

Computational complexity of iterated maps on points and sets (Extended Abstract)

Christoph Spandl

Computer Science Department, Universität der Bundeswehr München, 85577
Neubiberg, Germany
`christoph.spandl@unibw.de`

Consider a *discrete dynamical system* (M, f) where $M \subseteq \mathbb{R}^n$ is a *box*, that is a set of the form $M = I_1 \times \dots \times I_n$ where $I_i \subseteq \mathbb{R}$, $i = 1, \dots, n$, are compact intervals. Additionally, $f: M \rightarrow M$ is assumed to be a C^2 -diffeomorphism. The dynamics is governed by the *iteration equation*

$$\begin{aligned} x^{(k+1)} &= f(x^{(k)}) \\ x^{(0)} &\in M. \end{aligned} \tag{1}$$

The above equation describes the dynamics on points. However, in practice the initial condition is usually not given as a point $x^{(0)} \in M$ but may be given as a (small) box $I^{(0)} \subseteq M$ accounting for physical imprecision. To meet this, dynamics has to be extended to sets. Let a compact set $A \subseteq M$ be given, then the above iteration can be extended to sets by the iteration $A^{(k+1)} = f(A^{(k)})$, $A^{(0)} = A$. This approach is formally straightforward but complicated in detail, if computational complexity considerations are concerned. Therefore, in the present work, besides the dynamics of points, also the dynamics of boxes enclosing the concrete sets are considered. This means the following. Denote by $\mathbb{M}$ the set of all boxes $I \subseteq M$ and let $f_{\mathbb{M}}: \mathbb{M} \rightarrow \mathbb{M}$ be defined as follows: $I = [a_1, b_1] \times \dots \times [a_n, b_n] \mapsto f_{\mathbb{M}}(I) = [a'_1, b'_1] \times \dots \times [a'_n, b'_n]$ where $a'_i = \min(f_i(I))$, $b'_i = \max(f_i(I))$ for $i = 1, \dots, n$. Then the dynamical system $(\mathbb{M}, f_{\mathbb{M}})$ describes a dynamics of boxes $I \subseteq M$ and gives in each iteration step the tightest enclosure: $f_{\mathbb{M}}(I)$ is the smallest box with the property $f(I) \subseteq f_{\mathbb{M}}(I)$. Hence $f_{\mathbb{M}}$ is an inclusion function for f . If f is computable, then $f_{\mathbb{M}}$ also is computable (see [6], Lemma 5.2.6 and Example 5.2.11). However, the time complexity for computing the iteration in the case of chaotic dynamics (that is for systems with positive Lyapunov exponents, see below) may be exponential. This is not practical at all. To come to a practical iteration algorithm, a computable inclusion function $\bar{f}_{\mathbb{M}}: \mathbb{M} \rightarrow \mathbb{M}$ for f is needed. The presented one is a *centered form*, a notion coming from interval analysis [3]. Assume that f and Df are computable in polynomial time, and also a kind of Lipschitz constant L_{ij} is assumed to be known, which fulfills $\sum_k |(D^2 f)_{ijk}(M)| \leq L_{ij}$ ($D^2 f$ is the second derivative of f). Then there exists a polynomial time computable inclusion function $\bar{f}_{\mathbb{M}}$ having some nice properties.

The derivation of $\bar{f}_{\mathbb{M}}$ is based on a well known technique from stability analysis in dynamical systems theory based on a linearization of the dynamics [1].

