Lehrstuhl für Automatentheorie
Technische Universität Dresden
Prof. Dr. F. Baader

01062 Dresden
Tel.: 0351/463–39171
Tel.: 0351/463– 38237

3. Exercises for the lecture **Term Rewriting Systems**

Important

This exercise sheet is rather crowded, so it is highly probable that we will not finish its presentation in 90 minutes. This is not a problem: in this case, we will continue its presentation in the following week.

Exercise 1:

Consider the following reduction relations:

- Let M be a set and 2^M the set of all subsets of M. We define the reduction relation $\underset{M}{\longrightarrow}$ on 2^M as follows: $A \underset{M}{\longrightarrow} B$ iff $B \subsetneq A$.
- We define the reduction relation $\underset{p,q}{\longrightarrow}$ on the non-negative integers as follows:

$$\begin{array}{lll} n \underset{p,q}{\longrightarrow} n-p & \text{iff} & n > p, \text{ und} \\ n \underset{p,q}{\longrightarrow} n-q & \text{iff} & n > q. \end{array}$$

For each of the following properties, describe those sets M and those non-negative integers p, q such that $\underset{M}{\longrightarrow}$ and $\underset{p,q}{\longrightarrow}$ satisfy this property.

(a) terminating?

(b) Church-Rosser?

(c) normalising?

(d) confluent?

Exercise 2:

Which of the closure operator defined in the lecture commute? Prove or disprove the validity of each of the following equations:

(a) $(\xrightarrow{+})^{=} = (\xrightarrow{=})^{+}$

(b) $(\longrightarrow \cup \xrightarrow{-1})^{+} = (\xrightarrow{+}) \cup (\xrightarrow{+})^{-1}$

(c) $(\xrightarrow{-1})^{+} = (\xrightarrow{+})^{-1}$

Exercise 3:

In the lecture, we set $A := \mathbb{N} \setminus \{0, 1\}$ and considered the following "divisibility" reduction relation on A:

$$\longrightarrow := \{(m, n) \in A \times A \mid m > n \text{ and there is some } \ell \in A \text{ with } n \cdot \ell = m\}.$$

Prove that $\xleftrightarrow{*} = A \times A$.

Exercise 4:

Consider the Semi-Thue system $\underset{S}{\longrightarrow}$ from the lecture with $S := \{abb \longrightarrow aa, a \longrightarrow b\}$ (remember: $\underset{S}{\longrightarrow} = \{(uabbv, uaav) \mid u, v \in \{a,b\}^*\} \cup \{(uav, ubv) \mid u, v \in \{a,b\}^*\}$)

(a) Prove that $\underset{S}{\longrightarrow}$ is finitely branching.

(b) Prove that $\underset{S}{\longrightarrow}$ is terminating by defining a monotone mapping $\varphi : \{a,b\}^* \longrightarrow \mathbb{N}$, i.e., a mapping such that $u \underset{S}{\longrightarrow} v$ implies $\varphi(u) > \varphi(v)$.

Exercise 5:

Let $\mathbf{ack} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Ackermann function, i.e.,

$$\mathbf{ack}(n,m) = \begin{cases} m+1 & \text{, if } n = 0 \\ \mathbf{ack}(n-1, 1) & \text{, if } m = 0 \\ \mathbf{ack}(n-1, \mathbf{ack}(n, m-1)) & \text{, otherwise} \end{cases} .$$

Choose an appropriate order on pairs $(n,m) \in \mathbb{N} \times \mathbb{N}$ and use well-founded induction to prove that the function **ack** is well-defined (Hint: the Ackermann function is well-defined if it is well-defined for each pair of non-negative integers).

Exercise 6:

Disprove the following claim:

"If $(M, \longrightarrow)$ is a reduction system such that $x \longrightarrow y$ is decidable, then the set $\{x \in M \mid x \text{ is reducible}\}$ is also decidable."

Hint: define a reduction relation $\longrightarrow$ on the non-negative integers such

- $n \longrightarrow m$? is decidable, but
- is n reducible? is undecidable

Exercise 7:

Prove Lemma 2.21 from the lecture:
Lemma 2.21: If $>_A$ is a strict order on A and $>_B$ is a strict order on B, then the lexicographic product $>_{A \times B}$ is a strict order on $A \times B$.