfold}_Y \circ ev_\kappa \circ \bar{h} \\ &= F(\text{next}^\kappa) \circ F(ev_\kappa) \circ \text{unfold}_Z \circ \bar{h} \\ &= F(\text{next}^\kappa) \circ F(ev_\kappa) \circ F(\bar{h}) \circ f \\ &= F(\text{next}^\kappa) \circ F(h) \circ f \\ &= F(\blacktriangleright^\kappa(h)) \circ F(\text{next}^\kappa) \circ f \end{aligned}$$

□

We end the section with an example, showing how a simple non-causal stream function can be encoded in the type theory. For more examples of coding with guarded recursion and clock quantification, see [5].

Example 3. The type $S(\mathbb{N}) \stackrel{\text{def}}{=} \forall\kappa.Str^\kappa$ was seen above to be a coinductive type of streams. We now show how to encode the function $\text{odd}: S(\mathbb{N}) \rightarrow S(\mathbb{N})$ returning the stream of the elements at odd indices of the input stream. This function is not causal, and so there is no function $Str^\kappa \rightarrow Str^\kappa$ that does the same.

Following [8] we first define

$$\frac{\Delta; \Gamma, x: A \rightarrow \blacktriangleright^\kappa B \vdash t: A \rightarrow B}{\Delta; \Gamma \vdash \text{pfix}^\kappa x.t: A \rightarrow B}$$

as $\text{pfix}^\kappa x.t = \text{fix}^\kappa y: \blacktriangleright^\kappa(A \rightarrow B).t[\lambda a.y \circ \text{next}^\kappa(a)/x]$. Note that it satisfies

$$\text{pfix}^\kappa x.t = t[\text{next}^\kappa \circ (\text{pfix}^\kappa x.t)/x].$$

Next define

$$\kappa; f: S(\mathbb{N}) \rightarrow \blacktriangleright^\kappa Str^\kappa \vdash \text{odddrec}^\kappa(f): S(\mathbb{N}) \rightarrow Str^\kappa$$

as

$$\text{odddrec}^\kappa(f)(x :: y :: xs) \stackrel{\text{def}}{=} x :: \kappa f(xs)$$

Here the pattern matching syntax on the left hand side uses $S(\mathbb{N}) \cong \mathbb{N} \times S(\mathbb{N})$, and the cons operation $::^\kappa$ on the right hand side is the one of Str^κ , i.e. has type $\mathbb{N} \times \blacktriangleright^\kappa Str^\kappa \rightarrow Str^\kappa$. Now, define

$$\begin{aligned} \text{odd}^\kappa &\stackrel{\text{def}}{=} \text{pfix}^\kappa f.(\text{odddrec}^\kappa(f)): S(\mathbb{N}) \rightarrow Str^\kappa \\ \text{odd} &\stackrel{\text{def}}{=} \lambda xs: S(\mathbb{N}). (\Lambda\kappa.\text{odd}^\kappa(xs)): S(\mathbb{N}) \rightarrow S(\mathbb{N}) \end{aligned}$$

Intuitively, odd breaks the synchronisation between the input and output stream by using different clocks for them. This allows us to essentially ignore the timing on the input stream (treating it as an element of a coinductive type), but still use guarded recursion to construct the output stream.

To see that odd computes as expected, unfold the definition of the cons operation on $S(\mathbb{N})$ from the proof of Theorem 2:

$$x :: xs = \Lambda\kappa.(x :: \kappa \text{next}^\kappa(xs[\kappa]))$$

Using this we get

$$\begin{aligned}
\text{odd}(x :: y :: xs) &= \Lambda \kappa. \text{odd}^\kappa(x :: y :: xs) \\
&= \Lambda \kappa. \text{odddrec}^\kappa(\text{next}^\kappa \circ \text{odd}^\kappa)(x :: y :: xs) \\
&= \Lambda \kappa. (x :: {}^\kappa(\text{next}^\kappa(\text{oddd}^\kappa(xs)))) \\
&= x :: (\Lambda \kappa. \text{oddd}^\kappa(xs)) \\
&= x :: (\text{odd}(xs))
\end{aligned}$$

4. Model

We now construct a model of the calculus of Section 2. We start in this section, by describing a family of categories, and constructing a sketch of a model of the calculus, ignoring the coherence problem arising from interpreting dependent type theory in locally cartesian closed categories. We address this problem in Section 6.

