Corrections are displayed in Red

Lemma 3.34. *For any frame $\mathcal{F}$, $\Vdash_{\mathcal{F}} \vartheta$ implies $\Vdash_{\mathcal{F}} \chi$.*

Proof. Let $\mathcal{F} = (W, R)$ with $\Vdash_{\mathcal{F}} \vartheta$. Moreover, let $\mathcal{M} = (\mathcal{F}, V)$ and $w \in W$. We have to show that $\mathcal{M}, w \models \chi$.

We make a case distinction as follows:

1. $\mathcal{M}, w \not\models \Box\Diamond\top$.
2. $\mathcal{M}, w \models \Box\Diamond\top$.

In Case 1, $\mathcal{M}, w \models \Diamond\Box\bot$ and thus $\mathcal{M}, w \models \chi$ as required.

Now for Case 2. It is our aim to show that $\mathcal{M}, w \models \Box\bot$.

Assume to the contrary $\mathcal{M}, w \not\models \Box\bot$. Then there is a $v \in W$ with $(v, w) \in R$. Take a new model $\mathcal{M}' = (\mathcal{F}, V')$, where $V'(p) = W \setminus \{v\}$. Since $\mathcal{M}, w \models \Box\Diamond\top$, we have $\mathcal{M}', w \models \Box\Diamond\top$. Thus $\mathcal{M}, w \models \Box(\Box(\Box p \to p) \to p)$ and $\mathcal{M}', v \models (\Box(\Box p \to p) \to p)$.

Since $\mathcal{M}', w \not\models p$, it follows that $\mathcal{M}', v \not\models \Box(\Box p \to p)$. Hence there is a $u \in W$ such that $(v, u) \in R$ and $\mathcal{M}, u \not\models \Box p \to p$, i.e. $\mathcal{M}, u \models \Box p$ and $\mathcal{M}, u \models \neg p$. Since $V'(p) = W \setminus \{v\}$, $\mathcal{M}, u \models \neg p$ yields $u = w$. Thus $(u, u) \in R$. Since $\mathcal{M}, v \models \neg p$, we obtain a contradiction to $\mathcal{M}, v \models \Box p$. ❑