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1. Exercises for the lecture **Term Rewriting Systems**

Important

The exercises and other information can be found on the homepage of this lecture at `http://lat.inf.tu-dresden.de/teaching/ss2004/trs/`.
If you have any questions concerning the organisation and the subject of the lecture, the exercises, etc., please do not hesitate to ask me: in room 434, via email (morawska@tcs.inf.tu-dresden.de), by phone (463–39171).

The goal of these first exercises is to recall concepts from discrete algebra that will be important throughout the course. If terms are used whose definition is not clear to you, you can have a look at

Franz Baader and Tobias Nipkow: Term rewriting and all that. Cambridge University Press, 1998.

(which is available at the Computer Science Library) or in any other algebra textbook.

Exercise 1:

A structure $(M, \circ)$ for M a non-empty set and $\circ$ a binary function $M \times M \longrightarrow M$ is a *monoid* if

(a) $\circ$ is associative (i.e., for all $\ell, m, n \in M : \ell \circ (m \circ n) = (\ell \circ m) \circ n$) and

(b) M contains a neutral element (i.e., there is an $e \in M$ such that, for all $m \in M$: $e \circ m = m \circ e = m$).

Let T be the smallest set that contains 0 and, for each $x \in T$, also $s(x)$. We write $s^n(0)$ for $\underbrace{s(\ldots s}_{n-\text{times}}(0)$, and define the operation "+"on T inductively as follows: for each $m, n \geq 0$, we have

$$\begin{aligned} s^n(0) + 0 &= s^n(0) \text{ and} \\ s^n(0) + s^{m+1}(0) &= s(s^n(0) + s^m(0)) \end{aligned}$$

Prove that $(\{s^n(0) \mid n \geq 0\}, +)$ is a commutative monoid by induction on the structure of terms.

Exercise 2:

For $M = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, C_1, C_2, C_3, C_4, D, E\}$ consider the following relation $\rightarrow \subseteq M \times M$:

$$\begin{array}{l} A_i \rightarrow B_j \text{ for } i, j \in \{1, 2\} \\ A_i \rightarrow B_3 \text{ for } i \in \{3, 4\} \\ B_1 \rightarrow C_1,\ B_2 \rightarrow C_2,\ B_2 \rightarrow C_3,\ D \rightarrow C_4 \\ B_3 \rightarrow C_i \text{ for } i \in \{1, 2, 3, 4\} \\ C_i \rightarrow E \text{ for } i \in \{3, 4\} \end{array}$$

(a) Which of the following properties are satisfied by $\rightarrow$?

(i) finite

(ii) symmetric

(iii) antisymmetric

(iv) reflexive

(v) irreflexive

(vi) transitive

(b) A relation r is *smaller than* r' if $r \subseteq r'$. In this case, we also say that r' *contains* r. For p a property on relations and r a relation, the *p closure of* r is the smallest relation containing r that satisfies p.

Describe the following closures of $\rightarrow$:

(i) the symmetric closure of $\rightarrow$

(ii) the reflexive closure of $\rightarrow$

(iii) the transitive closure of $\rightarrow$

(iv) the reflexive-transitive closure of $\rightarrow$

(c) Characterise the properties p for which, for any relation r, the p closure of r exists.

(d) For $r, r' \subseteq M \times M$ relations, its *composition* $r \circ r'$ is defined as follows:

$$r \circ r := \{(\ell, n) \mid \text{there is } m \in M \text{ with } (\ell, m) \in r \text{ and } (m, n) \in r'\}$$

Moreover, we set

$$\begin{array}{rcl} r^0 & := & \{(m, m) \mid m \in M\} \\ r^1 & := & r \\ r^{i+1} & := & r \circ r^i \end{array}$$

Describe $\rightarrow^0$, $\rightarrow^1$, $\rightarrow^2$, and $\rightarrow^3$.

Exercise 3:

A *partial order* on M is a reflexive, transitive, and antisymmetric relation on M.
A *linear order* r on M is a partial order that is total, i.e., for any $m, n \in m$, either $(m, n) \in r$ or $(n, m) \in r$.
An *equivalence relation* is a reflexive, transitive, and symmetric relation.
For each of the following relations, say whether they are a partial or linear order or an equivalence relation. For the equivalence relations, describe the equivalence classes.

(a) “is divisible by” on the non-negative integers

(b) “is less than or equal” on the non-negative integers

(c) “occurs as a sub-string in” on words

(d) “is a sibling of” on human beings

(e) “is in the same course as” on students