The model is a generalisation of the topos-of-trees model of [6, 7] to more dimensions. Types and terms in clock context Δ are interpreted in the category

$$\text{GR}[\Delta] \stackrel{\text{def}}{=} \mathbf{Set}^{(\omega^\Delta)^{\text{op}}}$$

The objects of this category are families of sets X_δ indexed by maps $\delta: \Delta \rightarrow \omega$ (ω are the natural numbers with the usual order), plus maps $r_{\delta \leq \delta'}^X: X_{\delta'} \rightarrow X_\delta$ for each $\delta \leq \delta'$ (pointwise order) satisfying functoriality in δ . If, for example, $\Delta = \kappa, \kappa'$ just consists of two clock variables, an object X can be drawn as a 2-dimensional lattice of sets and maps

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
X_{1,2} & \leftarrow & X_{2,2} & \leftarrow & X_{3,2} \leftarrow \dots \\
\downarrow & & \downarrow & & \downarrow \\
X_{1,1} & \leftarrow & X_{2,1} & \leftarrow & X_{3,1} \leftarrow \dots
\end{array}$$

A morphism $f: X \rightarrow Y$ is a family of maps $f_\delta: X_\delta \rightarrow Y_\delta$ such that $f_\delta \circ r_{\delta \leq \delta'}^X = r_{\delta \leq \delta'}^Y \circ f_{\delta'}$ for each $\delta \leq \delta'$.

Since $\text{GR}[\Delta]$ is a topos it is in particular locally cartesian closed, i.e., a category with pullbacks, terminal object and, for all morphisms $f: X \rightarrow Y$, a functor $\prod_f: \text{GR}[\Delta]/X \rightarrow \text{GR}[\Delta]/Y$, right adjoint to the functor f^* given by pullback. (Recall that the slice category $\text{GR}[\Delta]/X$ has as object morphisms $Z \rightarrow X$ and commutative triangles as morphisms.) We say that $\prod_f(g)$ is the *fibred product* of g along f . Recall also that f^* always has a left adjoint $\sum_f$ given by composition with f . Since the categories $\text{GR}[\Delta]$ are toposes, they moreover model W-types [16].

Locally cartesian closed categories (lccc's) almost model dependent type theory (up to the coherence problem). In our setting the lccc structure lets us model contexts $(\Delta; \Gamma)$ as objects $\llbracket \Delta; \Gamma \rrbracket$ of $\text{GR}[\Delta]$, types $\Delta; \Gamma \vdash A: \text{Type}$ as objects of the slice category $\text{GR}[\Delta]/\llbracket \Delta; \Gamma \rrbracket$, i.e., as morphisms

$$p_{\Delta; \Gamma \vdash A}: \llbracket A \rrbracket \rightarrow \llbracket \Delta; \Gamma \rrbracket$$

and terms $\Delta; \Gamma \vdash t: A$ as sections of $p_{\Delta; \Gamma \vdash A}$, i.e., morphisms such that $p_{\Delta; \Gamma \vdash A} \circ \llbracket t \rrbracket = \text{id}_{\llbracket \Delta; \Gamma \rrbracket}$. Dependent products are modelled using fibred products and dependent sums by composition.

4.1 Modelling guarded recursion

The operator $\blacktriangleright^\kappa$, next^κ and fix and $\otimes$ can be modelled using generalisations of the constructions of [6]. For example, in an

empty context we define

$$(\blacktriangleright^\kappa A)(\delta) = \begin{cases} A(\delta[\kappa \mapsto (\delta(\kappa) - 1)]) & \text{if } \delta(\kappa) = 0 \\ 1 & \text{else} \end{cases} \quad (8)$$

and this is extended to a family of endofunctors on slices

$$\blacktriangleright_\Gamma^\kappa: \text{GR}[\Delta]/\Gamma \rightarrow \text{GR}[\Delta]/\Gamma$$

(see [6, 7]), as needed for modelling $\blacktriangleright^\kappa$ in non-empty contexts. In each slice there is a fixed point combinator $(\blacktriangleright_\Gamma^\kappa X \rightarrow X) \rightarrow X$ giving unique fixed points.