The result of the linearization is a second dynamical system, called the *system of variations*:

$$\begin{aligned} x^{(k+1)} &= f(x^{(k)}) \\ z^{(k+1)} &= (Df)(x^{(k)}) \cdot z^{(k)} \\ x^{(0)} &\in M, \quad z^{(0)} = \mathbb{1} \end{aligned}$$

where $z^{(k)}$ is an $n \times n$ -matrix. The systems of variations is a special case of a cocycle [1]. Also, the dynamics of boxes by $\bar{f}_{\mathbb{M}}$ can be formulated in terms of a modified version of a cocycle, here called the *inclusion cocycle*:

$$\begin{aligned} x^{(k+1)} &= f(x^{(k)}) \\ \mathbf{z}^{(k+1)} &= V(x^{(k)}, \mathbf{e}^{(k)}) \cdot \mathbf{z}^{(k)} \\ x^{(0)} &\in M, \quad \mathbf{z}^{(0)} = \mathbb{1} \end{aligned}$$

with

$$\begin{aligned} V(x, \mathbf{e}) &= (Df)(x) + \frac{1}{2} \|\mathbf{e}\|_{\infty} \operatorname{sgn}((Df)(x)) : L \\ \mathbf{e}^{(k)} &= \mathbf{z}^{(k)} \cdot \mathbf{e}^{(0)} \\ \mathbf{e}^{(0)} &\in \mathbb{IR}_s^n \text{ s.t. } x^{(0)} + \mathbf{e}^{(0)} \in \mathbb{M}. \end{aligned}$$

Here, $\mathbb{IR}_s^n$ denotes all symmetric boxes $\mathbb{IR}_s^n = \{[-a, a] \subseteq \mathbb{R}^n \mid a \in \mathbb{R}^n\}$, $\mathbf{e}^{(k)} \in \mathbb{IR}_s^n$ are symmetric boxes and $\mathbf{z}^{(k)}$ interval matrices. The matrix $\operatorname{sgn}((Df)(x)) : L \in \mathbb{R}^{n \times n}$ is defined by $(\operatorname{sgn}((Df)(x)) : L)_{ij} = \operatorname{sgn}((Df)_{ij}(x)) \cdot L_{ij}$ for all $i, j = 1, \dots, n$ and $\operatorname{sgn}(a)$ is the sign of $a \in \mathbb{R}$.

The inclusion cocycle is a certified generalization of the usual cocycle such that the enclosure

$$f^k(x^{(0)} + \mathbf{e}^{(0)}) \subseteq x^{(k)} + \mathbf{z}^{(k)} \cdot \mathbf{e}^{(0)}$$

holds exactly. To achieve a reduction of overestimation of the enclosing boxes, a technique is applied which is similar to one, when Lyapunov exponents [2, 1] of the dynamical system (M, f) are computed numerically. The system of variations gives

$$z^{(k)} = (Df)(x^{(k-1)})(Df)(x^{(k-2)}) \cdot \dots \cdot (Df)(x^{(0)}).$$

The idea now is to perform successively a QR -decomposition of the matrix $(Df)(x^{(k)})Q^{(k)}$, where $Q^{(k)}$ is the orthogonal matrix obtained in the previous step and starting with $Q^{(0)} = \mathbb{1}$. In detail:

$$\begin{aligned} Q^{(k+1)}R^{(k+1)} &= (Df)(x^{(k)})Q^{(k)} \\ Q^{(0)} &= \mathbb{1} \end{aligned}$$

where $Q^{(k)}$ is an orthogonal matrix and $R^{(k)}$ an upper triangular matrix with positive diagonal entries. Using this decomposition,

$$z^{(k)} = Q^{(k)}R^{(k)}R^{(k-1)} \cdot \dots \cdot R^{(1)}$$

is obtained. Let $\lambda_i^{(k)}$, $i = 1, \dots, n$, be the diagonal entries of $R^{(k)} R^{(k-1)} \dots R^{(1)}$. Then

$$\lim_{k \rightarrow \infty} \frac{1}{k} \ln(\lambda_i^{(k)})$$

exists for μ -almost all $x \in M$, if an ergodic invariant measure μ of (M, f) exists (for the basic concepts of ergodic theory see e.g. [4]). These n values are called the Lyapunov exponents. A similar procedure can also be applied in the case of an inclusion cocycle. Then the following proposition holds.

