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4. Exercises for the lecture **Term Rewriting Systems**

Exercise 1:

Prove that, for a strict order $\succ$, its induced multiset order $\succ_{mul}$ is a strict order.

Exercise 2:

Let $(A, \succ)$ be a strict order. Prove or disprove the following claims (remember that a is the *smallest* element of a set M iff $a \in M$ and, for all $b \neq a$ in M, we have $b \succ a$):

(a) If A has a smallest element, then $\succ$ is well-founded.

(b) If every subset of A has a smallest element, then $\succ$ is well-founded.

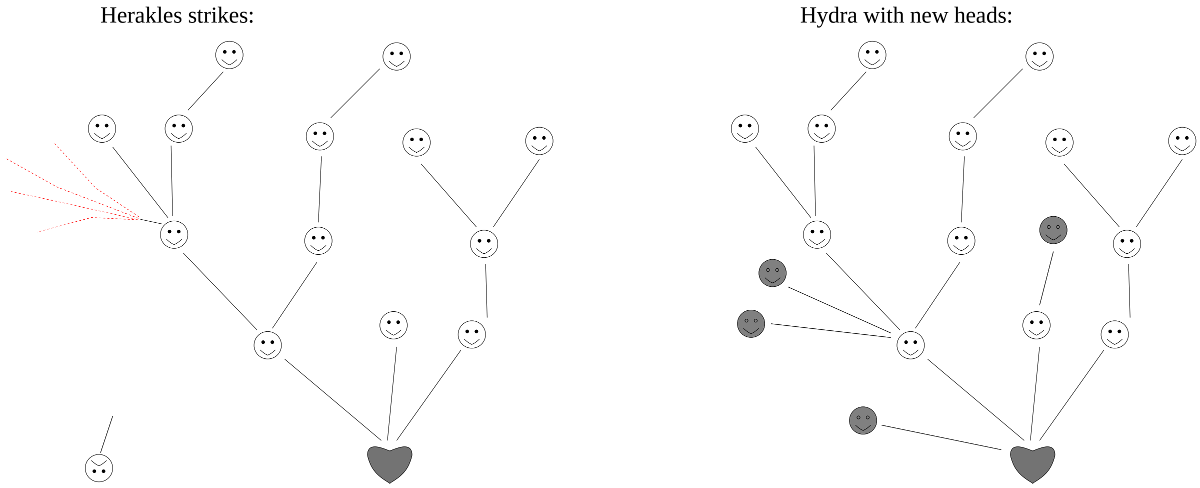
(c) If $\succ$ is well-founded, then, for each element $a \in A$, there are only finitely many b with $a \succ b$.

(d) If $\succ$ is well-founded, then, for each element $a \in A$, there is an $n_a \in \mathbb{N}$ such that each $\succ$-path starting at a is of length at most n_a.

(e) If $\succ$ is well-founded, then each subset of A has a smallest element.

Exercise 3:

Consider the following "tree rewrite system", determine whether it is terminating, and describe Herakles' chances to become immortal.
To become immortal, *Herakles* has to solve several tasks, one of which is to kill *Hydra*: Hydra is a strange being, having one heart and a variety of heads (but only finitely many), each of which is connected via a chain of heads to her heart (see the picture below). With one stroke, Herakles can cut off one of Hydra's heads—but Hydra grows, for each head cut off, an arbitrary, finite number of new heads! Each of the newly grown heads is, however, closer to the heart than the one that was cut off. The only way to kill Hydra is to cut off *all* heads.



Exercise 4:

Let $\succ$ be a strict total[1] order on A and let $\mathcal{M}(A)$ be the set of all finite multisets over A. Consider a mapping τ that maps two multisets $M_1, M_2 \in \mathcal{M}(A)$ to a tuple $\langle \widehat{M_1}, \widehat{M_2} \rangle$ over $A_\perp$ as follows:

- The length of each tupel $\widehat{M_i}$ is the maximum of the cardinalities of M_1 and M_2, where the cardinality of a multiset takes into account multiple occurrences of elements, i.e., $\text{card}(M) = \sum_{a \in A} M(a)$.
- The entries in the tupel $\widehat{M_1}$ (resp. $\widehat{M_2}$) are the elements from M_1 (resp. M_2) in descending order w.r.t. $\succ$. If $\text{card}(M_i) < \text{card}(M_j)$ for some $i, j \in \{1, 2\}$, then the remaining positions in $\widehat{M_i}$ are filled with $\perp$ (as usual, we set $a \succ \perp$ for each $a \in A$).

For example, $\tau(\langle\{3, 1, 1, 4, 4, 17\}, \{2, 1, 1, 8\}\rangle) = \langle\langle 17, 4, 4, 3, 1, 1\rangle, \langle 8, 2, 1, 1, \perp, \perp\rangle\rangle$.
Prove that, for any multisets $M_1, M_2 \in \mathcal{M}(A)$ and $\succ_{lex}$ the lexicographic order induced by $\succ$,

$$M_1 \succ_{mul} M_2 \text{ iff } \tau(M_1, M_2) \in \succ_{lex}$$

[1] An order $\succ$ is total if, for any two elements a, b either $a \succ b$ or $b \succ a$.