To model quantification over clocks, we consider functors induced by inclusions of the form $i: \Delta \rightarrow \Delta, \kappa$ between clock contexts. Such an i induces a functor

$$\text{GR}[\Delta] \xrightarrow{i^*} \text{GR}[\Delta, \kappa]$$

defined on objects as $i^*(X)(\delta) = X(\delta \circ i)$. In the simple case of Δ being empty, i^* is simply the constant set functor $\mathbf{Set} \rightarrow \text{GR}[\kappa]$. It is a crucial invariant of the model that the functor i^* corresponds to clock-weakening in the sense that, e.g.,

$$\llbracket \Delta, \kappa; \Gamma \vdash A: \text{Type} \rrbracket \cong i^* \llbracket \Delta; \Gamma \vdash A: \text{Type} \rrbracket \quad (9)$$

(in fact, this will be an equality in Lemma 12). Since, by Proposition 1 universal quantification over clocks is a right adjoint to weakening, we should model it using a right adjoint to i^* . Standard constructions from topos theory give both a right and a left adjoint to i^* .

Lemma 3. *The functor i^* has a right adjoint i_* , and a left adjoint $i_!$ defined on objects as*

$$\begin{aligned}
i_*(X)(\delta) &= \lim_{\leftarrow} (X(\delta[\kappa \mapsto -])) \\
i_!(X)(\delta) &= X(\delta[\kappa \mapsto 1])
\end{aligned}$$

The unit of the adjunction $i^ \dashv i_*$ is an isomorphism.*

For example, in the case of Δ being empty, $i_*(X)$ is the limit of the sequence $X_1 \leftarrow X_2 \leftarrow X_3 \dots$.

To interpret $\forall \kappa$ in contexts, we need to extend i_* to slice categories as in the following proposition.

Proposition 2. *The functor $(i^*)^\rightarrow: \text{GR}[\Delta]/X \rightarrow \text{GR}[\Delta, \kappa]/i^*X$, mapping an object $p: Y \rightarrow X$ to $i^*p: i^*Y \rightarrow i^*X$ has a right adjoint mapping $q: Z \rightarrow i^*X$ to $\eta_X^{-1} \circ i_*q: i_*Z \rightarrow X$ (where η is the unit of the adjunction). Moreover, the right adjoint commutes with reindexing between slices.*

5. Universes

To model the universes U_Δ we need semantic universes in the following standard sense.

Definition 2. *Let $\mathbb{C}$ be an lccc with finite coproducts and W-types, and let $E \rightarrow U$ be a morphism in $\mathbb{C}$. Say that $B \rightarrow A$ is small wrt $E \rightarrow U$ if it can be presented as a pullback of $E \rightarrow U$ along some map $\bar{B}: A \rightarrow U$. In this case we say that $\bar{B}$ is a code for $B \rightarrow A$. Say X is small if $X \rightarrow 1$ is small.*

A universe in $\mathbb{C}$ is a map $E \rightarrow U$ such that $0, 1, 1+1$ and $\mathbb{N}$ are small wrt $E \rightarrow U$ and moreover the induced notion of smallness is closed under composition, small fibred products, and W-types. The latter means that if $B \rightarrow A \rightarrow \Gamma$ are two small maps, then also the induced W-type (see [16]) is a small map into Γ .

We will often denote a universe $E \rightarrow U$ simply by U .

If U is a Grothendieck universe in $\mathbf{Set}$, the first projection $(\prod_{X \in U} X) \rightarrow U$ is a universe in the sense of Definition 2. A morphism $f: A \rightarrow B$ is small iff each fibre $f^{-1}(b)$ is isomorphic to an element of U . To construct universes in the categories $\text{GR}[\Delta]$ we make the following assumption.

Assumption 1. For the rest of the paper we will assume that we are given a Grothendieck universe U in $\mathbf{Set}$.

Using U one can construct universes in presheaf categories such as $\mathbf{GR}[\Delta]$ via a construction going back to [12] (see also [8, 19]). The universe V_Δ is defined as the object in $\mathbf{GR}[\Delta]$, whose value at $\delta \in \omega^\Delta$ is the set of diagrams of the form

$$\{\delta' \in \omega^{\Delta'} \mid \delta' \leq \delta\}^{\text{op}} \rightarrow \mathbb{U}$$

where the indexing set carries the restriction of the order from $(\omega^{\Delta'})^{\text{op}}$, and $\mathbb{U}$ is the full subcategory of $\mathbf{Set}$ on the objects from U . We introduce the notation $\downarrow \delta$ for the ordered set $\{\delta' \in \omega^\Delta \mid \delta' \leq \delta\}$, thus $V_\Delta(\delta)$ is the set of objects of the functor category $\mathbb{U}^{(\downarrow \delta)^{\text{op}}}$.