Proposition 1. *Consider the inclusion cocycle. Let $e^{(k)} = [-1, 1] \cdot e^{(k)}$ with $e^{(k)} \in \mathbb{R}_+^n$. Then*

$$e^{(k)} = |Q^{(k)}| \left(\prod_{i=1}^k |R^{(i)}| \cdot e^{(0)} \right)$$

holds, where $Q^{(k)}$ and $R^{(k)}$ are given by the decomposition

$$\begin{aligned} Q^{(k+1)} R^{(k+1)} &= V(x^{(k)}, e^{(k)}) Q^{(k)} \\ Q^{(0)} &= \mathbb{1}. \end{aligned}$$

In the present work, the space complexity of the above iteration scheme is analyzed. Therefore, a computational model is specified. Let $\widehat{\mathbb{R}}(p, \beta)$ be a *fixed point number system* where $\beta \geq 2$ is the *base* and $p \in \mathbb{Z}$ the *precision*: $\widehat{\mathbb{R}}(p, \beta) = \{x \in \mathbb{R} \mid \exists s \in \mathbb{Z} . x = s \cdot \beta^{-p}\}$. A fixed point number $\hat{x} \in \widehat{\mathbb{R}}(p, \beta)$ *approximates* a real number $x \in \mathbb{R}$, if the enclosure property $x \in \hat{x} + [-1, 1] \cdot \beta^{-p}$ holds. Then any real number $x \in \mathbb{R}$ can be represented as a sequence $(\hat{x}_n)_{n \in \mathbb{N}}$ of fixed point numbers $\hat{x}_n \in \widehat{\mathbb{R}}(p_n, \beta)$, each $\hat{x}_n$ approximating x and $\lim_{n \rightarrow \infty} p_n = \infty$. Such a sequence $(\hat{x}_n)_{n \in \mathbb{N}}$ representing x is called a *fixed point identifier* of x . Having names for real numbers, it is a straightforward task to characterize computable real functions $f: \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\text{dom}(f)$ open, by the concept of approximations. Let $\widehat{\mathbb{R}}(\beta) = \bigcup_{p \in \mathbb{Z}} \widehat{\mathbb{R}}(p, \beta)$ be the set of arbitrary precision fixed point numbers. If the base β is fixed, β is omitted in the notation and the above set is denoted by $\widehat{\mathbb{R}}$. A function $\hat{f}: \subseteq \widehat{\mathbb{R}}^n \rightarrow \widehat{\mathbb{R}}^m \cup \{\text{udf}, \text{usp}\}$ is called an *approximation function* of f , if whenever $\hat{x} \in \widehat{\mathbb{R}}^n$ approximates $x \in \text{dom}(f)$ and $\hat{f}(\hat{x}) \in \widehat{\mathbb{R}}^m$, then $\hat{f}(\hat{x})$ approximates $f(x)$. The interpretation of *usp* is ‘unspecified’, that is no approximation can be given. The interpretation of *udf* is ‘undefined’, that is $\hat{x} + [-1, 1] \cdot \beta^{-p}$ is not in the domain of definition of f . The function $\hat{f}$ is called *approximation continuous* with respect to f , if $\hat{f}$ approximates f and if for any fixed point identifier $(\hat{x}_k)_{k \in \mathbb{N}}$ of $x \in \text{dom}(f)$, for some $N \in \mathbb{N}$, $\hat{f}(\hat{x}_k) \in \widehat{\mathbb{R}}^m$ holds for all $k \geq N$ and $(\hat{f}(\hat{x}_k))_{k \geq N}$ is a fixed point identifier of $f(x)$. Since $\widehat{\mathbb{R}}$ is a countable set, the notion of computability for $\hat{f}: \subseteq \widehat{\mathbb{R}}^n \rightarrow \widehat{\mathbb{R}}^m \cup \{\text{udf}, \text{usp}\}$ can be defined generically using classical recursion theory. This can be used to define the notion of computability for functions $f: \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\text{dom}(f)$ open. The function f is called *computable* if it has a computable, approximation continuous approximation function. It is not hard to show that a real function $f: \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\text{dom}(f)$ open, is computable in the above definition iff it is computable and

has a recursively enumerable domain in the definition of computable Analysis [6].

