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2. Exercises for the lecture **Term Rewriting Systems**

Please note that all *emphasized* concepts have been defined in the lecture, i.e., before trying to solve an exercise, look up their definition in the notes you took in the lecture (or in the book I cited on the first exercise sheet).

Exercise 1:

Let $\longrightarrow$ be the *symbolic differentiation relation* introduced in the lecture, i.e.,

$$\mathbf{D}_X(X) \longrightarrow 1 \tag{D1}$$

$$\mathbf{D}_X(Y) \longrightarrow 0 \tag{D2}$$

$$\mathbf{D}_X(\alpha + \beta) \longrightarrow \mathbf{D}_X(\alpha) + \mathbf{D}_X(\beta) \tag{D3}$$

$$\mathbf{D}_X(\alpha \cdot \beta) \longrightarrow \alpha \cdot \mathbf{D}_X(\beta) + \mathbf{D}_X(\alpha) \cdot \beta \tag{D4}$$

(a) Compute the *normal forms* of the following terms:

(i) $\mathbf{D}_X(((X \cdot X) \cdot X) + (X \cdot X))$

(ii) $\mathbf{D}_X((X \cdot Y) + (Y \cdot Y))$.

(b) Prove that $\longrightarrow$ is *terminating*.

Exercise 2:

A **group** can be defined by the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{I1}$$

$$e \circ x \approx x \tag{I2}$$

$$i(x) \circ x \approx e \tag{I3}$$

(a) Prove that groups satisfy

$$x \circ e \approx x \tag{$*$}$$

by showing that $x \circ e$ can be transformed to x using the identities (I1), (I2), and (I3).

(b) Consider the following identity (I3′):

$$x \circ i(x) \approx e \tag{I3'}$$

Prove that (I1), (I2) and (I3′) does not imply $*$.

Hint: Give a counterexample, i.e., a model of (I1), (I2) and (I3'), in which ($*$) does not hold—such a model exists with only two elements.

Exercise 3:

Consider the following identities:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{R1}$$

$$(x \circ y) \circ x \approx x \tag{R2}$$

Prove or disprove whether the following identities are implied by (R1) and (R2):

(a) $x \circ x \approx x$?

(b) $(x \circ y) \circ z \approx x \circ z$?

Exercise 4:

Let Σ be a finite alphabet. A **word rewrite system** over Σ is a finite set of pairs of words

$$W = \{(u_1, v_1), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*.$$

For W a word rewrite system, we define $\xrightarrow[W]{} \subseteq \Sigma^* \times \Sigma^*$ as follows: $u \xrightarrow[W]{} v$ iff there are $x, y \in \Sigma^*$ and $(u_i, v_i) \in W$ such that $u = xu_iy$ and $v = xv_iy$. Finally, $\leftrightarrow_W$ is the symmetric closure of $\xrightarrow[W]{}$, $\leftrightarrow_W^*$ the reflexive-transitive closure of $\leftrightarrow_W$, and $\xrightarrow[W]{*}$ the transitive-reflexive closure of $\xrightarrow[W]{}$.
Now consider the word rewrite system $W = \{(ab, bb), (ab, a), (b, aba)\}$ over $\{a, b\}$. Decide whether

(a) $b \xrightarrow[W]{*} aaaa$,

(b) $abb \xrightarrow[W]{*} bbba$,

(c) $bbbb \xrightarrow[W]{*} aa$, and

(d) $bbbb \leftrightarrow_W^* aa$.

Exercise 5:

Consider the following identities E:

$$(x \circ y) \circ z \approx x \circ (y \circ z) \tag{W1}$$

$$(x \circ y) \circ x \approx x \circ y \tag{W2}$$

$$x \circ x \approx x \tag{W3}$$

Devise a decision procedure for the following problem: given two terms s and t over $\circ$, can they be rewritten to the same term using (W1), (W2), and (W3)?
Hint: (W1) implies that parentheses are obsolete, and we can thus work on words/strings rather than on terms. Then we can try to find a way to get short words by applying the identities (W2) and (W3). Using this knowledge, we can describe a procedure that compare such "shortest forms" of two terms.