The presheaf of elements is

$$E_\Delta^V(\delta) = \sum_{F \in V_\Delta(\delta)} F(\delta) \quad (10)$$

with the first projection into V_Δ .

Proposition 3 ([12]). Each $E_\Delta^V \rightarrow V_\Delta$ is a universe in $\mathbf{GR}[\Delta]$ in the sense of Definition 2. A morphism $A \rightarrow B$ is small if and only if each $A_\delta \rightarrow B_\delta$ is small.

Remark 1. Note that the family $(V_\Delta)_\Delta$ is not closed under reindexing, i.e., if $\Delta' \subset \Delta$, it is not the case that $i^*V_{\Delta'} \cong V_\Delta$, where $i: \Delta' \rightarrow \Delta$ is the inclusion. For example, in the case of $\emptyset \subset \kappa$, $i^*V_\emptyset(\delta) = U$, which is not the same as $V_\kappa(\delta)$. Since the property (9) is crucial to the model construction, this means that using a single universe in the calculus with $\llbracket \Delta, \kappa; - \vdash U : \text{Type} \rrbracket = V_\Delta$ is not an option. This is our semantic motivation for indexing universes by clock contexts.

In the calculus, in clock context Δ , there is a universe type $U_{\Delta'}$ for each $\Delta' \subseteq \Delta$. This will be modelled as $i^*V_{\Delta'}$, where $i: \Delta' \rightarrow \Delta$ is the inclusion. To model closure of $U_{\Delta'}$ under type forming operations, we need to know that $i^*V_{\Delta'}$ is a universe. This follows from the following lemma.

Lemma 4. Let $\mathbb{C}$ be a locally cartesian closed category with W -types and finite coproducts, and let $\mathbb{D}$ be a small category with a terminal object, such that the functor category $\mathbb{C}^{\mathbb{D}}$ is locally cartesian closed. Let $E \rightarrow U$ be a universe in $\mathbb{C}$, and let $D: \mathbb{C} \rightarrow \mathbb{C}^{\mathbb{D}}$ be the diagonal functor $D(C)(d) = C$. Then $D(E) \rightarrow D(U)$ is a universe in $\mathbb{C}^{\mathbb{D}}$.

Note that this proves that $i^*V_{\Delta'}$ is a universe, since up to the isomorphism $\mathbf{GR}[\Delta] \cong \mathbf{GR}[\Delta']^{(\omega^{\Delta'})^{\text{op}}}$ (where $\Delta = \Delta', \Delta''$), i^* is the diagonal functor.

For reasons of space we omit the proof of Lemma 4, but just state an important lemma on which the proof is based.

Lemma 5. Under the same assumptions as Lemma 4, a map $f: Y \rightarrow X$ in $\mathbb{C}^{\mathbb{D}}$ is small with respect to $D(E) \rightarrow D(U)$ if and only if f_1 is small with respect to $E \rightarrow U$ and for each d in $\mathbb{D}$ the naturality square below is a pullback diagram

$$\begin{array}{ccc} Y(d) & \longrightarrow & Y(1) \\ \downarrow & & \downarrow \\ X(d) & \longrightarrow & X(1) \end{array}$$

6. A category subsuming all $\mathbf{GR}[\Delta]$

We now turn to the coherence problem of interpretation of dependent type theory in locally cartesian closed categories, i.e., the problem that pullback, which models substitution, usually does not commute with constructions such as dependent products and is not

associative [11]. Here we follow a recent approach developed independently by Voevodsky [13] and Streicher [20] using universes, see also [8, 19]. Voevodsky and Streicher show that given a universe $E \rightarrow U$ in a locally cartesian closed category, one can construct a model of dependent type theory by modelling contexts as objects and types $\Gamma \vdash A : \text{Type}$ as morphisms $\llbracket \Gamma \rrbracket \rightarrow U$.

In our setting we have a whole family of locally cartesian closed categories $\mathbf{GR}[\Delta]$ (indexed over Δ) for which we need to simultaneously solve the coherence problem, and at the same time maintain the invariant that $\llbracket \Delta, \kappa; \Gamma \rrbracket \cong i^* \llbracket \Delta; \Gamma \rrbracket$. To do this we introduce a new category

$$\mathbf{GR} \stackrel{\text{def}}{=} \mathbf{Set}^{(\omega^{\text{CV}})^{\text{op}}}$$

Definition 3. An object X of $\mathbf{GR}$ is supported by Δ if whenever $\delta \leq \delta'$ and $\delta|_\Delta = \delta'|_\Delta$ (where $\delta|_\Delta$ is the restriction of δ to Δ), then $X_{\delta'} \rightarrow X_\delta$ is an isomorphism. An object X is strictly supported by Δ , if, under the conditions above, $X_{\delta'} \rightarrow X_\delta$ is an identity.