Now come back to the iteration (1). In a simulation of the dynamical system, only a finite segment $(x^{(k)})_{0 \leq k \leq N}$ for some $N \in \mathbb{N}$ of the true orbit is computed. Let $N \in \mathbb{N}$ be fixed. For the segment $(\hat{x}^{(k)})_{0 \leq k \leq N}$ of the computed pseudo orbit, a predefined accuracy is demanded:

$$\begin{aligned}\hat{x}^{(k)} &\in \widehat{\mathbb{R}}^n(p^o, \beta) \\ x^{(k)} &\in \hat{x}^{(k)} + [-1, 1] \cdot \beta^{-p^o}\end{aligned}$$

for all $k \leq N$, where $p^o \in \mathbb{Z}^n$, are fixed precisions. Here, β^{-p} is an abbreviation for the vector $(\beta^{-p_1}, \dots, \beta^{-p_n})^T$. In practice, this means that actually an iteration

$$\begin{aligned}\hat{x}^{(k+1)} &= \hat{f}(\hat{x}^{(k)}) \\ \hat{x}^{(0)} &\in \widehat{\mathbb{R}}^n(p^s, \beta)\end{aligned}\tag{2}$$

is performed, where $\hat{f}$ approximates f . Furthermore, an error term $\bar{e}^{(k)} \in \mathbb{R}_+^n$ is determined with

$$x^{(k)} \in \hat{x}^{(k)} + [-1, 1] \cdot \bar{e}^{(k)}\tag{3}$$

and the constraint

$$\bar{e}^{(k)} \leq \beta^{-p^o}.\tag{4}$$

The inequality above is interpreted each coordinate separately. The main results of this work are generalizations of that derived in [5]. The *loss of significance rate* defined in [5] is extended to self mappings in n dimensions:

Definition 2. Let $p_i^{min}(x, N, p^o)$, $i = 1, \dots, n$, be the minimal precision for p_i^s in (2) such that (3) and (4) hold for all $k \leq N$. Here, $x \in M$ is the initial condition. Then the growth rate of $p^{min}(x, N, p^o)$ is given by

$$\sigma(x, p^o) = \limsup_{N \rightarrow \infty} \frac{p^{min}(x, N, p^o)}{N}.\tag{5}$$

The loss of significance rates $\sigma: M \rightarrow \mathbb{R}^n$ are defined by

$$\sigma(x) = \lim_{p \rightarrow \infty} \sigma(x, p)\tag{6}$$

where $p \rightarrow \infty$ is interpreted each coordinate separately.

The loss of significance rates actually give, asymptotically, the loss of precision in each iteration step. Since the loss of precision is typically proportional to the number of iterations, the loss of significance rate is a natural complexity measure. Then the shown results are:

1. Lower bounds on the loss of significance rates for the iteration of points and boxes are given by the Lyapunov exponents of the dynamical system (M, f) .
2. There exists an algorithm computing segments of orbits of the dynamical system with loss of significance rates bounded from above by the Lyapunov exponents.

References

1. L. Barreira and Y. B. Pesin. *Introduction to Smooth Ergodic Theory*, volume 148 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2013.
2. P. Collet and J.-P. Eckmann. *Concepts and Results in Chaotic Dynamics*. Theoretical and Mathematical Physics. Springer-Verlag, Berlin, Heidelberg, 2006.
3. R. Krawczyk. Centered forms and interval operators. *Computing*, 34:243–259, 1985.
4. D. Ruelle. *Chaotic Evolution and Strange Attractors*. Cambridge University Press, Cambridge, 1989.
5. C. Spandl. Computational complexity of iterated maps on the interval. *Mathematics and Computers in Simulation*, 82:1459–1477, 2012.
6. K. Weihrauch. *Computable Analysis*. Springer-Verlag, Berlin Heidelberg New York, 2000.