Each of the categories $\mathbf{GR}[\Delta]$ is isomorphic to the full subcategory of $\mathbf{GR}$ consisting of the objects strictly supported by Δ and is equivalent to the category $\mathbf{GR}[\Delta]$ of objects supported by Δ . Note that up to the equivalence $\mathbf{GR}[\Delta] \simeq \mathbf{GR}[\Delta]$ the functors $i^*: \mathbf{GR}[\Delta] \rightarrow \mathbf{GR}[\Delta, \kappa]$ are simply inclusions of subcategories of $\mathbf{GR}$.

6.1 Extending operations to $\mathbf{GR}$

Lemma 6. The embeddings $\mathbf{GR}[\Delta] \rightarrow \mathbf{GR}$ preserve the locally cartesian closed structure and W -types, likewise do $\mathbf{GR}[\Delta'] \rightarrow \mathbf{GR}[\Delta]$ for $\Delta' \subseteq \Delta$.

The proof of Lemma 6 is similar to that of Lemma 4.

Also the operations $\blacktriangleright^\kappa$ and i_* can be extended to $\mathbf{GR}$. To state the properties of i_* , define the subcategory $\mathbf{GR}[\text{CV} \setminus \kappa]$ of $\mathbf{GR}$ on objects supported by $\text{CV} \setminus \kappa$.

Proposition 4. For any κ there are functors

$$\begin{aligned} \blacktriangleright^\kappa &: \mathbf{GR} \rightarrow \mathbf{GR} \\ \text{Lim}_\kappa &: \mathbf{GR} \rightarrow \mathbf{GR}[\text{CV} \setminus \kappa] \end{aligned}$$

such that Lim_κ is right adjoint to the inclusion, and such that the left square below commutes and the right is a map of adjunctions whenever $\kappa \notin \Delta$.

$$\begin{array}{ccc} \mathbf{GR}[\Delta, \kappa] & \xrightarrow{\blacktriangleright^\kappa} & \mathbf{GR}[\Delta, \kappa] \\ \downarrow & & \downarrow \\ \mathbf{GR} & \xrightarrow{\blacktriangleright^\kappa} & \mathbf{GR} \end{array} \quad \begin{array}{ccc} \mathbf{GR}[\Delta, \kappa] & \xleftarrow[i_*]{i^*} & \mathbf{GR}[\Delta] \\ \downarrow & & \downarrow \\ \mathbf{GR} & \xleftarrow[\text{Lim}_\kappa]{\perp} & \mathbf{GR}[\text{CV} \setminus \kappa] \end{array}$$

Moreover, the unit of the extended adjunction is still an isomorphism, and if X is in $\mathbf{GR}[\text{CV} \setminus \kappa]$ then the functor

$$\mathbf{GR}/X \rightarrow \mathbf{GR}[\text{CV} \setminus \kappa]/X$$

mapping $p_Z: Z \rightarrow X$ to $\eta_X^{-1} \circ \text{Lim}_\kappa p_Z$ is right adjoint to the inclusion. If X is in $\mathbf{GR}[\Delta]$ then this adjunction extends the one of Proposition 2.

6.2 Universes in $\mathbf{GR}$

The model construction uses universes for two purposes: to solve the coherence problem as indicated above and to model the universes of the calculus. We need different set theoretic universes for each purpose and so we strengthen Assumption 1.

Assumption 2. For the rest of the paper we will assume that we are given two Grothendieck universe U, U' in $\mathbf{Set}$, such that $U \subseteq U'$,

$U \in U'$. Moreover, we will assume that U' has unique names, i.e., if $X, Y \in U'$ and $X \cong Y$ then $X = Y$.

Using these universes we can construct the universes V and V_Δ (for each Δ) in GR:

$$\begin{aligned} V(\delta) &= \{X : (\downarrow \delta)^{\text{op}} \rightarrow U'\} \\ V_\Delta(\delta) &= \{X : (\downarrow \delta)^{\text{op}} \rightarrow U \mid X \text{ strictly supported by } \Delta\} \end{aligned}$$

where U' is the full subcategory of **Set** on objects of U' , X as usual is required to be a functor, $\downarrow \delta$ is defined as in Section 5, and X strictly supported by Δ should be understood as in Definition 3. The projections

$$\text{El}_V : E_V \rightarrow V \quad \text{El}_\Delta : E_\Delta \rightarrow V_\Delta$$

are defined as in (10). Note that $\text{El}_\Delta : E_\Delta \rightarrow V_\Delta$ is isomorphic to the universe obtained by including that of Proposition 3 into GR.

Proposition 5. *Both $E_V \rightarrow V$ and $E_\Delta \rightarrow V_\Delta$ are universes in GR.*

Proof. That V is a universe is proved in [12]. Since $\text{GR} \cong \text{GR}[\Delta]^{\mathbb{D}^{\text{op}}}$ (where $\mathbb{D} = \omega^{\text{CV} \setminus \Delta}$), Lemma 4 shows that the universe V_Δ of $\text{GR}[\Delta]$ is also a universe in GR. $\square$

Proposition 6. *The object V_Δ is an object of $\overline{\text{GR}}[\Delta]$.*

We will use the universe V for the coherent model, i.e., type judgements will be interpreted as morphisms with V as codomain, and V_Δ will be used to interpret U_Δ .

The requirement of uniqueness of names in Assumption 2 corresponds to the following category theoretic property.

Definition 4. *Let $E \rightarrow U$ be a universe in a category $\mathbb{C}$. We say that $E \rightarrow U$ has unique names if whenever $f, g : \Gamma \rightarrow U$ are codes for the same map $A \rightarrow \Gamma$ it is the case that $f = g$.*

Proposition 7. *Both $E_V \rightarrow V$ and $E_\Delta \rightarrow V_\Delta$ have unique names.*

Lemma 7. *Each V_Δ is small with respect to V . We write $\bar{V}_\Delta : 1 \rightarrow V$ for the code.*

7. Universe structures

It is essential to the coherent interpretation that type constructors can be encoded as morphisms on the universe V . For example, for binary product types we need a map $\bar{\times} : V \times V \rightarrow V$ such that if $\bar{A}, \bar{B} : X \rightarrow V$ are codes for A, B in the slice over X then $\bar{\times} \circ \langle \bar{A}, \bar{B} \rangle$ is a code for the product of A and B as computed in the slice category. In the cases of dependent products and sums the domains of the codes of these operations is the object $V^{(1)}$ defined as the exponential

$$(V \times V \rightarrow V)^{\text{El}_V}$$

in the category $\mathbb{C}/V$. This object is essentially the interpretation of the $\sum X : V. V^X$ and enjoys the universal property that morphisms $X \rightarrow V^{(1)}$ correspond naturally to pairs $\langle \bar{B}, \bar{A} \rangle$ where $\bar{B} : X \rightarrow V$ and $\bar{A} : B \rightarrow V$ where B is the pullback of E_V along $\bar{B}$.

Lemma 8 ([13, 19]). *There are unique maps $\prod, \sum, W : V^{(1)} \rightarrow V$ encoding dependent products, sums and W-types in the sense that if $p_A : A \rightarrow B$ and $p_B : B \rightarrow X$ are small with codes $\bar{B}$ and $\bar{A}$ respectively, then $\prod \circ \langle \bar{B}, \bar{A} \rangle, \sum \circ \langle \bar{B}, \bar{A} \rangle$ and $W \circ \langle \bar{B}, \bar{A} \rangle$ are codes of $\prod_{p_B} p_A, p_B \circ p_A$, and the W-type induced by p_A, p_B respectively.*

The uniqueness statement of Lemma 8 follows from Proposition 7. Likewise there are unique maps encoding binary products and sums, as well as $0, 1, 1 + 1$.

Definition 5. A universe structure is a universe together with a choice of codes encoding dependent products, sums, W-types, unit, $0, 1 + 1$. A universe embedding is a monomorphism $i : U \rightarrow U'$ between universes such that i is a code for $E \rightarrow U$, and which commutes with the universe structure.

Proposition 8. *The inclusions $\text{in}_\Delta : V_\Delta \rightarrow V$ and $\text{in}_{\Delta', \Delta} : V_{\Delta'} \rightarrow V_\Delta$ are universe embeddings.*

Proof. This follows from [19, Remark 9.5]. $\square$

Remark 2. As mentioned, types in context will be modelled as morphisms into V , and V_Δ will be used to model U_Δ . The universe structure on V will be used to model the type constructors and the universe structure on V_Δ will be used to model the corresponding terms acting on the universe type. Proposition 8 ensures that the type equality of Figure 5 hold.

Without the assumption of uniqueness of names in the set theoretic universes of Assumption 2 it would only be possible to model these type equalities as isomorphisms. In fact, without this assumption the canonical choice of encoding of function space as a universe map, does not commute with the universe inclusions.

7.1 Encoding operations for guarded recursion

We now show how to construct codes for the operations specific to our calculus.

Lemma 9. *There is a map $\triangleright^\kappa : \triangleright^\kappa V \rightarrow V$ encoding $\triangleright^\kappa$ in the sense that if Γ is an object of GR and $\bar{B} : \Gamma \rightarrow V$ is a code for $B \rightarrow \Gamma$ then $\triangleright^\kappa \circ \text{next}^\kappa \circ \bar{B}$ is a code for $\triangleright^\kappa B \rightarrow \Gamma$.*

The next lemma uses the notation $\hat{f} : X \rightarrow \text{Lim}_\kappa Y$ for the adjoint correspondent to $f : X \rightarrow Y$ under the adjunction of Proposition 4 whenever X in $\text{GR}[\text{CV} \setminus \kappa]$.

Lemma 10. *There is a map $\bar{\nabla} \kappa : \text{Lim}_\kappa(V) \rightarrow V$ encoding the functor of Proposition 2 for all Δ in the sense that if Γ is in $\text{GR}[\text{CV} \setminus \kappa]$ and if $p_B : B \rightarrow \Gamma$ is an object of GR/Γ and $\bar{B} : \Gamma \rightarrow V$ is a code for p_B then $\bar{\nabla} \kappa \circ \hat{\bar{B}}$ is a code for $\eta^{-1} \circ i_* p_B$.*

Proof. $\text{Lim}_\kappa V(\delta)$ is the set of diagrams of the form

$$F : \{\delta' \in \omega^{\text{CV}} \mid \delta' \upharpoonright_{\text{CV} \setminus \kappa} \leq \delta \upharpoonright_{\text{CV} \setminus \kappa}\}^{\text{op}} \rightarrow U'.$$

Define $\bar{\nabla} \kappa(F)(\delta')$ to be the limit of $F(\delta'[\kappa \mapsto -])$. $\square$

Both these maps restrict to V_Δ such that the following diagrams commute.

$$\begin{array}{ccc} \text{Lim}_\kappa V_\Delta & \xrightarrow{\bar{\nabla} \kappa} & V_\Delta \\ \text{Lim}_\kappa(\text{in}_\Delta) \downarrow & & \downarrow \text{in}_\Delta \\ \text{Lim}_\kappa(V) & \xrightarrow{\bar{\nabla} \kappa} & V \end{array} \quad \begin{array}{ccc} \triangleright^\kappa V_\Delta & \xrightarrow{\triangleright^\kappa} & V_\Delta \\ \triangleright^\kappa(\text{in}_\Delta) \downarrow & & \downarrow \text{in}_\Delta \\ \triangleright^\kappa V & \xrightarrow{\triangleright^\kappa} & V \end{array}$$

8. Interpreting the calculus

We have now established the ingredients and can define the interpretation. The basic principles of the interpretation are

- Contexts $\Delta; \Gamma$ are interpreted as objects of GR
- Type judgements in context $\Delta; \Gamma \vdash A$: Type are interpreted as morphisms $\llbracket A \rrbracket : \llbracket \Delta; \Gamma \rrbracket \rightarrow V$ in GR
- The empty context is interpreted as the terminal object, and comprehension, i.e., the rule

$$\frac{\Delta; \Gamma \vdash \quad \Delta; \Gamma \vdash A : \text{Type}}{\Delta; \Gamma, x : A \vdash}$$

$$\begin{aligned}
\llbracket \Delta; \Gamma \vdash X : \text{Type} \rrbracket &= \overline{X} \circ ! \\
\llbracket \Delta; \Gamma \vdash A \times B : \text{Type} \rrbracket &= \overline{\times} \circ \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle \\
\llbracket \Delta; \Gamma \vdash A + B : \text{Type} \rrbracket &= \overline{+} \circ \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle \\
\llbracket \Delta; \Gamma \vdash \prod x : A. B : \text{Type} \rrbracket &= \prod \circ \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle \\
\llbracket \Delta; \Gamma \vdash \sum x : A. B : \text{Type} \rrbracket &= \sum \circ \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle \\
\llbracket \Delta; \Gamma \vdash Wx : A. B : \text{Type} \rrbracket &= W \circ \langle \llbracket A \rrbracket, \llbracket B \rrbracket \rangle \\
\llbracket \Delta; \Gamma \vdash \text{next}^\kappa A : \text{Type} \rrbracket &= \text{next}^\kappa \circ \llbracket A \rrbracket \\
\llbracket \Delta; \Gamma \vdash \forall \kappa. A : \text{Type} \rrbracket &= \overline{\forall \kappa} \circ \widehat{\llbracket A \rrbracket} \\
\llbracket \Delta; \Gamma \vdash U_{\Delta'} : \text{Type} \rrbracket &= \overline{V_{\Delta'}} \circ !
\end{aligned}$$

Figure 6. Interpretation of types. In the first line X ranges over 0, 1, 2, and ! is the unique map to the terminal object

is interpreted using pullback:

$$\begin{array}{ccc}
\llbracket \Delta; \Gamma, x : A \rrbracket & \longrightarrow & \text{El}_V \\
\downarrow p_{\Delta; \Gamma, x : A} & \lrcorner & \downarrow \\
\llbracket \Delta; \Gamma \rrbracket & \xrightarrow{\llbracket A \rrbracket} & V
\end{array}$$

- Term judgements $\Delta; \Gamma \vdash t : A$ are interpreted as sections of $p_{\Delta; \Gamma, x : A}$ in the sense of Section 4.

Figure 6 summarizes the interpretation of types using the universe structure on V constructed in Section 7.

Theorem 3 (Soundness). *Under Assumption 2, the interpretation of types and contexts defined above can be extended to an interpretation of the type theory which is well-defined and sound with respect to the equality theory of Section 2. Moreover, all the terms of Figure 4 and (5) are interpreted as isomorphisms.*

For reasons of space, we omit the interpretation of terms, most of which is standard. For example, the rules for forming terms in the universe of Figure 5 are interpreted using the universe structure on V_{Δ} . The interpretation of a term $\llbracket \Delta; \Gamma \vdash t : U_{\Delta'} \rrbracket$ corresponds to a map $\llbracket t \rrbracket : \llbracket \Delta; \Gamma \rrbracket \rightarrow V_{\Delta'}$ and $\text{El}(t)$ is interpreted as $\text{in}_{\Delta'} \circ \llbracket t \rrbracket : \llbracket \Delta; \Gamma \rrbracket \rightarrow V$. The type equalities of Figure 5 follow from the map $\text{in}_{\Delta'}$ preserving the universe structure and $\triangleright^\kappa, \overline{\forall \kappa}$.

We end by stating the following two lemmas which are crucial to the interpretation.

Lemma 11. *If $\Delta; \Gamma$ is a well-formed context then $\llbracket \Delta; \Gamma \rrbracket$ is an object of $\text{GR}[\Delta]$.*

Lemma 12. *Let $\kappa \notin \Delta$, then*

- if $\Delta; \Gamma$ is a well-formed context then $\llbracket \Delta; \Gamma \rrbracket = \llbracket \Delta, \kappa; \Gamma \rrbracket$
- If $\Delta; \Gamma \vdash A : \text{Type}$ then
$$\llbracket \Delta; \Gamma \vdash A : \text{Type} \rrbracket = \llbracket \Delta, \kappa; \Gamma \vdash A : \text{Type} \rrbracket$$
- If $\Delta; \Gamma \vdash t : A$ then $\llbracket \Delta; \Gamma \vdash t : A \rrbracket = \llbracket \Delta, \kappa; \Gamma \vdash t : A \rrbracket$

9. Conclusions and future work

Guarded recursive types are a powerful tool, whose properties are still to be fully understood. This paper not only shows how they can be used for productive coprogramming with coinductive types in a dependently typed setting, but may also be useful for development of reasoning principles for guarded recursive types.

For future work, I would like to extend the present work to the case of intensional type theory, building on the results of [8] which show how to construct intensional models of guarded recursion. Note that the model construction used in this paper constructs locally cartesian closed categories which are all models of extensional type theory. In the setting of intensional type theory it would be interesting to see if the constructions of this paper are compatible with other principles of type theory, in particular the univalence axiom.

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